L’Hôpital’s Rule and Partial Fractions

NOTE: “PFE” is an acronym for “Partial Fraction Expansion”.

1 Using L’Hôpital’s Rule for PFE’s

You are given a rational function $\frac{P(x)}{Q(x)}$, with $\deg(Q) > \deg(P)$, common factors having been cancelled.

1. Let $Q(a) = 0$, $Q'(a) \neq 0$. Then the PFE looks like

$$\frac{P(x)}{Q(x)} = \frac{A}{x-a} + \cdots$$

2. If $Q(a) = 0$, $Q'(a) = 0, \ldots, Q^{(p-1)}(a) = 0, Q^{(p)}(a) \neq 0$, then we have

$$\frac{P(x)}{Q(x)} = \frac{A}{(x-a)^p} + \cdots$$

We will use L’Hôpital’s Rule to find the coefficient $A$ in each case.

1. Multiply across by $x - a$ and notice that setting $x = a$ results in the indeterminate form $\frac{0}{0}$. Pulling off the factor $P(a)$, using L’Hôpital’s Rule, we find

$$A = P(a) \lim_{x \to a} \frac{x - a}{Q(x)} = \frac{P(a)}{Q'(a)}$$

2. In this case, multiplying across by $(x - a)^p$ requires L’Hôpital’s Rule to be repeated $p$ times, yielding

$$A = P(a) \lim_{x \to a} \frac{(x - a)^p}{Q(x)} = P(a) \frac{p!}{Q^{(p)}(a)}$$
2. Examples

1. \[
\frac{x}{x^2 - 3x + 2} = \frac{A}{x - 1} + \ldots \\
\text{Ans. } -1
\]

2. \[
\frac{x^2 + 1}{x^3 - 4x^2 + 5x - 2} = \frac{A}{(x - 1)^2} + \frac{B}{x - 2} + \ldots \\
\text{Ans. } -2, 5
\]

3. \[
\frac{x^2 - x + 1}{x^4 - 5x^3 + 9x^2 - 7x + 2} = \frac{A}{(x - 1)^3} + \frac{B}{x - 2} + \ldots \\
\text{Ans. } -1, 3
\]