Vector fields, Lagrange Inversion, and Random Walks

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We indicate the relation between vector fields and operator calculus — dual vector fields. After presenting the approach to Lagrange inversion in general in this context, we specialize to the positive definite case and give a formula for the coefficients of the inverse function in terms of an associated random walk.

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1 Classical Lagrange Inversion

If $y = f(z)$ is analytic around z_0 with $y_0 = f(z_0)$, $f'(z_0) \neq 0$, then for analytic g, with $z = f^{-1}(y)$,

$$g(z) = g(z_0)$$

$$+ \sum_{n=1}^{\infty} \frac{(y - y_0)^n}{n!} \left(\frac{d}{dz}\right)^{n-1} \left(g'(z) \left[\frac{z - z_0}{f(z) - f(z_0)} \right]^n \right) \Bigg|_{z=z_0}$$

Note that the expression involving the $(n-1)^{\text{st}}$ derivative is the coefficient of $(z - z_0)^{n-1}$ in the expansion of $[(z - z_0)/(f(z) - f(z_0))]^n$.

From now on we are in a neighborhood of the origin in $\mathbf{C}$. We take a "normalized" analytic function, $V(z)$, with $V(0)=0$, $V'(0)=1$. Its inverse is U, i.e., $v=V(z) \Leftrightarrow z=U(v)$. The Lagrange formula may be written

$$g(U)=g(0) + \sum_{n=1}^{\infty} \frac{v^n}{n!} \left(\frac{d}{dU}\right)^{n-1} \left(g'(U) \left[\frac{U}{V(U)} \right]^n \right) \Bigg|_{U=0}$$

We will derive a formula for $g(U)=\exp(xU)$. In this case, the expansion takes the form

$$e^{xU(v)} = \sum_{n=0}^{\infty} \frac{v^n}{n!} \, y_n(x)$$

where y_n are polynomials in x, called *basic polynomials*.

2 Analytic representations of the Heisenberg-Weyl algebra

A function $V(z)$ analytic in a neighborhood of the origin in $\mathbf{C}$ yields a generalized differential operator $V(D)$ acting on functions of x of the form $\sum p_j(x)\,\exp(a_j x)$, where a_j are in the domain of V. D is the operator d/dx. With X denoting the operator of multiplication by x, we have the commutation relations

$$[D, X] = I$$

with I the identity operator. This extends to

$$[V(D), X] = V'(D)$$

Introduce the operator

$$W(D) = 1/V'(D)$$

Define $\xi = XW(D)$. Then

$$[V, \xi] = I$$

The operator $\xi = XW(D)$ now plays the rôle of the variable x, with corresponding differentiation operator V.

3 Vector fields and dual vector fields

Let A be local coordinates and let $\hat{\xi}$ be the vector field $\hat{\xi} = W(A)\partial_A$. Then the main observation is the relation

$$\xi\, e^{Ax} = \hat{\xi}\, e^{Ax}$$

since both evaluate to $xW(A)\,\exp(Ax)$. We say that the operator ξ is *dual* to the vector field $\hat{\xi}$.

Now we can use the vector field to exponentiate the operator ξ. Since ξ and $\hat{\xi}$ commute, we may iterate the above relation to

$$e^{t\xi} e^{Ax} = e^{t\hat{\xi}} e^{Ax}$$

3.1 Integral curves

Now we exponentiate by solving the equation for the characteristics: $\dot{A} = W(A)$. Recalling that $W(A) = 1/V'(A)$, we integrate to get

$$A(t) = U(t + V(A))$$

In other words, the solution to the initial-value problem

$$\frac{\partial u}{\partial t} = \hat{\xi}\, u\,, \qquad u(0) = f$$

is $u = f(U(t + V(A)))$, for any smooth f.

We write this as

$$e^{t\hat{\xi}} f(A) = f(U(t + V(A)))$$

With $f = \exp(Ax)$, we thus get

$$e^{t\xi} e^{Ax} = e^{t\hat{\xi}} e^{Ax} = \exp\left(xU(t + V(A))\right)$$

Setting $A = 0$ we get

$$e^{t\xi} 1 = e^{xU(t)}$$

Thus we have the expansion

$$e^{xU(t)} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, \xi^n 1 = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, y_n(x)$$

In other words

$$y_n(x) = \xi^n 1$$

Since every y_n, $n \geq 1$, has a common factor of x, let $\theta_n(x) = y_{n+1}(x)/x, n \geq 0$.

Now form $(\exp(xU(t)) - 1)/x$ and take the limit $x \to 0$ to get

$$U(t) = \sum_{n=1}^{\infty} \frac{t^n}{n!}\, \theta_{n-1}(x)$$

4 Action of a generalized differential operator

Write

$$W(z) = \sum_{n=0}^{\infty} \frac{\mu_n}{n!} z^n$$

Generally, the only conditions on W are coming from the relation $W = 1/V'$. However, as suggested by the notation, if W is the moment generating function for a probability distribution, then μ_n are the corresponding moments.

Thus, write in the general case $\langle\langle X^n \rangle\rangle = \mu_n$ and in the positive definite, probabilistic, case: $\langle X^n \rangle = \mu_n$.

For an analytic function f, we expand

$$f(x+X) = \sum_{n=0}^{\infty} \frac{X^n}{n!} f^{(n)}(x)$$

Taking (generalized) expected value, we see that

$$W(D)\, f(x) = \sum_{n=0}^{\infty} \frac{\mu_n}{n!} f^{(n)}(x) = \langle\langle f(x+X) \rangle\rangle$$

5 Basic polynomials and random walks

We extend the generalized averaging to several variables by taking them to be effectively independent:

$$\langle\langle X_1^{n_1} X_2^{n_2} \ldots X_m^{n_m} \rangle\rangle = \mu_{n_1} \mu_{n_2} \cdots \mu_{n_m}$$

Then we have

Theorem 5.1 *The basic polynomials are given in the form of generalized factorials by*

$y_n(x) =$

$$\langle\langle x(x+X_1)(x+X_1+X_2) \cdots (x+X_1+X_2+\cdots+X_{n-1}) \rangle\rangle$$

In the probabilistic case, we denote the random walk generated by the underlying distribution by $S_n = X_1 + X_2 + \cdots + X_n$, where the X_i are independent, identically distributed random variables with moment generating function equal to W. With $S_0 = x$, the corresponding expectation value is denoted by $\langle \cdot \rangle_x$. Then the Theorem yields

$$\theta_n = \langle S_1 S_2 \cdots S_n \rangle_x$$

6 W as a convolution operator

Writing, in the probabilistic case, $W(D) = \int e^{uD}\, p(du)$, we have

$(XW(D))^n =$

$$x \int e^{u_n D}\, p(du_n) \cdots x \int e^{u_1 D}\, p(du_1)$$

With $\exp(uD)f(x) = f(x+u)$, we get

$(XW(D))^n =$

$$\int x(x+u_1)(x+u_1+u_2)\cdots(x+u_1+\cdots+u_{n-1})$$

$$\cdot \exp\Big((\sum_{j=1}^{n} u_j)D\Big)\, p(du_1)\cdots p(du_n)$$

This is a formula for the operator ξ^n. I.e.,

$$\xi^n = \langle S_0 S_1 S_2 \cdots S_{n-1} e^{S_n D} \rangle_x$$

Applying this to the constant function 1 yields the formula of the Theorem.

We thus have

$$e^{xU(v)} = 1 + x\sum_{n=0}^{\infty} \frac{v^n}{n!} \langle \prod_{j=1}^{n-1} (x + S_j) \rangle_0$$

and

$$U(v) = \sum_{n=1}^{\infty} \frac{v^n}{n!} \langle \prod_{j=1}^{n-1} S_j \rangle_0$$

Note that given an analytic moment generating function $W(z)$, we can form

$$V(z) = \int_0^z \frac{du}{W(u)}$$

And the inverse of V is given by the above formula.

7 Examples

Example 1. Gaussian random walk

With $W = \exp(z^2/2)$, we get V as the distribution function of a standard Gaussian, modulo a factor of $\sqrt{2\pi}$. Thus, we get the expansion of the inverse Gaussian distribution.

Example 2. Exponential random walk

With $W = (1 - qz)^{-1}$, an exponential distribution with mean q, we get

$$V = z - qz^2/2\,, \qquad U = \frac{1 - \sqrt{1 - 2qv}}{q}$$

Thus, with T_1, T_2,. . . ,T_n,. . . independent exponentials with mean q we have

$$\langle T_1(T_1{+}T_2)\cdots(T_1{+}T_2{+}\cdots{+}T_n)\rangle = n!\binom{2n}{n}\left(\frac{q}{2}\right)^n$$

Example 3. Cayley example

With $V(z) = z\,e^{-z}$, we get $W(z) = e^{z}(1-z)^{-1}$, so that the corresponding probability distribution is an exponential with mean 1 shifted by 1.

Checking that

$$y_n(x) = x(x+n)^{n-1}$$

we find

$$n^{n-1} = \langle (1+T_1)(2+T_1+T_2)\cdots(n-1+T_1+T_2+\cdots+T_{n-1})\rangle$$

8 Conclusions. Further work.

1. Note that we are indeed able to recover the more general $g(U(v))$ by the relation

$$g(U(v)) = \sum_{n=0}^{\infty} \frac{v^n}{n!}\, g(D)\, y_n(x) \Big|_{x=0}$$

2. The original application involved $W = (1 + z\, \tan z)^2$ which arises in finding the critical points of the function $(\sin x)/x$. The idea was to develop a recursive method suitable for efficient symbolic computation. Thus the techniques presented here were developed.

3. Further work involves

   a. Multivariate case: the analytic HW version has been available for some time, but the corresponding random walk formulation is yet to be completed

   b. An interesting project would be to develop a dual version of differential geometry

   c. What about multiplication in a group?

9 References

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