

New Graph

[5, 5, 1, 5, 2], [3, 4, 5, 2, 1]

$$\pi = [2, 2, 1, 1, 3]$$

POSSIBLE RANKS

$$\begin{matrix} 1 \times 9 \\ 3 \times 3 \end{matrix}$$

BASE DETERMINANT 351/4096, .8569335938e-1

NullSpace of Δ

{1, 2, 3, 4, 5}

Range of Δ : $[\lambda_3, \lambda_4, \lambda_2, \lambda_1, -\lambda_3 - \lambda_4 - \lambda_2 - \lambda_1]$

1. Coloring, {}

R: [5, 5, 1, 5, 2] **B**: [3, 4, 5, 2, 1]

' See graph

' ' See pair graph

'

Ω for $A+\tau\Delta$:

$$\begin{bmatrix} -18(\tau - 1 + \tau^2) & (\tau - 3 + \tau^2) & -18(\tau^3 + \tau^2) & 9(\tau - 1 + \tau^2)^2 & (\tau - 3 + \tau^2) & 9(\tau^3 + \tau^2) \\ (\tau - 1 + \tau^2) & 9(\tau^3 + \tau^2) & (\tau - 3 + \tau^2) & & & \end{bmatrix}$$

For $\tau=1/2$, [-20, -52, -5, -13, -65] . FixedPtCheck, [20, 52, 5, 13, 65]

$$\det(A + \tau \Delta) = 1(\tau^3 + 3\tau^2)(\tau - 1 + \tau^2)^2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	3 vs 3	4 vs 5

bi =

$$\begin{aligned} & \$ [[0, 0, 3/4, 0, 1/4], [0, 0, 0, 3/4, 1/4], [1/4, 0, 0, 0, 3/4], [0, 3/4, 0, 0, 1/4], [3/4, 1/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[-5/76, -65/228, -37/19, 44/171, 368/171], [53/38, 119/114, -6/19, -568/171, 224/171], [-12/19, \\ & -23/57, 44/57, -64/171, 128/171], [-27/76, 409/228, 115/57, 116/171, -688/171], [-17/76, -221/228, \\ & 11/19, 332/171, -208/171]] \$ \times \$ [[5/2, 3/2, 3/2, 3/2, 2], [15/8, 13/8, 15/8, 9/8, 5/2], [75/32, 47/32, \\ & 45/32, 39/32, 41/16], [291/128, 199/128, 225/128, 141/128, 37/16], [1113/512, 719/512, 873/512, \\ & 597/512, 653/256]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{2, 5}}, net cycles: 0 . order: 2

$$[y_3, y_2, 0, 0, y_1]$$

$$\begin{aligned} R = & \$ [[0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, -4/9, 5/9], [0, -4/9, 5/9], [1, 5/9, -13/9] \\ & , [0, -4/9, 5/9], [0, 5/9, -4/9]] \$ \times \$ [[1, 3, 0, 0, 5], [0, 5, 0, 0, 4], [0, 4, 0, 0, 5]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 3, 5}, {2, 4}}, net cycles: 2 .

$$\$ [[3, 1, 2, 2, 1], [1, 2, 3, 1, 2], [2, 1, 1, 2, 3], [3, 2, 2, 1, 1], [1, 1, 3, 2, 2]] \$$$

$$[2y_1 - y_2 + 2y_3 - y_4, y_1, y_2, y_3, y_4]$$

$$p' = -1 - s + s^3 + s^4$$

Â» SYNC'D 45/512, 0.08789062500

2 . Coloring, {2}

R: [5, 4, 1, 5, 2] **B:** [3, 5, 5, 2, 1]

‘ See graph

‘ ‘ See pair graph

‘

Ω for A+τΔ :

$$[' -6' (' - 1 + \tau ')', 6' (' 1 + \tau ')', 3' (' - 1 + \tau ')'^2, 3' (' 1 + \tau ')'^2, 3' (' 3 + \tau^2 ')']'$$

For τ=1/2, [4, 12, 1, 9, 13] . FixedPtCheck, [4, 12, 1, 9, 13]

$$\det(A + \tau \Delta) = 1' (' - 1 + \tau ')'' (' 1 + 3\tau^2 ')'' (' 1 + \tau ')'$$

Delta Range : $[y_3, y_4, -y_3 - y_4 - y_1 - y_2, y_1, y_2]$

$[2, 2, 1, 1, 3]$

+ \ ; - \ ; Δ

$\$ [[1, 3, 0, 2, 3], [3, 3, 3, 3, 6], [9, 7, 5, 3, 12], [17, 17, 7, 7, 24]] \$ [[3, 1, 2, 0, 3], [5, 5, 1, 1, 6], [7, 9, 3, 5, 12], [15, 15, 9, 9, 24]] \$ [[-1, 1, -1, 1, 0], [-1, -1, 1, 1, 0], [1, -1, 1, -1, 0], [1, 1, -1, -1, 0]] \$$

$[-y_2, -y_1, y_1, y_2, 0]$

$$p = s + 4s^3$$

S+ \ ; S- \ ; NM

$\$ [[0, 1, 2, 1, 1], [0, 1, 0, 1, 3], [1, 1, 0, 0, 3], [0, 2, 1, 1, 1], [3, 1, 0, 0, 1]] \$ [[1, 0, 1, 0, 3], [1, 0, 1, 2, 1], [2, 0, 1, 1, 1], [1, 1, 0, 0, 3], [1, 3, 0, 0, 1]] \$ [[4, 2, 1, 2, 3], [2, 4, 2, 1, 3], [2, 4, 2, 1, 3], [4, 2, 1, 2, 3], [2, 2, 1, 1, 6]] \$$

CmmCk true, true, true

$$p' = s + 4s^3$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
2 vs 4	3 vs 5	3 vs 5	2 vs 4	2 vs 4

Omega Rank for R : cycles: $\{\{2, 4, 5\}\}$, net cycles: 0 . order: 3

$\$ [[1, 3, 0, 2, 3], [0, 3, 0, 3, 3], [0, 3, 0, 3, 3], [0, 3, 0, 3, 3]] \$$

$[y_2, y_1, 0, -y_2 + y_1, y_1]$

$$p = -s^2 + s^3 \quad p = -s^2 + s^4$$

Omega Rank for B : cycles: $\{\{1, 3, 5\}\}$, net cycles: 0 . order: 3

$\$ [[3, 1, 2, 0, 3], [3, 0, 3, 0, 3], [3, 0, 3, 0, 3], [3, 0, 3, 0, 3]] \$$

$[y_2, y_2 - y_1, y_1, 0, y_2]$

$$p = -s^2 + s^3 \quad p = -s^2 + s^4$$

Â« NOT SYNC'D Â»

Nullspace of $\{\Omega\Delta^i\}$:

$[x_1, x_2, 4x_1, 4x_2]$

For $A+2\Delta$: $[y_1, -y_1 - y_2, -y_1 - y_2, y_1, y_2]$

For $A-2\Delta$: $[-y_1 - y_2, y_1, y_1, -y_1 - y_2, y_2]$

Range of $\{\Omega\Delta^i\}$: $[-\mu_2, -\mu_1, \mu_1, \mu_2, 0]$

rank of M is 5 , rank of N is 3

M N

$\$ [[0, 1, 1, 0, 2], [1, 0, 0, 1, 2], [1, 0, 0, 0, 1], [0, 1, 0, 0, 1], [2, 2, 1, 1, 0]] \$$ $\$ [[0, 1, 1, 0, 1], [1, 0, 0, 1, 1], [1, 0, 0, 1, 1], [0, 1, 1, 0, 1], [1, 1, 1, 1, 0]] \$$

Check is $\Omega\Delta N$ zero? *true*, $\pi\Delta = [-1, 1, -1, 1, 0]$

ker M, $[0, 0, 0, 0, 0]$

Range M, $[x_1, x_2, x_3, x_4, x_5]$

$\tau = 9$, $r' = 2/3$

Ranges

Action of R on ranges, $[[3], [1], [3]]$

Action of B on ranges, $[[2], [2], [1]]$

$\beta(\{1, 2, 5\}) = 1/3$

$\beta(\{1, 3, 5\}) = 1/3$

$\beta(\{2, 4, 5\}) = 1/3$

ker N, $[-\mu_2, -\mu_1, \mu_1, \mu_2, 0]$

Range of N

$[y_2, y_1, y_1, y_2, y_3]$

Partitions

$\alpha(\{\{2, 3\}, \{1, 4\}, \{5\}\}) = 1/1$

$b_1 = \{2, 3\}$, $b_2 = \{1, 4\}$, $b_3 = \{5\}$

Action of R and B on the blocks of the partitions: $\$ [[0, 1, 1], [1, 0, 1], [1, 1, 0]] \$ = \$ [[0, 0, 1], [1, 0, 0], [0, 1, 0]] \$ + \$ [[0, 1, 0], [0, 0, 1], [1, 0, 0]] \$$

$['3', '1', '2', '2', '3', '1']$ with invariant measure $[1, 1, 1]$

N by blocks, check: true . ' See partition graph.

' ' See level-3 partition graph.

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Right Group	
Coloring	{2}
Rank	3
R,B	[5, 4, 1, 5, 2], [3, 5, 5, 2, 1]
π_2	[1, 1, 0, 2, 0, 1, 2, 0, 1, 1]
u_2	[1, 1, 0, 1, 0, 1, 1, 1, 1, 1] (dim 1)
wpp	[2, 2, 2, 2, 1]
π_3	[0, 0, 1, 0, 1, 0, 0, 0, 1, 0]
u_3	[0, 0, 1, 0, 1, 0, 0, 0, 1, 1]

3. Coloring, {3}

R: [5, 5, 5, 5, 2] **B:** [3, 4, 1, 2, 1]

' See graph

' ' See pair graph

,

Ω for $A+\tau\Delta$:

' ['18' (' - 1 + τ ')', -18' (' 1 + τ ')', -9' (' - 1 + τ ')², 9' (' 1 + τ ')'' (' - 1 + τ ')', 9' (' 1 + τ ')'' (' - 3 + τ ')'']'

For $\tau=1/2$, [-4, -12, -1, -3, -15] . FixedPtCheck, [4, 12, 1, 3, 15]

$\det(A + \tau \Delta) = 1' (' 1 + \tau ')'' (' - 1 + \tau ')'$ ³

Delta Range : [$y_3, y_4, -y_3 - y_4 - y_1 - y_2, y_1, y_2$]

[2, 2, 1, 1, 3]

+ \ ; - \ ; Δ

\$ [[0, 3, 0, 0, 6] , [2, 8, 4, 1, 3] , [9, 6, 6, 0, 15] , [11, 23, 7, 10, 21]] \$ \$ [[4, 1, 2, 2, 0] , [6, 0, 0, 3, 9] , [7, 10, 2, 8, 9] , [21, 9, 9, 6, 27]] \$ \$ [[-2, 1, -1, -1, 3] , [-2, 4, 2, -1, -3] , [1, -2, 2, -4, 3] , [-5, 7, -1, 2, -3]] \$

[$y_1, y_3, 6y_1 + 5y_3 + 2y_2, -7y_1 - 6y_3 - 3y_2, y_2$]

$$p = s + 2s^2 - 4s^3 - 8s^4$$

S+ \ ; S- \ ; NM

\$ [[4, 3, 2, 1, 5], [3, 4, 1, 2, 5], [4, 3, 2, 1, 5], [3, 4, 1, 2, 5], [3, 3, 2, 2, 5]] \$ \$ [[4, 3, 2, 1, 5], [3, 4, 1, 2, 5], [4, 3, 2, 1, 5], [3, 4, 1, 2, 5], [3, 3, 2, 2, 5]] \$ \$ [[0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0]] \$

CmmCk true, true, true

Δ -Rank	A+(1/2) Δ	A-(1/2) Δ	R	B
3 vs 4	4 vs 5	4 vs 5	2 vs 2	2 vs 4

Omega Rank for R : cycles: {{2, 5}}, net cycles: 1 . order: 2

$$[0, y_1, 0, 0, y_2]$$

R = \$ [[0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0]] \$ x \$ [[0, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[2/9, -1/9], [2/9, -1/9], [2/9, -1/9], [2/9, -1/9], [-1/9, 2/9]] \$ x \$ [[0, 3, 0, 0, 6], [0, 6, 0, 0, 3]] \$

Omega Rank for B : cycles: {{1, 3}, {2, 4}}, net cycles: 2 . order: 2

$$[[4, 1, 2, 2, 0], [2, 2, 4, 1, 0], [4, 1, 2, 2, 0], [2, 2, 4, 1, 0]] $$$

$$[2 y_2, y_1, 2 y_1, y_2, 0]$$

$$p = -s + s^3 \quad p' = -s + s^3$$

Â» SYNC'D 9/128 , 0.07031250000

4 . Coloring, {4}

R: [5, 5, 1, 2, 2] B: [3, 4, 5, 5, 1]

' See graph

' ' See pair graph

'

Ω for A+ $\tau\Delta$:

$$[(-18(-1 + \tau)^4, 18(1 + \tau)^4, 9(-1 + \tau)^2, -9(-1 + \tau)^4(1 + \tau)^4, 9(3 + \tau^2)^4]$$

For $\tau=1/2$, [4, 12, 1, 3, 13] . FixedPtCheck, [4, 12, 1, 3, 13]

$$\det(A + \tau \Delta) = (-1 + \tau)^3 (1 + \tau)$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	3 vs 3	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 0, 3/4, 0, 1/4], [0, 0, 0, 3/4, 1/4], [1/4, 0, 0, 0, 3/4], [0, 1/4, 0, 0, 3/4], [3/4, 1/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[-93/1001, -262/3003, -2752/3003, 12928/9009, -2048/9009], [1104/1001, -1249/3003, \\ & -5692/3003, -6224/9009, 18112/9009], [142/1001, -235/3003, -996/1001, -13712/9009, 23104/9009], \\ & [-173/1001, 3818/3003, 736/3003, -8672/9009, -2432/9009], [-30/91, -17/273, 580/273, 272/819, \\ & -1600/819]] \$ \times \$ [[5/2, 1, 3/2, 3/2, 5/2], [9/4, 1, 15/8, 3/4, 25/8], [45/16, 31/32, 27/16, 3/4, 89/32], \\ & [321/128, 113/128, 135/64, 93/128, 355/128], [1335/512, 7/8, 963/512, 339/512, 1523/512]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{2, 5}}, net cycles: 0 . order: 2

$$[y_1, y_2, 0, 0, y_3]$$

$$\begin{aligned} R = & \$ [[0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, 5/9, -4/9], [0, 5/9, -4/9], [1, -4/9, -4/9], \\ & [0, -4/9, 5/9], [0, -4/9, 5/9]] \$ \times \$ [[1, 4, 0, 0, 4], [0, 4, 0, 0, 5], [0, 5, 0, 0, 4]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 3, 5}}, net cycles: 0 . order: 3

$$[y_1, 0, y_3, y_2, y_4]$$

$$\begin{aligned} B = & \$ [[0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 0, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, 1/27, -8/27, 10/27], [1/2, 1/27, -8/27, \\ & -7/54], [0, 10/27, 1/27, -8/27], [0, 10/27, 1/27, -8/27], [0, -8/27, 10/27, 1/27]] \$ \times \$ [[3, 0, 2, 2, 2], [2, \\ & 0, 3, 0, 4], [4, 0, 2, 0, 3], [3, 0, 4, 0, 2]] \$ \end{aligned}$$

Â» SYNC'D 9/128 , 0.07031250000

5 . Coloring, {5}

R: [5, 5, 1, 5, 1] **B:** [3, 4, 5, 2, 2]

‘ See graph

‘ ‘ See pair graph

‘

Ω for $A+\tau\Delta$:

$$\begin{aligned} & [[-18^{\prime} ((1 + \tau^{\prime})^{\prime})^2 ((-3 + \tau^{\prime})^{\prime}) , -18^{\prime} ((-1 + \tau^{\prime})^{\prime}) ((3 + \tau^2)^{\prime}) , 9^{\prime} ((-1 + \tau^{\prime})^{\prime}) ((1 + \tau^{\prime})^{\prime})^2 \\ & ((-3 + \tau^{\prime})^{\prime}) , 9^{\prime} ((-1 + \tau^{\prime})^{\prime})^2 ((3 + \tau^2)^{\prime}) , -9^{\prime} ((1 + \tau^{\prime})^{\prime}) ((3 + \tau^2)^{\prime}) ((-3 + \tau^{\prime})^{\prime})] \end{aligned}$$

For $\tau=1/2$, [180, 52, 45, 13, 195] . FixedPtCheck, [180, 52, 45, 13, 195]

$$\det(A + \tau \Delta) = 1^4 (-1 + \tau)^3 (1 + \tau)^4$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	2 vs 2	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 0, 3/4, 0, 1/4], [0, 0, 0, 3/4, 1/4], [1/4, 0, 0, 0, 3/4], [0, 3/4, 0, 0, 1/4], [1/4, 3/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[49/37, -8/111, -1180/333, -160/333, 320/111], [-13/111, -16/333, -140/999, 32/37, -448/999], \\ & [-28/111, 401/333, 40/999, -112/111, 128/999], [8/37, 7/111, -392/333, -304/333, 640/333], [-17/37, \\ & -38/111, 944/333, 128/333, -256/111]] \$ \times \$ [[1, 3, 3/2, 3/2, 2], [7/8, 21/8, 3/4, 9/4, 5/2], [13/16, 57/16, \\ & 21/32, 63/32, 2], [85/128, 381/128, 39/64, 171/64, 133/64], [43/64, 57/16, 255/512, 1143/512, 521/256]] \$ \\ & \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 5}}, net cycles: 1 . order: 2

$$[y_2, 0, 0, 0, y_1]$$

$$\begin{aligned} R = & \$ [[0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[5/9, -4/9], [5/9, -4/9], [-4/9, 5/9], [5/9, \\ & -4/9], [-4/9, 5/9]] \$ \times \$ [[4, 0, 0, 0, 5], [5, 0, 0, 0, 4]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{2, 4}}, net cycles: 0 . order: 4

$$[0, y_1, y_2, y_3, y_4]$$

$$\begin{aligned} B = & \$ [[0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0], [0, 1, 0, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[1/2, -1/4, -5/18, 5/36], [0, 0, -1/9, 2/9], \\ & [0, 1/2, -1/9, -5/18], [0, 0, 2/9, -1/9], [0, 0, 2/9, -1/9]] \$ \times \$ [[0, 4, 2, 2, 1], [0, 3, 0, 4, 2], [0, 6, 0, 3, 0], \\ & [0, 3, 0, 6, 0]] \$ \end{aligned}$$

Â» SYNC'D 1/16 , 0.06250000000

6 . Coloring, {2, 3}

R: [5, 4, 5, 5, 2] **B:** [3, 5, 1, 2, 1]

' See graph

' ' See pair graph

Ω for $A+\tau\Delta$:

$$[18(-1+\tau)^2(3+\tau^2), 18(1+\tau)^2(-3+\tau), -9(-1+\tau)^2(3+\tau^2), 9(1+\tau)^3(-3+\tau), 9(3+\tau^2)(1+\tau)(-3+\tau)]$$

For $\tau=1/2$, [-52, -180, -13, -135, -195] . FixedPtCheck, [52, 180, 13, 135, 195]

$$\det(A + \tau \Delta) = (-1 + \tau)^2 (1 + \tau)^2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	3 vs 3	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 0, 3/4, 0, 1/4], [0, 0, 0, 1/4, 3/4], [3/4, 0, 0, 0, 1/4], [0, 3/4, 0, 0, 1/4], [3/4, 1/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[1/2, -19/18, -38/27, 88/27, -32/27], [2/3, 7, -380/27, -272/27, 448/27], [1, 16/9, -200/27, \\ & -128/27, 256/27], [37/6, -43/6, -362/27, 232/27, 160/27], [-17/6, -13/6, 466/27, 88/27, -416/27]] \$ \times \$ [\\ & [3, 3/2, 3/2, 1/2, 5/2], [3, 1, 9/4, 3/8, 19/8], [111/32, 7/8, 9/4, 1/4, 69/32], [423/128, 93/128, 333/128, \\ & 7/32, 275/128], [57/16, 359/512, 1269/512, 93/512, 1063/512]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{2, 4, 5}}, net cycles: 1 . order: 3

$$[0, y_1, 0, y_2, y_3]$$

$$\begin{aligned} R = & \$ [[0, 0, 0, 0, 1], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[10/27, -8/27, 1/27], [-8/27, 1/27, 10/27], \\ & [10/27, -8/27, 1/27], [10/27, -8/27, 1/27], [1/27, 10/27, -8/27]] \$ \times \$ [[0, 3, 0, 2, 4], [0, 4, 0, 3, 2], [0, 2, \\ & 0, 4, 3]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 3}}, net cycles: 0 . order: 4

$$[y_1, y_2, y_3, 0, y_4]$$

$$\begin{aligned} B = & \$ [[0, 0, 1, 0, 0], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, 0, -4/9, 5/9], [0, 1, -4/9, -4/9], [0, 0, \\ & 5/9, -4/9], [1, -2, -4/9, 14/9], [0, 0, 5/9, -4/9]] \$ \times \$ [[4, 1, 2, 0, 2], [4, 0, 4, 0, 1], [5, 0, 4, 0, 0], [4, 0, 5, \\ & 0, 0]] \$ \end{aligned}$$

\hat{A} » SYNC'D 7/64 , 0.1093750000

7 . Coloring, {2, 4}

R: [5, 4, 1, 2, 2] **B:** [3, 5, 5, 5, 1]

' See graph

' ' See pair graph

Ω for $A+\tau\Delta$:

$$\left[-18 \binom{3+\tau}{-1+\tau}^2, -18 \binom{1+\tau}{3+\tau^2}, 9 \binom{3+\tau}{-1+\tau}^3, -9 \binom{1+\tau}{3+\tau^2} \binom{3+\tau}{-1+\tau} \right]$$

For $\tau=1/2$, [-28, -156, -7, -117, -91] . FixedPtCheck, [28, 156, 7, 117, 91]

$$\det(A + \tau \Delta) = 1 \binom{1+\tau}{-1+\tau}^2 \binom{-1+\tau}{3+\tau^2}^2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	4 vs 4	3 vs 3

bi =

$$\begin{aligned} & \$ [[0, 0, 3/4, 0, 1/4], [0, 0, 0, 1/4, 3/4], [1/4, 0, 0, 0, 3/4], [0, 1/4, 0, 0, 3/4], [3/4, 1/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[-172/197, 871/591, 2468/591, 112/1773, -8384/1773], [799/394, -4723/1182, -3334/591, \\ & -4136/1773, 17824/1773], [41/394, -841/1182, 38/591, 3496/1773, -2336/1773], [-487/197, 3844/591, \\ & 7304/591, 10048/1773, -38912/1773], [139/394, -295/1182, -1870/591, -1832/1773, 7456/1773]] \$ \times \$ [\\ & [5/2, 1, 3/2, 1/2, 7/2], [3, 1, 15/8, 1/4, 23/8], [21/8, 25/32, 9/4, 1/4, 99/32], [369/128, 107/128, 63/32, \\ & 25/128, 399/128], [1449/512, 53/64, 1107/512, 107/512, 1521/512]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{2, 4}}, net cycles: 0 . order: 4

$$[y_1, y_2, 0, y_3, y_4]$$

$$\begin{aligned} R = & \$ [[0, 0, 0, 0, 1], [0, 0, 0, 1, 0], [1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, 1, -4/9, -4/9], [0, 0, -4/9, 5/9], [1, -2, \\ & -4/9, 14/9], [0, 0, 5/9, -4/9], [0, 0, 5/9, -4/9]] \$ \times \$ [[1, 4, 0, 2, 2], [0, 4, 0, 4, 1], [0, 5, 0, 4, 0], [0, 4, 0, \\ & 5, 0]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 3, 5}}, net cycles: 1 . order: 3

$$[y_1, 0, y_3, 0, y_2]$$

$$\begin{aligned} B = & \$ [[0, 0, 1, 0, 0], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 0, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[-8/27, 1/27, 10/27], [10/27, -8/27, 1/27], \\ & [10/27, -8/27, 1/27], [10/27, -8/27, 1/27], [1/27, 10/27, -8/27]] \$ \times \$ [[3, 0, 2, 0, 4], [4, 0, 3, 0, 2], [2, 0, \\ & 4, 0, 3]] \$ \end{aligned}$$

Â» SYNC'D 7/64 , 0.1093750000

8. Coloring, {2, 5}

R: [5, 4, 1, 5, 1] **B:** [3, 5, 5, 2, 2]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

' [-18' (' 1 + τ ')' , 18' (' - 1 + τ ')' , 9' (' 1 + τ ')'' (' - 1 + τ ')' , 9' (' 1 + τ ')'' (' - 1 + τ ')' , -9' (' 3 + τ ² ')'']'

For τ=1/2, [-12, -4, -3, -3, -13] . FixedPtCheck, [12, 4, 3, 3, 13]

$$\det(A + \tau \Delta) = 1' (' 1 + \tau ')' ^2 (' - 1 + \tau ')' ^2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	5 vs 5	5 vs 5	3 vs 3	3 vs 3

bi =

\$ [[0, 0, 3/4, 0, 1/4] , [0, 0, 0, 1/4, 3/4] , [1/4, 0, 0, 0, 3/4] , [0, 3/4, 0, 0, 1/4] , [1/4, 3/4, 0, 0, 0]] \$ x \$ [[1, 0, 0, 0, 0] , [0, 1, 0, 0, 0] , [0, 0, 1, 0, 0] , [0, 0, 0, 1, 0] , [0, 0, 0, 0, 1]] \$ =
 \$ [[15/26, -7/78, 4/3, 56/117, -256/117] , [-1/6, -1/6, -4/9, 8/9, 0] , [7/39, 46/39, -10/9, -112/117, 32/39] , [9/13, -38/39, -14/3, -16/117, 608/117] , [-3/13, 4/39, 4/3, -64/117, -64/117]] \$ x \$ [[1, 3, 3/2, 1/2, 3] , [9/8, 21/8, 3/4, 3/4, 15/4] , [9/8, 27/8, 27/32, 21/32, 3] , [123/128, 351/128, 27/32, 27/32, 231/64] , [285/256, 855/256, 369/512, 351/512, 201/64]] \$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 5}}, net cycles: 0 . order: 2

$$[y_1, 0, 0, y_2, y_3]$$

R = \$ [[0, 0, 0, 0, 1] , [0, 0, 0, 1, 0] , [1, 0, 0, 0, 0] , [0, 0, 0, 0, 1] , [1, 0, 0, 0, 0]] \$ x \$ [[1, 0, 0, 0, 0] , [0, 0, 0, 0, 0] , [0, 0, 0, 0, 0] , [0, 0, 0, 1, 0] , [0, 0, 0, 0, 1]] \$ = \$ [[0, 2/9, -1/9] , [1/2, -1/9, -5/18] , [0, -1/9, 2/9] , [0, 2/9, -1/9] , [0, -1/9, 2/9]] \$ x \$ [[4, 0, 0, 2, 3] , [3, 0, 0, 0, 6] , [6, 0, 0, 0, 3]] \$

Omega Rank for B : cycles: {{2, 5}}, net cycles: 0 . order: 2

$$[0, y_1, y_2, 0, y_3]$$

$$B = \$ [[0, 0, 1, 0, 0], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0], [0, 1, 0, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[1/2, -1/9, -5/18], [0, 2/9, -1/9], [0, 2/9, -1/9], [0, -1/9, 2/9], [0, -1/9, 2/9]] \$ \times \$ [[0, 4, 2, 0, 3], [0, 3, 0, 0, 6], [0, 6, 0, 0, 3]] \$$$

Â» SYNC'D 3/16 , 0.1875000000

9. Coloring, {3, 4}

R: [5, 5, 5, 2, 2] **B:** [3, 4, 1, 5, 1]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$['-18' (' - 1 + \tau ')'' (' 3 + \tau ^ 2 ')', -18' (' 1 + \tau ')'^ 2 (' - 3 + \tau ')', 9' (' - 1 + \tau ')'^ 2 (' 3 + \tau ^ 2 ')', 9' (' - 1 + \tau ')'' (' 1 + \tau ')'^ 2 (' - 3 + \tau ')', -9' (' 3 + \tau ^ 2 ')'' (' 1 + \tau ')'' (' - 3 + \tau ')'']'$$

For τ=1/2, [52, 180, 13, 45, 195] . FixedPtCheck, [52, 180, 13, 45, 195]

$$\det(A + \tau \Delta) = 1' (' - 1 + \tau ')'^ 3 (' 1 + \tau ')'$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	5 vs 5	5 vs 5	2 vs 2	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 0, 3/4, 0, 1/4], [0, 0, 0, 3/4, 1/4], [3/4, 0, 0, 0, 1/4], [0, 1/4, 0, 0, 3/4], [3/4, 1/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[-13/111, -16/333, -140/999, 32/37, -448/999], [49/37, -8/111, -1180/333, -160/333, 320/111], \\ & [8/37, 7/111, -392/333, -304/333, 640/333], [-28/111, 401/333, 40/999, -112/111, 128/999], [-17/37, \\ & -38/111, 944/333, 128/333, -256/111]] \$ \times \$ [[3, 1, 3/2, 3/2, 2], [21/8, 7/8, 9/4, 3/4, 5/2], [57/16, 13/16, \\ & 63/32, 21/32, 2], [381/128, 85/128, 171/64, 39/64, 133/64], [57/16, 43/64, 1143/512, 255/512, 521/256]] \\ & \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{2, 5}}, net cycles: 1 . order: 2

$$[0, y_2, 0, 0, y_1]$$

$$R = \$ [[0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0], [0, 1, 0, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[5/9, -4/9], [5/9, -4/9], [5/9, -4/9], [-4/9, 5/9], [-4/9, 5/9]] \$ \times \$ [[0, 4, 0, 0, 5], [0, 5, 0, 0, 4]] \$$$

Omega Rank for B : cycles: {{1, 3}}, net cycles: 0 . order: 4

$$[y_1, 0, y_2, y_3, y_4]$$

$$B = \$ [[0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [1, 0, 0, 0, 0], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, 0, -1/9, 2/9], [1/2, -1/4, -5/18, 5/36], [0, 0, 2/9, -1/9], [0, 1/2, -1/9, -5/18], [0, 0, 2/9, -1/9]] \$ \times \$ [[4, 0, 2, 2, 1], [3, 0, 4, 0, 2], [6, 0, 3, 0, 0], [3, 0, 6, 0, 0]] \$$$

Â» SYNC'D 1/16 , 0.06250000000

10 . Coloring, {3, 5}

R: [5, 5, 5, 5, 1] **B:** [3, 4, 1, 2, 2]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$['-18' (' 1 + \tau ')', 18' (' - 1 + \tau ')', 9' (' - 1 + \tau ')'' (' 1 + \tau ')', -9' (' - 1 + \tau ')'^ 2, 9' (' 1 + \tau ')'' (' - 3 + \tau ')'']'$$

For τ=1/2, [-12, -4, -3, -1, -15] . FixedPtCheck, [12, 4, 3, 1, 15]

$$\det(A + \tau \Delta) = 1' (' - 1 + \tau ')'^ 3 ' (' 1 + \tau ')'$$

Delta Range : [y₃, y₄, -y₃ - y₄ - y₁ - y₂, y₁, y₂]

$$[2, 2, 1, 1, 3]$$

$$+ \quad \backslash ; - \quad \backslash ; \Delta$$

$$\$ [[3, 0, 0, 0, 6], [8, 2, 1, 4, 3], [6, 9, 0, 6, 15], [23, 11, 10, 7, 21]] \$ \quad \$ [[1, 4, 2, 2, 0], [0, 6, 3, 0, 9], [10, 7, 8, 2, 9], [9, 21, 6, 9, 27]] \$ \quad \$ [[1, -2, -1, -1, 3], [4, -2, -1, 2, -3], [-2, 1, -4, 2, 3], [7, -5, 2, -1, -3]] \$$$

$$[y_1, y_2, -6y_1 - 7y_2 - 3y_3, 5y_1 + 6y_2 + 2y_3, y_3]$$

$$p = s + 2s^2 - 4s^3 - 8s^4$$

S+ \ ; S- \ ; NM

$$\$ [[4, 3, 2, 1, 5], [3, 4, 1, 2, 5], [4, 3, 2, 1, 5], [3, 4, 1, 2, 5], [3, 3, 2, 2, 5]] \$ \quad \$ [[4, 3, 2, 1, 5], [3, 4, 1, 2, 5], [4, 3, 2, 1, 5], [3, 4, 1, 2, 5], [3, 3, 2, 2, 5]] \$ \quad \$ [[0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0]] \$$$

CmmCk true, true, true

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
3 vs 4	4 vs 5	4 vs 5	2 vs 2	2 vs 4

Omega Rank for R : cycles: $\{\{1, 5\}\}$, net cycles: 1 . order: 2

$$[y_1, 0, 0, 0, y_2]$$

$$R = \$ [[0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[2/9, -1/9], [2/9, -1/9], [2/9, -1/9], [2/9, -1/9], [-1/9, 2/9]] \$ \times \$ [[3, 0, 0, 0, 6], [6, 0, 0, 0, 3]] \$$$

Omega Rank for B : cycles: $\{\{2, 4\}, \{1, 3\}\}$, net cycles: 2 . order: 2

$$\$ [[1, 4, 2, 2, 0], [2, 2, 1, 4, 0], [1, 4, 2, 2, 0], [2, 2, 1, 4, 0]] \$$$

$$[y_2, 2 y_1, y_1, 2 y_2, 0]$$

$$p = -s + s^3 \quad p' = -s + s^3$$

Â» SYNC'D 9/128 , 0.07031250000

11 . Coloring, {4, 5}

R: [5, 5, 1, 2, 1] **B:** [3, 4, 5, 5, 2]

‘ See graph

‘ ‘ See pair graph

‘

Ω for $A+\tau\Delta$:

$$['18' ('1 + \tau')', -18' ('-1 + \tau')', -9' ('-1 + \tau')' ('1 + \tau')', 9' ('-1 + \tau')'^2, 9' ('3 + \tau'^2)']'$$

For $\tau=1/2$, [12, 4, 3, 1, 13] . FixedPtCheck, [12, 4, 3, 1, 13]

$$\det(A + \tau \Delta) = 1' ('-1 + \tau')'^3 ('1 + \tau')'$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	3 vs 3	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 0, 3/4, 0, 1/4], [0, 0, 0, 3/4, 1/4], [1/4, 0, 0, 0, 3/4], [0, 1/4, 0, 0, 3/4], [1/4, 3/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[1104/1001, -1249/3003, -5692/3003, -6224/9009, 18112/9009], [-93/1001, -262/3003, \\ & -2752/3003, 12928/9009, -2048/9009], [-173/1001, 3818/3003, 736/3003, -8672/9009, -2432/9009], \\ & [142/1001, -235/3003, -996/1001, -13712/9009, 23104/9009], [-30/91, -17/273, 580/273, 272/819, \\ & -1600/819]] \$ \times \$ [[1, 5/2, 3/2, 3/2, 5/2], [1, 9/4, 3/4, 15/8, 25/8], [31/32, 45/16, 3/4, 27/16, 89/32], \\ & [113/128, 321/128, 93/128, 135/64, 355/128], [7/8, 1335/512, 339/512, 963/512, 1523/512]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 5}}, net cycles: 0 . order: 2

$$[y_1, y_2, 0, 0, y_3]$$

$$\begin{aligned} R = \$ [[0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0], [0, 1, 0, 0, 0], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ 1, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, 5/9, -4/9], [0, 5/9, -4/9], [0, -4/9, 5/9], \\ [1, -4/9, -4/9], [0, -4/9, 5/9]] \$ \times \$ [[4, 1, 0, 0, 4], [4, 0, 0, 0, 5], [5, 0, 0, 0, 4]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{2, 4, 5}}, net cycles: 0 . order: 3

$$[0, y_1, y_2, y_3, y_4]$$

$$\begin{aligned} B = \$ [[0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0], [0, \\ 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[1/2, 1/27, -8/27, -7/54], [0, 1/27, -8/27, \\ 10/27], [0, 10/27, 1/27, -8/27], [0, 10/27, 1/27, -8/27], [0, -8/27, 10/27, 1/27]] \$ \times \$ [[0, 3, 2, 2, 2], [0, \\ 2, 0, 3, 4], [0, 4, 0, 2, 3], [0, 3, 0, 4, 2]] \$ \end{aligned}$$

Â» SYNC'D 9/128 , 0.07031250000

12 . Coloring, {2, 3, 4}

R: [5, 4, 5, 2, 2] **B:** [3, 5, 1, 5, 1]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$\begin{aligned} & ['-18' (' - 1 + \tau ')^2 (' 3 + \tau ')^2, 18' (' 1 + \tau ')^2 (' - 3 + \tau ')^2, 9' (' - 1 + \tau ')^3 (' 3 + \tau ')^3, \\ & 9' (' 1 + \tau ')^3 (' - 3 + \tau ')^3, -9' (' 1 + \tau ')^2 (' - 1 + \tau ')^2 (' 3 + \tau ')^2 (' - 3 + \tau ')^2]' \end{aligned}$$

For τ=1/2, [-28, -180, -7, -135, -105] . FixedPtCheck, [28, 180, 7, 135, 105]

$$\det(A + \tau \Delta) = 1' (' 1 + \tau ')^2 (' - 1 + \tau ')^2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	3 vs 3	3 vs 3

bi =

$$\begin{aligned} & \$ [[0, 0, 3/4, 0, 1/4], [0, 0, 0, 1/4, 3/4], [3/4, 0, 0, 0, 1/4], [0, 1/4, 0, 0, 3/4], [3/4, 1/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[1/2, 1/2, -32/9, -40/9, 64/9], [-1/2, -7/6, 8, 8/9, -64/9], [0, -1, 10/9, 32/9, -32/9], [4, 1/3, -22, \\ & -64/9, 224/9], [-1, 2/3, 4, 32/9, -64/9]] \$ \times \$ [[3, 1, 3/2, 1/2, 3], [27/8, 7/8, 9/4, 1/4, 9/4], [27/8, 5/8, \\ & 81/32, 7/32, 9/4], [459/128, 79/128, 81/32, 5/32, 135/64], [891/256, 145/256, 1377/512, 79/512, 135/64] \\ &] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{2, 4}}, net cycles: 0 . order: 2

$$[0, y_2, 0, y_1, y_3]$$

$$\begin{aligned} R = & \$ [[0, 0, 0, 0, 1], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0], [0, 1, 0, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[1/3, -4/9, 2/9], [0, -4/9, 5/9], [1/3, -4/9, \\ & 2/9], [0, 5/9, -4/9], [0, 5/9, -4/9]] \$ \times \$ [[0, 4, 0, 2, 3], [0, 5, 0, 4, 0], [0, 4, 0, 5, 0]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 3}}, net cycles: 0 . order: 2

$$[y_1, 0, y_2, 0, y_3]$$

$$\begin{aligned} B = & \$ [[0, 0, 1, 0, 0], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 0, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, -4/9, 5/9], [1/3, -4/9, 2/9], [0, 5/9, -4/9] \\ & , [1/3, -4/9, 2/9], [0, 5/9, -4/9]] \$ \times \$ [[4, 0, 2, 0, 3], [5, 0, 4, 0, 0], [4, 0, 5, 0, 0]] \$ \end{aligned}$$

Â» SYNC'D 3/16 , 0.1875000000

13 . Coloring, {2, 3, 5}

R: [5, 4, 5, 5, 1] **B:** [3, 5, 1, 2, 2]

' See graph

' ' See pair graph

'

Ω for $A+\tau\Delta$:

$$\begin{aligned} & ['-18' (' 3 + \tau^2 ') , -18' (' - 1 + \tau ') ' (' - 3 + \tau ') , 9' (' 3 + \tau^2 ') ' (' - 1 + \tau ') , -9' (' 1 + \tau ' \\ &) ' (' - 1 + \tau ') ' (' - 3 + \tau ') , 9' (' 3 + \tau^2 ') ' (' - 3 + \tau ') '] \end{aligned}$$

For $\tau=1/2$, $[-52, -20, -13, -15, -65]$. FixedPtCheck, $[52, 20, 13, 15, 65]$

$$\det(A + \tau \Delta) = 1 \cdot (1 + \tau)^2 \cdot (-1 + \tau)^2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	3 vs 3	2 vs 4

bi =

$$\begin{aligned} & \$ [[0, 0, 3/4, 0, 1/4], [0, 0, 0, 1/4, 3/4], [3/4, 0, 0, 0, 1/4], [0, 3/4, 0, 0, 1/4], [1/4, 3/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[3/5, 62/15, 48/5, -128/45, -512/45], [-4/15, -19/5, -212/45, 208/45, 64/15], [14/15, 39/5, \\ & 172/45, -368/45, -64/15], [3, -14/3, -16, 32/9, 128/9], [-6/5, -19/15, 4/5, 16/45, 64/45]] \$ \times \$ [[3/2, 3, \\ & 3/2, 1/2, 5/2], [7/4, 9/4, 9/8, 3/4, 25/8], [13/8, 93/32, 21/16, 9/16, 83/32], [209/128, 303/128, 39/32, \\ & 93/128, 391/128], [859/512, 363/128, 627/512, 303/512, 1367/512]] \$ \end{aligned}$$

Check x AllOnes: $[1, 1, 1, 1, 1]$

Omega Rank for R : cycles: $\{\{1, 5\}\}$, net cycles: 0 . order: 2

$$[y_1, 0, 0, y_2, y_3]$$

$$\begin{aligned} \mathbf{R} = & \$ [[0, 0, 0, 0, 1], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0], [0, \\ & 0, 0, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \$ [[0, 5/9, -4/9], [1/2, -4/9, 1/18], [0, 5/9, \\ & -4/9], [0, 5/9, -4/9], [0, -4/9, 5/9]] \$ \times \$ [[3, 0, 0, 2, 4], [4, 0, 0, 0, 5], [5, 0, 0, 0, 4]] \$ \end{aligned}$$

Omega Rank for B : cycles: $\{\{1, 3\}, \{2, 5\}\}$, net cycles: 2 . order: 2

$$\$ [[1, 4, 2, 0, 2], [2, 2, 1, 0, 4], [1, 4, 2, 0, 2], [2, 2, 1, 0, 4]] \$$$

$$[y_1, 2y_2, y_2, 0, 2y_1]$$

$$p' = -s + s^3 \quad p = -s + s^3$$

Â» SYNC'D 3/128 , 0.02343750000

14 . Coloring, $\{2, 4, 5\}$

R: $[5, 4, 1, 2, 1]$ **B:** $[3, 5, 5, 5, 2]$

' See graph

' ' See pair graph

,

Ω for $A+\tau\Delta$:

$$[18(1+\tau)^2(3+\tau), 18(3+\tau^2), -9(1+\tau)(3+\tau)(-1+\tau), 9(1+\tau)(3+\tau^2), 9(3+\tau)(3+\tau^2)]$$

For $\tau=1/2$, [84, 52, 21, 39, 91] . FixedPtCheck, [84, 52, 21, 39, 91]

$$\det(A + \tau \Delta) = 1(1 + \tau)^2(-1 + \tau)^2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	2 vs 4	3 vs 3

bi =

$$\begin{aligned} & \$ [[0, 0, 3/4, 0, 1/4], [0, 0, 0, 1/4, 3/4], [1/4, 0, 0, 0, 3/4], [0, 1/4, 0, 0, 3/4], [1/4, 3/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0], [0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1]] \$ = \\ & \$ [[14/27, -7/243, 596/243, 16/243, -704/243], [-7/27, -118/243, 80/243, 640/243, -512/243], \\ & [5/27, 254/243, -304/243, 160/243, -128/243], [20/27, -253/243, -1324/243, 208/243, 1216/243], [-4/27, \\ & 83/243, 92/243, -560/243, 448/243]] \$ \times \$ [[1, 5/2, 3/2, 1/2, 7/2], [5/4, 11/4, 3/4, 5/8, 29/8], [35/32, \\ & 23/8, 15/16, 11/16, 109/32], [139/128, 349/128, 105/128, 23/32, 467/128], [143/128, 1493/512, 417/512, \\ & 349/512, 1777/512]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 5}, {2, 4}}, net cycles: 2 . order: 2

$$\$ [[4, 1, 0, 2, 2], [2, 2, 0, 1, 4], [4, 1, 0, 2, 2], [2, 2, 0, 1, 4]] \$$$

$$[2y_2, y_1, 0, y_2, 2y_1]$$

$$p' = s - s^3 \quad p = -s + s^3$$

Omega Rank for B : cycles: {{2, 5}}, net cycles: 0 . order: 2

$$[0, y_1, y_2, 0, y_3]$$

$$\begin{aligned} B = & \$ [[0, 0, 1, 0, 0], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 0, 0, 0, 1], [0, 1, 0, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0], [0, \\ & 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 0, 0], [0, 0, 0, 0, 1]] \$ = \$ [[1/2, -4/9, 1/18], [0, 5/9, -4/9], [0, 5/9, \\ & -4/9], [0, 5/9, -4/9], [0, -4/9, 5/9]] \$ \times \$ [[0, 3, 2, 0, 4], [0, 4, 0, 0, 5], [0, 5, 0, 0, 4]] \$ \end{aligned}$$

Â» SYNC'D 3/128 , 0.02343750000

15 . Coloring, {3, 4, 5}

R: [5, 5, 5, 2, 1] **B:** [3, 4, 1, 5, 2]

' See graph

' ' See pair graph

,

Ω for $A+\tau\Delta$:

$$[-18' (' 3 + \tau ^ 2 ') ' , -18' (' - 1 + \tau ') ' (' - 3 + \tau ') ' , 9' (' - 1 + \tau ') ' (' 3 + \tau ^ 2 ') ' , 9' (' - 1 + \tau ') ' ^ 2 ' (' - 3 + \tau ') ' , 9' (' 3 + \tau ^ 2 ') ' (' - 3 + \tau ') '] ' '$$

For $\tau=1/2$, [-52, -20, -13, -5, -65] . FixedPtCheck, [52, 20, 13, 5, 65]

$$\det(A + \tau \Delta) = 1' (' 1 + 3\tau ^ 2 ') ' (' - 1 + \tau ') ' ^ 2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
4 vs 4	5 vs 5	5 vs 5	3 vs 3	4 vs 5

bi =

$$\begin{aligned} & \$ [[0, 0, 3/4, 0, 1/4] , [0, 0, 0, 3/4, 1/4] , [3/4, 0, 0, 0, 1/4] , [0, 1/4, 0, 0, 3/4] , [1/4, 3/4, 0, 0, 0]] \$ \times \$ [[1, \\ & 0, 0, 0, 0] , [0, 1, 0, 0, 0] , [0, 0, 1, 0, 0] , [0, 0, 0, 1, 0] , [0, 0, 0, 0, 1]] \$ = \\ & \$ [[53/38, 119/114, -6/19, -568/171, 224/171] , [-5/76, -65/228, -37/19, 44/171, 368/171] , [-27/76, \\ & 409/228, 115/57, 116/171, -688/171] , [-12/19, -23/57, 44/57, -64/171, 128/171] , [-17/76, -221/228, \\ & 11/19, 332/171, -208/171]] \$ \times \$ [[3/2, 5/2, 3/2, 3/2, 2] , [13/8, 15/8, 9/8, 15/8, 5/2] , [47/32, 75/32, \\ & 39/32, 45/32, 41/16] , [199/128, 291/128, 141/128, 225/128, 37/16] , [719/512, 1113/512, 597/512, \\ & 873/512, 653/256]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 5}}, net cycles: 0 . order: 2

$$[y_3, y_1, 0, 0, y_2]$$

$$\begin{aligned} R = & \$ [[0, 0, 0, 0, 1] , [0, 0, 0, 0, 1] , [0, 0, 0, 0, 1] , [0, 1, 0, 0, 0] , [1, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0] , [0, \\ & 1, 0, 0, 0] , [0, 0, 0, 0, 0] , [0, 0, 0, 0, 1]] \$ = \$ [[0, -4/9, 5/9] , [0, -4/9, 5/9] , [0, -4/9, 5/9] , \\ & [1, 5/9, -13/9] , [0, 5/9, -4/9]] \$ \times \$ [[3, 1, 0, 0, 5] , [5, 0, 0, 0, 4] , [4, 0, 0, 0, 5]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 3}, {2, 4, 5}}, net cycles: 2 .

$$\$ [[1, 3, 2, 2, 1] , [2, 1, 1, 3, 2] , [1, 2, 2, 1, 3] , [2, 3, 1, 2, 1] , [1, 1, 2, 3, 2]] \$$$

$$[y_4, y_3, y_2, y_1, 2y_4 - y_3 + 2y_2 - y_1]$$

$$p' = -1 - s + s^3 + s^4$$

Â» SYNC'D 45/512 , 0.08789062500

16. Coloring, {2, 3, 4, 5}

R: [5, 4, 5, 2, 1] **B:** [3, 5, 1, 5, 2]

' See graph

' ' See pair graph

,

Ω for $A+\tau\Delta$:

' ['-6' (' 3 + τ ')', 6' (' - 3 + τ ')', 3' (' - 1 + τ ')'' (' 3 + τ ')', 3' (' 1 + τ ')'' (' - 3 + τ ')', 3' (' 3 + τ ')'' (' - 3 + τ ')'']'

For $\tau=1/2$, [-28, -20, -7, -15, -35] . FixedPtCheck, [28, 20, 7, 15, 35]

$\det(A + \tau \Delta) = 1' (' - 1 + \tau ')'' (' 1 + 3\tau^2 ')'' (' 1 + \tau ')'$

Delta Range : [y₃, y₄, -y₃ - y₄ - y₁ - y₂, y₁, y₂]

[2, 2, 1, 1, 3]

+ \ ; - \ ; Δ

\$ [[3, 1, 0, 2, 3] , [5, 5, 1, 1, 6] , [9, 7, 3, 5, 12] , [17, 17, 7, 7, 24]] \$ \$ [[1, 3, 2, 0, 3] , [3, 3, 3, 3, 6] , [7, 9, 5, 3, 12] , [15, 15, 9, 9, 24]] \$ \$ [[1, -1, -1, 1, 0] , [1, 1, -1, -1, 0] , [1, -1, -1, 1, 0] , [1, 1, -1, -1, 0]] \$

[-y₂, -y₁, y₂, y₁, 0]

$p' = s - 4s^3$ $p = s - 4s^3$

S+ \ ; S- \ ; NM

\$ [[1, 0, 2, 1, 1] , [1, 0, 0, 1, 3] , [2, 0, 1, 1, 1] , [1, 1, 0, 0, 3] , [1, 3, 0, 0, 1]] \$ \$ [[0, 1, 1, 0, 3] , [0, 1, 1, 2, 1] , [1, 1, 0, 0, 3] , [0, 2, 1, 1, 1] , [3, 1, 0, 0, 1]] \$ \$ [[4, 2, 2, 1, 3] , [2, 4, 1, 2, 3] , [4, 2, 2, 1, 3] , [2, 4, 1, 2, 3] , [2, 2, 1, 1, 6]] \$

CmmCk true, true, true

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
2 vs 4	3 vs 5	3 vs 5	2 vs 4	2 vs 4

Omega Rank for R : cycles: {{1, 5}, {2, 4}}, net cycles: 2 . order: 2

\$ [[3, 1, 0, 2, 3] , [3, 2, 0, 1, 3] , [3, 1, 0, 2, 3] , [3, 2, 0, 1, 3]] \$

[y₂, y₂ - y₁, 0, y₁, y₂]

$$p = -s + s^3 \quad p' = -s + s^3$$

Omega Rank for B : cycles: $\{\{1, 3\}, \{2, 5\}\}$, net cycles: 2 . order: 2

$$\$ [[1, 3, 2, 0, 3], [2, 3, 1, 0, 3], [1, 3, 2, 0, 3], [2, 3, 1, 0, 3]] \$$$

$$[y_2 - y_1, y_2, y_1, 0, y_2]$$

$$p = -s + s^3 \quad p' = -s + s^3$$

Â« NOT SYNC'D Â»

Nullspace of $\{\Omega\Delta^i\}$:

$$[x_1, x_2, -4x_1, -4x_2]$$

$$\text{For } A+2\Delta : [-y_1 - y_2, y_1, -y_1 - y_2, y_1, y_2]$$

$$\text{For } A-2\Delta : [y_2, y_1, y_2, y_1, -y_1 - y_2]$$

$$\text{Range of } \{\Omega\Delta^i\} : [\mu_2, -\mu_1, -\mu_2, \mu_1, 0]$$

rank of M is 5 , rank of N is 3

M N

$$\$ [[0, 1, 0, 1, 2], [1, 0, 1, 0, 2], [0, 1, 0, 0, 1], [1, 0, 0, 0, 1], [2, 2, 1, 1, 0]] \$ \quad \$ [[0, 1, 0, 1, 1], [1, 0, 1, 0, 1], [0, 1, 0, 1, 1], [1, 0, 1, 0, 1], [1, 1, 1, 1, 0]] \$$$

Check is $\Omega\Delta N$ zero? *true*, $\pi\Delta = [1, -1, -1, 1, 0]$

$$\text{ker M, } [0, 0, 0, 0, 0]$$

$$\text{Range M, } [x_1, x_4, x_5, x_2, x_3]$$

$$\tau = 9, r' = 2/3$$

Ranges

$$\text{Action of R on ranges, } [[2], [1], [2]]$$

$$\text{Action of B on ranges, } [[3], [3], [1]]$$

$$\beta(\{1, 2, 5\}) = 1/3$$

$$\beta(\{1, 4, 5\}) = 1/3$$

$$\beta(\{2, 3, 5\}) = 1/3$$

$$\text{ker N, } [-\mu_1, -\mu_2, \mu_1, \mu_2, 0]$$

Range of N

$$[y_1, y_2, y_1, y_2, y_3]$$

Partitions

$$\alpha(\{\{1, 3\}, \{2, 4\}, \{5\}\}) = 1/1$$

$$b1 = \{1, 3\}, b2 = \{2, 4\}, b3 = \{5\}$$

Action of R and B on the blocks of the partitions: $\$ [[1, 0, 1], [0, 1, 1], [1, 1, 0]] \$ = \$ [[0, 0, 1], [0, 1, 0], [1, 0, 0]] \$ + \$ [[1, 0, 0], [0, 0, 1], [0, 1, 0]] \$$
 $['3', '2', '1'], ['1', '3', '2']$ with invariant measure $[1, 1, 1]$

N by blocks, check: true . ' See partition graph.

' ' See level-3 partition graph.

'

Right Group	
Coloring	{2, 3, 4, 5}
Rank	3
R,B	[5, 4, 5, 2, 1], [3, 5, 1, 5, 2]
π_2	[1, 0, 1, 2, 1, 0, 2, 0, 1, 1]
u_2	[1, 0, 1, 1, 1, 0, 1, 1, 1, 1] (dim 1)
wpp	[2, 2, 2, 2, 1]
π_3	[0, 0, 1, 0, 0, 1, 0, 1, 0, 0]
u_3	[0, 0, 1, 0, 0, 1, 0, 1, 0, 1]

SUMMARY	
Graph Type	NOT CC
$v(A)$	0
$v(\Delta)$	1
π	[2, 2, 1, 1, 3]
Dbly Stoch	false

SANDWICH		Total 0
No .	Coloring	Rank

RT GROUPS		Total 2	
No .	Coloring	Rank	Solv
1	{2, 3, 4, 5}	3	Not Solvable
2	{2}	3	Not Solvable

Δ-RANK'D	SC'D !RK'D	τ-RANK'D	R/B RANK'D	NOT SYNC'D	Total Runs	2^{n-1}
12	0	12 , 12	13 , 9	2	16	16
