# Numeration,

# Trigometric Identities,

# and

# Cantor-Type Distributions

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Starting from numeration, we derive a family of trigonometric identities. These in turn have a probabilistic interpretation that is reminiscent of Cantor-type distributions, but in fact, we will *avoid* such singular distributions.

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# 1 Numeration in base $b$

Let $b$ be the base of a **system of numeration**, $b \in \mathbb{N}$, $b \geq 2$.

The basic feature of numeration in base $b$ is that every positive integer has a unique expression using "digits" from 0 to $b-1$ in positional notation as expressed by the identity

$$
\begin{aligned}
&1 + t + t^2 + \ldots + t^{b^N - 1} \\
&\quad = (1 + t + t^2 + \ldots + t^{b-1})(1 + t^b + t^{2b} + \ldots + t^{b(b-1)}) \\
&\qquad \cdot (1 + t^{b^2} + t^{2b^2} + \ldots + t^{(b-1)b^2}) \cdots \\
&\qquad \cdot (1 + t^{b^{N-1}} + t^{2b^{N-1}} + \ldots + t^{(b-1)b^{N-1}})
\end{aligned}
$$

**Proof**

$$\frac{1-t^{b^k}}{1-t^{b^{k-1}}} = 1 + t^{b^{k-1}} + t^{2b^{k-1}} + \ldots + t^{(b-1)b^{k-1}} \quad (1)$$

is a geometric series in $t^{b^{k-1}}$ so

$$\frac{1-t^{b^N}}{1-t} = \prod_{k=1}^{N} \frac{1-t^{b^k}}{1-t^{b^{k-1}}} \quad (2)$$

which expanded returns to the numeration identity.

# 2 Identities

## 2.1 Identities for base $b$

In eq. (2), substitute $t^2$ for $t$ and symmetrize:

$$\frac{t^{b^N} - t^{-b^N}}{t - t^{-1}} = \prod_{k=1}^{N} \frac{t^{b^k} - t^{-b^k}}{t^{b^{k-1}} - t^{-b^{k-1}}}$$

Expand the geometric series' in the product and substitute $t = e^{i\theta}$. Taking real parts of each term of the product yields the **base $b$ identity**

$$\frac{\sin b^N \theta}{\sin \theta} = \prod_{k=1}^{N} \left( \sum_{j=0}^{b-1} \cos(b - 2j - 1) b^{k-1} \theta \right) \quad (3)$$

## 2.2 Morrie's identity for base b

Specializing so that the left-hand side becomes equal to 1, we have the class of **Morrie's Identities**. The original Morrie Identity mentioned by Feynman is

$$\cos(20^\circ) \cdot \cos(40^\circ) \cdot \cos(80^\circ) = \frac{1}{8}$$

This corresponds to $b = 2$, $N = 3$.

Setting $\theta = \dfrac{\pi}{b^N + 1}$,

$$1 = \prod_{k=1}^{N} \left( \sum_{j=0}^{b-1} \cos(b - 2j - 1) b^{k-1} \frac{\pi}{b^N + 1} \right)$$

## 2.3 Examples

- **Base 2** yields the version of [Louck, et al.]

$$\frac{\sin 2^N \theta}{\sin \theta} = \prod_{k=1}^{N} \left(2 \cos 2^{k-1}\theta\right) = 2^N \prod_{k=1}^{N} \cos 2^{k-1}\theta \tag{4}$$

And **setting** $\theta = \pi/(1+2^N)$ yields the original family of Morrie identities of [Louck, et al.]:

$$1 = 2^N \prod_{k=1}^{N} \cos \pi \frac{2^{k-1}}{2^N+1}$$

- **Base 3** is interesting as well

$$\frac{\sin 3^N \theta}{\sin \theta} = \prod_{k=1}^{N} 2\left(\frac{1}{2} + \cos 2 \cdot 3^{k-1}\theta\right) \tag{5}$$

$$= 2^N \prod_{k=1}^{N} \left(\frac{1}{2} + \cos 2 \cdot 3^{k-1}\theta\right) \tag{6}$$

- **Base 8**

$$\frac{\sin 8^N\,\theta}{\sin\theta} =$$

$$2^N \prod_{k=1}^{N} \Big(\cos 8^{k-1}\theta + \cos 3\cdot 8^{k-1}\theta + \cos 5\cdot 8^{k-1}\theta + \cos 7\cdot 8^{k-1}\theta\Big) \qquad (7)$$

## 3 Infinite Products. Series of random variables.

The distribution of a sum of independent random variables of the form

$$\sum_{i=1}^{\infty} \alpha_i \varepsilon_i$$

where $\alpha_i$ are constants and $\varepsilon_i$ are independent Bernoulli random variables taking values $\pm 1$ with equal probability is called a **Cantor-type distribution**, as it includes the distribution of the Cantor function on the interval $[0, 1]$.

Here we will show how the trigonometric identities of $\S 3$ may be termed **avoiding** Cantor distributions.

In every case, we recover the **uniform distribution** on $[-1, 1]$ via an appropriate limiting scheme.

For **base 2**,

$$\frac{\sin\theta}{2^N \sin(\theta/2^N)} = \prod_{k=1}^{N} \cos\,\frac{\theta}{2^k} \qquad (8)$$

For **base 3**,

$$\frac{\sin\theta}{3^N \sin(\theta/3^N)} = \prod_{k=1}^{N} \left(\frac{1}{3} + \frac{2}{3}\,\cos\frac{2\theta}{3^k}\right) \qquad (9)$$

For **base 8**,

$$\frac{\sin\theta}{8^N \sin(\theta/8^N)} =$$

$$\prod_{k=1}^{N} \frac{1}{4}\,\left(\cos\frac{\theta}{8^k} + \cos\frac{3\theta}{8^k} + \cos\frac{5\theta}{8^k} + \cos\frac{7\theta}{8^k}\right) \qquad (10)$$

## 3.1 Some probability

The **characteristic function** of the distribution of a random variable $X$ is the expected value:

$$\phi(\theta) = \langle e^{i\theta X} \rangle$$

The characteristic function of the **sum of independent random variables** is the product of their characteristic functions:

$$\phi_1(\theta)\phi_2(\theta) = \langle e^{i\theta(X_1+X_2)} \rangle$$

And **Lévy's continuity theorem** says a sequence of characteristic functions converge to the characteristic function of a probability distribution if and only if the corresponding random variables converge in distribution.

Consider a **Bernoulli random variable** $\varepsilon$ taking values $\pm a$, each with probability $1/2$, then it has characteristic function

$$\langle e^{i\varepsilon} \rangle = \cos a\theta$$

On the other hand, a random variable uniformly distributed on the interval $[-T, T]$ has characteristic function

$$\frac{1}{2T} \int_{-T}^{T} e^{i\theta x}\, dx = \frac{\sin T\theta}{T\theta}$$

## 3.2 Infinite Products

**Base 2**: Take $N \to \infty$ in (8). We have

$$\frac{\sin\theta}{\theta} = \prod_{k=1}^{\infty} \cos\frac{\theta}{2^k} \tag{11}$$

The right-hand side are the characteristic functions of random variables of the form $\varepsilon_k/2^k$ where the $\varepsilon_k$ are independent **Bernoulli variables**

$$\varepsilon_k = \begin{cases} +1, & \text{with probability } 1/2 \\ -1, & \text{with probability } 1/2 \end{cases}$$

These correspond to choosing the successive binary digits $0$ or $1$ for an arbitrary number in the unit interval.

The distribution is spread over the interval $[-1, 1]$.

The left-hand side says that in the limit the random variable

$$U = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^k}$$

is **uniformly distributed** on $[-1, 1]$.

The total amount added is

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

**Base 3**: Taking $N \to \infty$ in (9) yields the infinite product

$$\frac{\sin\theta}{\theta} = \prod_{k=1}^{\infty}\left(\frac{1}{3} + \frac{2}{3}\cos\frac{2\theta}{3^k}\right)$$

The random variables are of the form $\varepsilon_k/3^k$ with

$$\varepsilon_k = \begin{cases} 0, & \text{with probability } 1/3 \\ \pm 2, & \text{with probability } 1/3 \text{ each} \end{cases}$$

First we choose equally among $0$, $\pm 2/3$. Next add in $0$, $\pm 2/9$, etc. to get a total of $\sum_{k=1}^{\infty}\frac{2}{3^k} = 1$.

Again, the distribution is **uniform** on $[-1, 1]$. We have *avoided* Cantor-type distributions.

**Base 8**: Taking $N \to \infty$ in (10) yields the infinite product

$$\frac{\sin\theta}{\theta} = \prod_{k=1}^{\infty} \frac{1}{4}\left(\cos\frac{\theta}{8^k} + \cos\frac{3\theta}{8^k} + \cos\frac{5\theta}{8^k} + \cos\frac{7\theta}{8^k}\right)$$

The **random variables** are of the form $\varepsilon_k/8^k$ where the $\varepsilon_k$ are independent variables with the distribution

$$\varepsilon_k = \begin{cases} \pm 1, & \text{with probability } 1/8 \text{ each} \\ \pm 3, & \text{with probability } 1/8 \text{ each} \\ \pm 5, & \text{with probability } 1/8 \text{ each} \\ \pm 7, & \text{with probability } 1/8 \text{ each} \end{cases}$$

First we choose equally among $\pm 1$, $\pm 3$, $\pm 5$, $\pm 7$. Rescale by a factor of $1/8$, add, and iterate. We have the sum $\sum_{k=1}^{\infty} 7/8^k = 1$.

The distribution of the infinite series is **uniform** on $[-1, 1]$.

**Rewriting equations** (8), (9), and (10):

$$\frac{\sin\theta}{\theta} = \frac{\sin(\theta/2^N)}{\theta/2^N} \prod_{k=1}^{N} \cos\frac{\theta}{2^k}$$

$$\frac{\sin\theta}{\theta} = \frac{\sin(\theta/3^N)}{\theta/3^N} \prod_{k=1}^{N} \left(\frac{1}{3} + \frac{2}{3}\cos\frac{2\theta}{3^k}\right)$$

$$\frac{\sin\theta}{\theta} = \frac{\sin(\theta/8^N)}{\theta/8^N} \cdot \prod_{k=1}^{N} \frac{1}{4}\left(\cos\frac{\theta}{8^k} + \cos\frac{3\theta}{8^k} + \cos\frac{5\theta}{8^k} + \cos\frac{7\theta}{8^k}\right)$$

The **finite versions** give uniform distributions. Choose random digits forming a "decimal" in the given base $b$ up to order $b^{-N}$, then choose the tail uniformly on $[-b^{-N}, b^{-N}]$ to fill out the interval $[-1, 1]$.

## 4 Conclusion

• **At each stage**, judicious choices of $\theta$ lead to interesting numerical identities.

The infinite products are rich with possibilities.

For example, equation (11). Starting with $\theta = \pi/2$ and iterating the half-angle formula for cosine gives the formula of Viète

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \; \frac{\sqrt{2+\sqrt{2}}}{2} \; \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

which starts the book of Kac, to follow up with after our conclusion here.

● **Identities** involving mixed radix numeration have corresponding infinite product formulas that should be interesting to explore. The factorial mixed radix is the subject of the Wikipedia article “Factoradic” where, among other features, it is shown how to use this system to enumerate permutations.

● **Series’** associated to the mixed radix cases turn out to in fact correspond to the generalized Cantor sets formed by iteratively extracting sets of various proportions from the unit interval. So perhaps there is an approach to those distributions starting from the present viewpoint.

# 5

# References

[1] W.A. Beyer, J.D. Louck, and D. Zeilberger, *A Generalization of a Curiosity that Feynman Remembered All His Life*, Math. Mag. 69, 43-44, 1996.

[2] Mark Kac, *Statistical independence in probability, analysis and number theory.*, The Carus Mathematical Monographs, No. 12, Published by the Mathematical Association of America. Distributed by John Wiley and Sons, Inc., New York, 1959. MR0110114 (22 #996)

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