1. a. State the Cauchy-Goursat Theorem (fill in):
If $f$ is analytic at all points interior to and on a simple closed contour $C$, then

b. State the Residue Theorem (fill in):
Let $C$ be a positively-oriented simple closed contour. If a function $f$ is analytic inside and on $C$ except for a finite number of singular points $z_k$ ($k = 1, 2, \ldots, n$) inside $C$, then

2. Let

$$f(z) = \frac{5}{z} + \frac{3 - 2i}{z - 4} - \frac{2}{(z - 4)^2} + \frac{i}{(z + i)^3} + \frac{2i}{(z + i)^5} + (z + 1)^3 - ze^z$$

Locate each singularity, state the order and the residue there.

<table>
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<tr>
<th>Pole</th>
<th>Order</th>
<th>Residue</th>
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3. Let \( f(z) = \frac{1}{z^4 + 4z^2} \).

a. Find the poles of \( f \).

In parts b. and c., evaluate \( \int_C f(z) \, dz \) for each given contour \( C \) using the calculus of residues.

b. Let \( C = \) the unit circle \( \{ z : |z| = 1 \} \). \( \int_C f(z) \, dz = \)

c. Let \( C = \) the circle \( \{ z : |z - 4i| = 3 \} \). \( \int_C f(z) \, dz = \)

4. Let \( C \) denote the rectangle with corners \(-1 - i, -1 + i, 2 + i, 2 - i\). Sketch.

Evaluate \( \int_C f(z) \, dz \) for each given \( f \).

a. \( f(z) = \frac{8z^2 + 1}{z - 3 - i} \). \( \int_C f(z) \, dz = \)

b. \( f(z) = \frac{8z^2 + 1}{(z - 1)^3} \). \( \int_C f(z) \, dz = \)

c. \( f(z) = \frac{8z^2 + 1}{(z - 1)^n} \) for \( n > 3 \). \( \int_C f(z) \, dz = \)
5. a. Show that $\int_{|z|=1} \tan \pi z \, dz = -4i$. (Hint: write as sine over cosine.)

b. Use residue techniques to show that $\int_{-\infty}^{\infty} \frac{dx}{x^2 + 9} = \frac{\pi}{3}$.

6. Let $l$ denote the straight line segment from 2 to $i$ in the complex plane. Let $u(z) = \text{Re } z$.

a. Find $M = \max_{z \in l} |u(z)|$.

b. Using the standard estimate, show that $\left| \int_{l} u(z) \, dz \right| \leq 2\sqrt{5}$.
7. Let \( \phi(z) = \bar{z} \). Let \( C_R \) denote the semicircle \( \{ z : |z| = R, \ 0 \leq \theta \leq \pi \} \).

   a. Using the standard estimate, show that \( \left| \int_{C_R} \phi(z) \, dz \right| \leq \pi R^2 \).

   b. Let \( C \) denote the simple closed contour obtained by adding the line segment \([ -R, R ]\) on the real axis to \( C_R \). Would you expect that \( \int_{C} \phi(z) \, dz = 0 \)? Why/Why not?

8. Let \( g(z) = \left( \frac{z + 1}{z - 1} \right)^5 \). Let \( C \) denote the circle \( \{ z : |z| = 2 \} \). Let, for \( |z_0| < 2 \),

   \[
   f(z_0) = \frac{1}{2\pi i} \int_{C} \frac{g(z)}{(z - z_0)^6} \, dz.
   \]

   a. In the integral for \( f \), substitute in for \( g \) to get a formula for \( f(z_0) \).

   b. Show that \( f(1) = 0 \).

   c. Show that \( f(-1) = -\frac{1}{32} \).