Second Quantization
and
Recurrences

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Via recurrences, we find the
matching polynomials of cyclically labelled paths, cycles, and trees.
The technique is to use
trace formulas for matrices acting on the space of symmetric tensors.

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### 1 Matching polynomials

**One-variable path**

\[ x \quad x \quad x \]

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\]

\[ 1 + 3x + x^2 \]

**Multi-variable path**

\[ x_1 \quad x_2 \quad x_3 \]

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array}
\]

\[ 1 + x_1 + x_2 + x_3 + x_1 x_3 \]

**nc-function:** \( \phi_n \) is the sum of all *nonconsecutive* monomials in the variables \( x_1, x_2, \ldots, x_n \).

**Reciprocal-Chebyshev 2\textsuperscript{nd} kind:** \( \phi_{n-1}(x) = \sum_k \binom{n - k}{k} x^k \)
Cycle

\[ x_2 \]

\[ x_1 \]

\[ x_3 \]

\[ x_4 \]

\[ 1 + x_1 + x_2 + x_3 + x_4 + x_1 x_3 + x_2 x_4 \]

**ncc-function:** \( \tau_n \) is the sum of all *nonconsecutive, cyclic* monomials in the variables \( x_1, x_2, \ldots, x_n \).

**Reciprocal-Chebyshev 1\textsuperscript{st} kind:**

\[
\tau_n(x) = \sum_{k=0}^{n} \binom{n-k}{k} \frac{n}{n-k} x^k
\]

**Multi-variable cyclic path**

1 \( \rightarrow \) 2 \( \rightarrow \) 3 \( \rightarrow \) 4 \( \rightarrow \) 5 \( \rightarrow \) 6 \( \rightarrow \) 7

\[ 1 + 2x_1 + 2x_2 + 2x_3 + x_1^2 + 2x_1 x_2 + 3x_1 x_3 + x_2^2 + 2x_2 x_3 + x_3^2 + x_1^2 x_3 + 2x_1 x_2 x_3 + x_1 x_3^2 \]

This is the question
2 Recurrences and matrices

\* nc-Recurrence

\[ \phi_n = \phi_{n-1} + x_n \phi_{n-2} \]

The nc-function \( \phi_n \) satisfies this recurrence with I.C.'s \( \phi_{-1} = 1, \phi_0 = 1 \).

Denoting by \( f_n \) and \( g_n \) the fundamental solutions to this recurrence, we have \( \phi_n = f_n + g_n \).

\* Matrices

\[ X = X_n = \begin{pmatrix} g_{n-1} & f_{n-1} \\ g_n & f_n \end{pmatrix} \]

The ncc-function \( \tau_n = g_{n-1} + f_n \) is the trace of \( X_n \).

The matrix factors as

\[ X = \begin{pmatrix} 0 & 1 \\ x_n & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x_{n-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ x_1 & 1 \end{pmatrix} \]
2.1 Tau-Delta recurrence

Any matrix element $\psi_N = \langle u, X^N v \rangle$, $u, v \in \mathbb{R}^2$, satisfies the **tau-Delta recurrence**

$$\psi_N = \tau \psi_{N-1} - \Delta \psi_{N-2}$$

where $\tau = \text{tr } X$ and $\Delta = \det X = (-1)^n x_1 x_2 \cdots x_n$.

- **First fundamental solution**

$$G_N = \sum_{k=0}^{\lfloor N/2 \rfloor} \binom{N-k}{k} \tau^{N-2k} (-\Delta)^k$$

- **Generating function**

$$\frac{1}{\det(I - tX)} = \sum_{N=0}^{\infty} t^N G_N$$

Powers of $X$ correspond to cyclic repetition of the initial path with $n$ edges.
Second quantization and trace formulas

• **Symmetric representation** of a $d \times d$ matrix $A$

With $\mathbf{u} = (u_1, \ldots, u_d)^T$, $\mathbf{v} = (v_1, \ldots, v_d)^T$,

$$\mathbf{v} = A \mathbf{u}$$

For given homogeneous degree $N$, define **matrix elements** by

$$v_{1}^{m_1} \cdots v_{d}^{m_d} = \sum_{n_1,\ldots,n_d} \left\langle \begin{array}{c} m_1, \ldots, m_d \\ n_1, \ldots, n_d \end{array} \right\rangle_A u_{1}^{n_1} \cdots u_{d}^{n_d}$$

$$v^m = \sum_{n} \left\langle \begin{array}{c} m \\ n \end{array} \right\rangle_A u^n$$

This is a representation of the multiplicative semigroup of matrices. In other words, we have the

• **Homomorphism property**

$$\left\langle \begin{array}{c} m \\ n \end{array} \right\rangle_{AB} = \sum_{k} \left\langle \begin{array}{c} m \\ k \end{array} \right\rangle_A \left\langle \begin{array}{c} k \\ n \end{array} \right\rangle_B$$
3.1 Symmetric traces

- The action defined here on polynomials is equivalent to the action on symmetric tensor powers, as in classical invariant theory. See Fulton and Harris [Representation theory, a first course, pp. 472-5].

- **boson Fock space** over the $d$-dimensional vector space is the space of symmetric tensor powers.

- **Symmetric trace**: for fixed homogeneous degree $N$ the symmetric trace of $A$ in degree $N$

  \[
  \operatorname{tr}_{\text{Sym}}^N(A) = \sum_{|m|=N} \left\langle m \right|_A
  \]

- **Symmetric trace theorem**
  
  (See Springer [Invariant theory, LNM 585, pp. 51-2].)

  \[
  \frac{1}{\det(I - tA)} = \sum_{N=0}^{\infty} t^N \operatorname{tr}_{\text{Sym}}^N(A).
  \]
• **Tau-Delta recurrence revisited**

For $G_N$, the first fundamental solution to the $\tau-\Delta$ recurrence, the Symmetric Trace Theorem says

$$G_N = \text{tr}^N_{\text{Sym}}(X) = \sum_{|m|=N} \langle m \rangle_X$$

$$= \sum_{|m|=N} \langle m \rangle \xi_n \xi_{n-1} \cdots \xi_1$$

By the Homomorphism Property, we calculate the matrix elements for each factor $\xi_i$.

• **Matrix elements** for $\xi_i = \begin{pmatrix} 0 & 1 \\ x_i & a_i \end{pmatrix}$. The mapping induced on polynomials is

$$v_1 = u_2, \quad v_2 = x_i u_1 + a_i u_2$$

And we find, for fixed homogeneous degree $N$,

$$\langle m \rangle_{\xi_i} = \begin{pmatrix} N - m \\ n \end{pmatrix} x_i^n a_i^{N-m-n}$$
4 Cyclic binomial identity

\[ G_N = \sum_{k_1, \ldots, k_n} \binom{N - k_2}{k_1} \binom{N - k_3}{k_2} \cdots \binom{N - k_n}{k_{n-1}} \binom{N - k_1}{k_n} \times x_1^{k_1} \cdots x_n^{k_n} a_1^{N-k_1-k_2} a_2^{N-k_2-k_3} \cdots a_n^{N-k_n-k_1} \]

\[ = \Delta^{N/2} U_N \left( \frac{\tau}{2\sqrt{\Delta}} \right) \]

\[ = \sum_{k=0}^{[N/2]} \binom{N - k}{k} \tau^{N-2k} (-\Delta)^k \]

\[ = \sum_{m,k} \binom{m}{k} \binom{N - m}{m - k} f_n^{N-2m+k} g_{n-1}^k (f_{n-1} g_n)^{m-k} \]

where \( U_N \) denotes the Chebyshev polynomial of the second kind.

Recall \( f_n \) and \( g_n \) are the fundamental solutions to the initial \( n \)-step recurrence.
5 Comments

• Special functions interest

\( n=2 \) finite \( _2F_1 \) summation or \( Chu-Vandermonde \) sum

\( n=3 \) gives \( _3F_2 \) \( Pfaff-Saalschütz \) sum

\( n \geq 4 \) gives a multivariate summation formula that requires further investigation

• Matching polynomials

\( G_N + (\phi_n - \tau_n)G_{N-1} \) is the matching polynomial for the \( N \)-fold repeated path of length \( n \)

\( 2 \Delta^{N/2} T_N \left( \frac{\tau}{2\sqrt{\Delta}} \right) \) is for the corresponding cycle.

Formulas for trees.
6 Conclusion

- **Second quantization of a recurrence** which is the periodic extension [constant coefficients] of a given recurrence [non-constant coefficients] yields identities in the underlying variables by interpreting the fundamental solution in various ways.

- **Hierarchy of hierarchies of identities** since for fixed $r$, an $r$-step recurrence gives a hierarchy of identities. Now vary $r$.

- **Relation** with mathematical objects such as multivariate Chebyshev polynomials?