Krawtchouk Polynomials
Matrices and Transforms

Philip Feinsilver

Southern Illinois University

Carbondale, Illinois USA 62901

René Schott

Nancy-Université, Université Henri Poincaré

54506 Vandoeuvre-lès-Nancy, France

Krawtchouk polynomials are formulated as matrices and properties of Krawtchouk transforms explored.

10^{th} International Conference on Artificial Intelligence and Symbolic Computation

AISC 2010

Paris, France

5–6 July 2010

1 Introduction

➨ Krawtchouk polynomials appear in a variety of contexts, most notably as orthogonal polynomials with respect to the binomial distribution.

- **Krawtchouk transform** on vectors.

- **Algorithm** for the Krawtchouk transform on vectors.

- **Krawtchouk expansions** of functions.

- **Operator calculus formulation** for the coefficients of Krawtchouk expansions.

- **Applications** of Krawtchouk transforms.

2 Krawtchouk Polynomials, Kravchuk Matrices

One may view Kravchuk matrices as an extension of the binomial coefficients. Consider the “degree-two algebraic rules” and translate them into a “second-degree Kravchuk matrix”:

$$
\begin{aligned}
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)(a-b) &= a^2 \qquad\quad - b^2 \\
(a-b)^2 &= a^2 - 2ab + b^2
\end{aligned}
$$

read off

$$
K^{(2)} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}
$$

The expansion coefficients make up the columns of the matrix.

2.1 Generating function

- The entries are determined by the expansion:

$$G(v; j, N) = (1 + v)^{N-j} \, (1 - v)^{j} = \sum_{i=0}^{N} v^{i} K_{ij}^{(N)}$$

- Expanding gives the explicit values of the matrix entries:

$$K_i(j, N) = K_{ij}^{(N)} = \sum_{k} (-1)^k \binom{j}{k} \binom{N-j}{i-k}$$

where matrix indices run from 0 to N.

➧ **Here** are the Kravchuk matrices of orders zero, one, and three:

$$K^{(0)} = \begin{bmatrix} 1 \end{bmatrix}$$

$$K^{(1)} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$K^{(3)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

3 Interpretations

● **As** polynomials in j they are orthogonal with respect to the binomial distribution, $\binom{N}{j}$.

● **These** correspond to functionals of a random walk moving ± 1 with equal probabilities.

● **Transforms** of vectors correspond to expansions via matrices.

● **Transforms** of functions correspond to expansions in terms of polynomials.

4 Transform on Vectors

➨ Multiplying on the right by K gives the transform of $\mathbf{f} = (f(0), f(1), \ldots, f(N))$. Multiply again by K using $K^2 = 2^N\,\mathrm{I}$ to get the inverse transform.

$$\hat{\mathbf{f}} = \mathbf{f}\,K \qquad \text{implies} \qquad \mathbf{f} = 2^{-N}\,\hat{\mathbf{f}}\,K$$

● Explicitly, this is the expansion of the vector $\mathbf{f}$ in terms of Krawtchouk polynomials in the variable j.

$$f(j) = 2^{-N} \sum_i \hat{f}(i)\,K_i(j, N)$$

● We have developed an algorithm for carrying out the transform.

4.1 Algorithm

➨ Given $N > 0$. Do the following for $n = 0$ to N :

➧ **Step 0.** Given a row vector of length $N + 1$.

➧ **Step n.** You have n **current** rows.

Form n **new** rows by summing adjacent values.

Form the $n + 1^{\mathrm{st}}$ row by differencing adjacent values of the current n^{th} row.

- At step n, you have $n + 1$ rows and $N + 1 - n$ columns.

- After step N, you have a single column of $N + 1$ values.

Transposed it is the Krawtchouk transform of the original row.

➨ Take the column that resulted from applying the algorithm as your new row. Apply the algorithm again. Divide the result by 2^N and you recover your original values.

♫ Examples

- Let $N = 3$. Start with $4, 2, 0, -3$. Then we have

$$[4 \quad 2 \quad 0 \quad -3] \Rightarrow \begin{bmatrix} 6 & 2 & -3 \\ 2 & 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 8 & -1 \\ 4 & 5 \\ 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 \\ 9 \\ -1 \\ 1 \end{bmatrix}$$

- Start with a row of $K^{(N)}$, you get 2^N times a vector with 1 in the corresponding spot.

$$[3 \quad 1 \quad -1 \quad -3] \Rightarrow 2^3 \, [0 \quad 1 \quad 0 \quad 0]$$

- Take a vector that starts with a binomial row, $\binom{n}{i}$.

Multiply on the left by $K^{(N)}$. It produces 2^n times a binomial row with index $N - n$.

$$K^{(5)} \, [1 \quad 3 \quad 3 \quad 1 \quad 0 \quad 0]^t = 2^3 \, [1 \quad 2 \quad 1 \quad 0 \quad 0 \quad 0]^t$$

5 Expansions of Functions

In the random walk interpretation, j is the number of jumps to the left. The position $x = N - 2j$.

- The **generating function** for functions of x is

$$(1+v)^{(N+x)/2}\,(1-v)^{(N-x)/2} = \sum_{n\geq 0} \frac{v^n}{n!}\,K_n(x,N)$$

- A polynomial function of x of degree at most N has an expansion

$$f(x) = \sum_{0\leq n\leq N} \tilde{f}(n)\,K_n(x,N)$$

- The coefficient $\tilde{f}(n)$ has the operator calculus expression

$$\tilde{f}(n) = \frac{1}{n!}\,(\cosh D)^{N-n}(\sinh D)^n f(0)$$

where $e^{\pm D}f(x) = f(x \pm 1)$, shift operators on functions of x.

5.1 Operator calculus via Matrices

We can use the matrix of the operator D acting on the powers of x for symbolic calculation. Since D is nilpotent acting on polynomials in x, the exponentials reduce to finite sums. For example, take $N = 4$.

$$\hat{D} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We have $\cosh \hat{D}$ and $\tanh \hat{D}$ respectively:

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 & -8 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

♫ Example

For N=4, let $f(x) = x^4 + 2x^3 - x^2 + 5x$.
We find, with $N = 4$, that

$$f = K_4 + 2K_3 + 15K_2 + 25K_1 + 36$$

where $K_0 = 1$,

$$K_1 = x, \qquad K_3 = x^3 - 10x$$
$$K_2 = x^2 - 4, \qquad K_4 = x^4 - 16x^2 + 24$$

This is obtained by multiplying the column of coefficients of $f(x)$ by the matrix Y formed by the top rows of $(\cosh^N \hat{D})(\tanh^n \hat{D})/n!$, for $0 \le n \le N$, which are readily computed iteratively. In this example we have

$$Y = \begin{bmatrix} 1 & 0 & 4 & 0 & 40 \\ 0 & 1 & 0 & 10 & 0 \\ 0 & 0 & 1 & 0 & 16 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

6 Further aspects

• **General** p, q. Polynomials with parameters p and q arise from Bernoulli trials where the probability of success is p, with $q = 1 - p$. They arise as well when working over finite fields, in which case q is the number of elements of the field.

• **Multivariate** polynomials are orthogonal with respect to corresponding multinomial distributions. Functions of several variables correspond to random walks in higher dimensions.

• **Positivity** results hold for transforms of polynomial functions.

• **Variety** of applications is seen in the references.

References

[1] N.M. Atakishiyev, G.S. Pogosyan, and K.B. Wolf, *Finite Models of the Oscillator*, Physics of Particles and Nuclei, Suppl. 3, **36**:521-555, 2005.

[2] N. Botros, J. Yang, P. Feinsilver, and R. Schott, *Hardware Realization of Krawtchouk Transform using VHDL Modeling and FPGAs*, IEEE Transactions on Industrial Electronics, **49** 6 (2002)1306–1312.

[3] Ph. Feinsilver and R. Schott, *Finite-Dimensional Calculus*, Journal of Physics A: Math.Theor., **42**:375214, 2009.

[4] F.J. MacWilliams, N.J.A. Sloane, *Theory of error-correcting codes*, North-Holland, 1977.

[5] G. Szëgo, *Orthogonal Polynomials*, AMS, 1955.

[6] P.-T. Yap and R. Paramesran, *Image analysis by Krawtchouk moments*, IEEE Transactions on image processing, **12**:1367-1377, 2003.