Zeons:
Some Properties and Applications

Philip Feinsilver

Southern Illinois University

Carbondale, Illinois USA 62901

Zeons are defined.
Their properties are illustrated and some applications presented.

32^{nd} International Conference on Quantum Probability and Related Topics
Levico Terme, Italy
29 May – 4 June 2011

1 What are Zeons?

➨ Zeons appear in a variety of contexts, but they are often not recognized nor explicitly acknowledged.

Definition 1.1 *A set of commuting elements, $\{x_i\}$, in an algebra, that individually square to zero are called zeons.*

- **Orthofermion generators**

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

➧ all products are zero

- **Even part** of Grassmann algebra generated by e_i,

$$x_{ij} = e_i \wedge e_j$$

➧ there are many relations among the generators, including zero products like $x_{12}x_{23} = 0$, etc.

2 Standard zeon algebra

● **Start** with a vector space $\mathcal{V}$ of dimension n, with basis $\{e_i\}$. A basis for a standard zeon algebra is

$$e_1, \ldots, e_i, \ldots, e_i e_j, \ldots, e_{\mathrm{I}}, \ldots, e_1 e_2 \cdots e_n$$

indexed by subsets I of $\{1, 2, \ldots, n\}$.

➧ **Construction** Start with $\mathcal{V}^{(1)}$, the algebra of `dual numbers` with basis 1 and e, where $e^2 = 0$. Continuing, for $n \geq 2$, set

$$\mathcal{V}^{(n)} = \mathcal{V}^{(1)} \otimes \cdots \otimes \mathcal{V}^{(1)}$$

n copies. Then define

$$e_i = 1 \otimes \cdots \otimes e \otimes \cdots \otimes 1$$

with e in the i^{th} place.

● These generate a standard zeon algebra.

2.1 sl(2)

- The **matrix of** e is $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = R$

- For a $*$**-algebra**, introduce $L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

- With $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ we have

$$[L, R] = H\,, \quad [R, H] = 2R\,, \quad [H, L] = 2L$$

an sl(2) standard triple.

3 Representations of semigroups

• **Start** with a finite set S and consider the semigroup of functions $S \to S$ under composition.

• **Associate** to each $f\colon S \to S$, the matrix X_f with

$$(X_f)_{ij} = \delta_{f(i)j}$$

so the entry in row i is 1 exactly in column $f(i)$.

• **Composing** on the right $i \to if \to ifg = g(f(i))$, we have

$$X_f X_g = X_{fg}$$

a representation of the semigroup.

➧ **Standard constructions** are

- Tensor powers

- Symmetric tensor powers a.k.a. boson Fock space

- Grassmann representations.

The first two approaches work. On the other hand, the representations on Grassmann algebra introduce minus signs, so that the matrices no longer represent functions.

3.1 Representations via zeons

➧ For fixed level ℓ, $1 \leq \ell \leq n$, we have the induced action on the basis e_{I}, $|\mathrm{I}| = \ell$

$$((X_f)^{\vee \ell})_{\mathrm{IJ}} = 1 \quad \text{if } f(\mathrm{I}) = \mathrm{J}, \text{ with } |\mathrm{I}| = |\mathrm{J}| = \ell$$

where row I has all zeros if $|f(\mathrm{I})| < |\mathrm{I}|$.

➧ The matrix elements are **permanents** of the corresponding submatrices, with rows indexed by I and columns by J.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\vee 2} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where rows and columns at level ℓ are labelled using dictionary ordering.

For each ℓ, we have a representation

$$(X_f)^{\vee\ell}(X_g)^{\vee\ell} = (X_{fg})^{\vee\ell}$$

These representations can be used to provide information about the asymptotic behavior of certain random walks on semigroups.

- **Application to Markov chains**

For a stochastic matrix A that generates a Markov chain with no transient states, the second zeon power $A^{\vee 2}$ allows you to determine ergodicity of the chain. For example if the chain is irreducible, periodic, one can immediately determine the periodic classes from the fixed points of $A^{\vee 2}$.

4 Representations of sl(2) on the Boolean lattice

- **Fix n.** Let $\mathcal{B} = \{\mathrm{I}\colon \mathrm{I} \subset \{1, 2, \ldots, n\}\}$ with

$$\mathcal{B}_\ell = \{\mathrm{I} \in \mathcal{B}\colon |\mathrm{I}| = \ell\}$$

denoting the ℓ^{th} layer, for $0 \le \ell \le n$. Define the inclusion operator with rows and columns indexed by elements of $\mathcal{B}$,

$$T_{\mathrm{IJ}} = 1 \quad \text{if } \mathrm{I} \supset \mathrm{J}, |\mathrm{J}| = |\mathrm{I}| - 1$$

and T^* its transpose.

T is the sum $\sum\limits_i \hat{e}_i$ where $\hat{e}_i$ is the operator of multiplication by e_i in the standard zeon algebra.

- **Define** the layer operator $\mathcal{L}$ by

$$\mathcal{L}_{\mathrm{IJ}} = |\mathrm{I}|\,\delta_{\mathrm{IJ}} = \ell \quad \text{if } |\mathrm{I}| = \ell, \mathrm{I} = \mathrm{J}$$

Then we have the commutator

$$U = [T^*, T] = nI - 2\mathcal{L}$$

and (T, T^*, U) are a standard sl(2) triple.

4.1 Boolean incidence matrix

➨ T is the inclusion operator for sets differing by 1 element. Then $T^k/k!$ is the inclusion operator for sets differing by k elements. Thus

$$(e^T)_{\mathrm{IJ}} = 1 \quad \text{if } \mathrm{I} \supset \mathrm{J}$$

and

$$(e^{T^*})_{\mathrm{IJ}} = 1 \quad \text{if } \mathrm{I} \subset \mathrm{J}$$

the incidence matrix for the Boolean lattice.

➧ We immediately have the Moebius function for the Boolean lattice: e^{-T^*}.

4.2 Group elements I

• **Working** on $\mathcal{B}$ we find the matrices for group elements generated by elements of the Lie algebra. First, we have

$$(e^{tT}e^{tT^*})_{\mathrm{IJ}} = t^{|\mathrm{I}\,\Delta\,\mathrm{J}|}\,(1+t^2)^{|\mathrm{I}\cap\mathrm{J}|}$$

with Δ denoting symmetric difference.

• **Restricting** to layer ℓ, we consider the Johnson metric

$$\text{dist}_{\mathrm{JS}}(\mathrm{I},\mathrm{J}) = |\mathrm{I}\setminus\mathrm{J}| = |\mathrm{J}\setminus\mathrm{I}| = \tfrac{1}{2}\,|\mathrm{I}\Delta\mathrm{J}|$$

• Setting $t=1$, on layer ℓ, we get

$$2^{|\mathrm{I}\cap\mathrm{J}|} = 2^{\ell-j}\,\mathrm{JS}_j^{n\ell}$$

where JS denotes the indicator matrix for the Johnson metric on layer ℓ.

➧ **So** $e^{T}e^{T^*}$ has a binary expansion with coefficients the matrices for the Johnson metric.

♫ `Example for` $n = 3$

$$
\left(\begin{array}{c|ccc|ccc|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\cline{1-4}
1 & 2 & 1 & 1 & 2 & 2 & 1 & 2 \\
1 & 1 & 2 & 1 & 2 & 1 & 2 & 2 \\
1 & 1 & 1 & 2 & 1 & 2 & 2 & 2 \\
\cline{2-7}
1 & 2 & 2 & 1 & 4 & 2 & 2 & 4 \\
1 & 2 & 1 & 2 & 2 & 4 & 2 & 4 \\
1 & 1 & 2 & 2 & 2 & 2 & 4 & 4 \\
\cline{5-8}
1 & 2 & 2 & 2 & 4 & 4 & 4 & 8
\end{array}\right)
$$

The blocks along the diagonal are the matrices of the restrictions at each level $0 \leq \ell \leq 3$.

4.3 Group elements II

For the general group elements we have

$$(e^{sT}\,u^{\mathcal{L}}\,e^{tT^*})_{\mathrm{IJ}} = s^{|\mathrm{I}\setminus\mathrm{J}|}(u+st)^{|\mathrm{I}\cap\mathrm{J}|}t^{|\mathrm{J}\setminus\mathrm{I}|}$$

This is illustrated by the following diagram

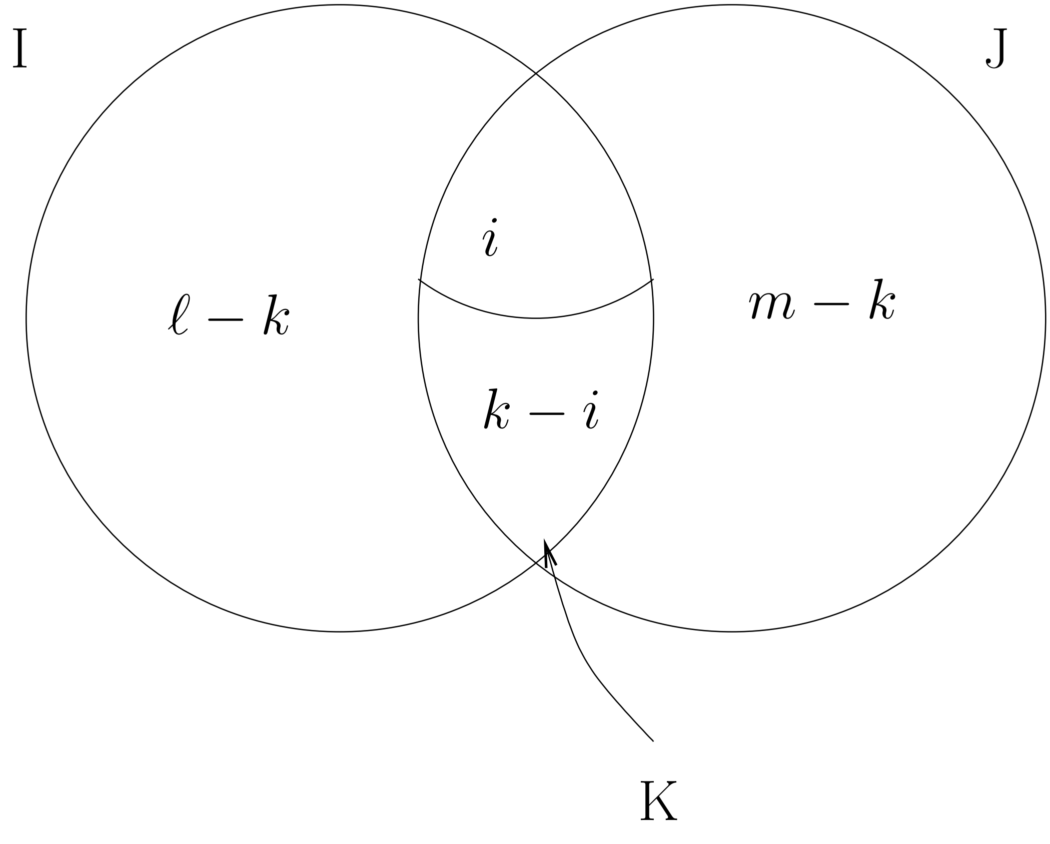

Figure 1: $\sum_i \binom{k}{i}\,s^{\ell-k+i}\,u^{k-i}\,t^{m-k+i}{=}s^{\ell-k}(u{+}st)^k t^{m-k}$

5 Further aspects

● **One can find** the complete decomposition into irreducible representations of the Lie algebra, providing an orthogonal basis of states for the Boolean system.

● The **Leibniz Rule** for moving the lowering operator past the raising operator can be computed using the action on the Boolean lattice.

● The **exponential formula** for a group element in coordinates of the first kind can be found using zeon algebra.

● The relation with **Krawtchouk polynomials** and the Hamming scheme follows from the exponential formula.

● **Further connections** with the Johnson scheme can be found, including finding the spectrum of the Johnson matrices.

6 Connections

● **Meyer's** discussion of the "toy Fock space" in his lecture notes on Quantum Probability are another way of approaching the Boolean action.

● **Sirugue, Sirugue-Collin, et al.**, have discussed many aspects in terms of spin systems.

● **Ceccherini-Silberstein, et al.** in their book Harmonic Analysis on Finite Groups present an approach focussing on the symmetric group.

● **Proctor** has shown how the Sperner property of a poset is equivalent to its carrying a representation of sl(2).

● **Accardi-Bach** have discussed limit theorems for Bernoulli processes which is closely related to this approach.