1. Solve the IVP

\[ y'' - 20y' + 100y = 100t, \quad y(0) = y'(0) = 0 \]

\[ y'' - 20y' + 100y = 0 \]

\[ (y - 10)^2 = 0 \]

2) \[ y = 10, 10 \]

3) Basis \( e^{10t}, te^{10t} \)

Put \( A \)

\[ y = At + B \]

2) \[ y' = A \rightarrow -20A \]

3) \[ y'' = 0 \]

\[ A = 1 \]

\[ B = \frac{1}{5} \]

5) \[ y = C_1 e^{10t} + C_2 te^{10t} + t + \frac{1}{5} \]

From \[ D = C_1 + \frac{1}{5} \Rightarrow C_1 = -\frac{1}{5} \]

\[ B' = 10C_1 e^{10t} + 10C_2 te^{10t} + C_2 e^{10t} + 1 \]

\[ D = 10(-\frac{1}{5}) + C_2 + 1 = -2 + C_2 + 1 \]

\[ C_2 = 1 \]

\[ y = -\frac{1}{5} e^{10t} + te^{10t} + t + \frac{1}{5} \]
2. Find the general solution to \( u'' + 10u' + 9u = 150 \cos 3t \).

Indicate the transient and the steady-state.

\[
\begin{align*}
\text{1)} & \quad r^2 + 10r + 9 = 0 \\
\text{2)} & \quad (r+1)(r+9) = 0 \quad \Rightarrow \quad r = -1, -9 \\
\text{3)} & \quad e^t, e^{-9t} \\
\text{PuA.} & \quad u = A \cos 3t + B \sin 3t \\
& \quad u' = -3A \sin 3t + 3B \cos 3t \\
& \quad u'' = -9A \cos 3t - 9B \sin 3t \\
\text{4)} & \quad u'' + 9u = 0 \quad \Rightarrow \quad 10u' = 150 \cos 3t \\
& \quad 30B = 150 \quad \Rightarrow \quad B = 5 \\
& \quad u = C_1 e^t + C_2 e^{-9t} + 5 \sin 3t \\
& \quad \text{transient} \quad \text{steady-state}
\end{align*}
\]
3. Solve $u'' + 100u = 198 \cos t$, with $u(0) = 0, u'(0) = 0$.

Write the solution as the product of sines and discuss whether you expect to observe beats.

3) \[ r^2 + 100 = 0 \]
   \[ r = \pm 10 i \]

For $(0,0)$ solution, use cosines

\[ u = A (\cos t - \cos 10t) \quad \text{e.g.} \]

\[ u' = -10 A \sin 10t \]
\[ u'' = -A \cos t + 100 A \cos 10t \]
\[ + 100u = 100A \cos t - 100A \cos 10t \]

3) \[ 198 \cos t = 99A \cos t \]
   \[ 2 = A \]

5) \[ u = 2(\cos t - \cos 10t) \]

\[ = -4 \sin \frac{11t}{2} \sin \frac{9t}{2} \]

3) Ratio $\frac{11}{9}$ near $1$ does not exhibit beats.
4. Find the characteristic roots for the homogeneous equation. For homogeneous equations and for item b., write the general solution. For item f., write the form of the general solution. In every case, classify each equation according to one of the following categories:

A. Solutions are oscillatory and bounded, possibly, but not necessarily, periodic and not converging to 0 as \( t \to \infty \).
B. Solutions are oscillatory with amplitude going to 0 as \( t \to \infty \).
C. Solutions are oscillatory with amplitude going to \( \infty \) as \( t \to \infty \).
D. Solutions are non-oscillatory and converge to 0 as \( t \to \infty \).
E. Solutions are non-oscillatory and are either constant or grow at most polynomially as \( t \to \infty \).
F. All nontrivial solutions diverge exponentially as \( t \to \infty \).
G. There exist both bounded as well as unbounded solutions as \( t \to \infty \).

\[ a. \quad u'' - 81u = 0 \quad r^2 - 81 = 0 \quad r = \pm 9 \]
\[ b. \quad u'' = \sin 3t \quad u' = \frac{1}{2} \cos 3t + C_1 \quad u = -\frac{1}{9} \sin 3t + C_1 t + C_2 \]
\[ c. \quad u'' - 7u' + 13u = 0 \quad r^2 - 7r + 13 = 0 \quad e^{7t/2} \left( C_1 \cos \left( \frac{\sqrt{33}}{2} t \right) + C_2 \sin \left( \frac{\sqrt{33}}{2} t \right) \right) \]
\[ d. \quad u'' - 7u' + 4u = 0 \quad r^2 - 7r + 4 = 0 \quad r = \frac{7 \pm \sqrt{49 - 16}}{2} = \frac{7 \pm 3}{2} = \frac{10}{2} = 5 \quad e^{7t/2} \left( C_1 e^{-\frac{3t}{2}} + C_2 e^{\frac{3t}{2}} \right) \]
\[ e. \quad u'' + 8u' + 7u = 0 \quad (r + 1)(r + 7) = 0 \quad C_1 e^{-t} + C_2 e^{-7t} \]
\[ f. \quad u'' + 49u = \cos t \quad r^2 + 49 = 0 \quad u = \frac{1}{4} \left( C_1 e^{-7t} + C_2 e^{7t} \right) + A \cos t + B \sin t \]

\[ \text{Or:} \quad -A \cos t + \frac{9}{4} A \sin t = \cos t \quad A = \frac{\sqrt{8}}{2} \text{ \cos t = point sol,} \]
E. $u'' + 256u = \cos(3t)$ \hspace{1cm} \text{forced}

beats

D. $u'' + 256u = 2 \cos(15.5t)$

C. $u'' + 256u = 2 \cos(16t)$ \hspace{1cm} \text{res.}

A. $u'' + 0.5u' + 9u = 0$ \hspace{1cm} \text{damped}

B. $u'' + u' + 16u = 2 \cos(4t)$ \hspace{1cm} \text{st. state}