L'Hôpital's Rule and Partial Fractions

NOTE: "PFE" is an acronym for "Partial Fraction Expansion".

1 Using L'Hôpital's Rule for PFE's

You are given a rational function $\dfrac{P(x)}{Q(x)}$, with $\deg(Q) > \deg(P)$, common factors having been cancelled.

1. Let $Q(a) = 0$, $Q'(a) \neq 0$. Then the PFE looks like

$$\frac{P(x)}{Q(x)} = \frac{A}{x-a} + \cdots$$

2. If $Q(a) = 0$, $Q'(a) = 0$, $\ldots$,$Q^{(p-1)}(a) = 0$, $Q^{(p)}(a) \neq 0$, then we have

$$\frac{P(x)}{Q(x)} = \frac{A}{(x-a)^p} + \cdots$$

We will use L'Hôpital's Rule to find the coefficient A in each case.

1. Multiply across by $x - a$ and notice that setting $x = a$ results in the indeterminate form $\dfrac{0}{0}$. Pulling off the factor $P(a)$, using L'Hôpital's Rule, we find

$$A = P(a) \lim_{x \to a} \frac{x-a}{Q(x)} = \frac{P(a)}{Q'(a)}$$

2. In this case, multiplying across by $(x-a)^p$ requires L'Hôpital's Rule to be repeated p times, yielding

$$A = P(a) \lim_{x \to a} \frac{(x-a)^p}{Q(x)} = P(a) \frac{p!}{Q^{(p)}(a)}$$

2 Examples

1.

$$\frac{x}{x^2 - 3x + 2} = \frac{A}{x - 1} + \dots$$

Ans. -1

2.

$$\frac{x^2 + 1}{x^3 - 4x^2 + 5x - 2} = \frac{A}{(x - 1)^2} + \frac{B}{x - 2} + \dots$$

Ans. $-2, 5$

3.

$$\frac{x^2 - x + 1}{x^4 - 5x^3 + 9x^2 - 7x + 2} = \frac{A}{(x - 1)^3} + \frac{B}{x - 2} + \dots$$

Ans. $-1, 3$