

## New Graph

[3, 3, 1, 1, 7, 7, 5, 5], [6, 8, 8, 6, 2, 4, 4, 2]

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$$\pi = [1, 1, 1, 1, 1, 1, 1, 1]$$

POSSIBLE RANKS

$$\begin{array}{l} 1 \times 8 \\ 2 \times 4 \end{array}$$

BASE DETERMINANT 4236243/134217728, .3156246990e-1

*NullSpace* of  $\Delta$

{1, 3, 6, 8}, {2, 4, 5, 7}

*Range* of  $\Delta$ :  $[\lambda_4, \lambda_5, \lambda_6, -\lambda_5 - \lambda_1 - \lambda_2, \lambda_1, -\lambda_4 - \lambda_6 - \lambda_3, \lambda_2, \lambda_3]$

1 . Coloring, {}

**R**: [3, 3, 1, 1, 7, 7, 5, 5]   **B**: [6, 8, 8, 6, 2, 4, 4, 2]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A + \tau \Delta$  :

‘ [ ‘1‘ (‘1 +  $\tau$ ‘)‘, -1‘ (‘-1 +  $\tau$ ‘)‘, 1‘ (‘1 +  $\tau$ ‘)‘, -1‘ (‘-1 +  $\tau$ ‘)‘, 1‘ (‘1 +  $\tau$ ‘)‘, -1‘ (‘-1 +  $\tau$ ‘)‘  
, 1‘ (‘1 +  $\tau$ ‘)‘, -1‘ (‘-1 +  $\tau$ ‘)‘ ] ‘

For  $\tau=1/2$ , [3, 1, 3, 1, 3, 1, 3, 1] . FixedPtCheck, [3, 1, 3, 1, 3, 1, 3, 1]

$\det(A + \tau \Delta) = 0$

Delta Range :  $[y_2, -y_1 - y_3 - y_5, -y_2 - y_4 - y_6, y_1, y_3, y_4, y_5, y_6]$

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

\$ [ [2, 0, 2, 0, 2, 0, 2, 0], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$ \$ [ [0, 2, 0, 2, 0, 2, 0, 2], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$ \$ [ [1, -1, 1, -1, 1, -1, 1, -1], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$

$$[-y_1, y_1, -y_1, y_1, -y_1, y_1, -y_1, y_1]$$

$$p = s^2$$

S+ \ ; S- \ ; NM  
 \$ [ [1, 0, 0, 1, 0, 0, 0, 0], [1, 1, 0, 0, 0, 0, 0, 0], [0, 1, 1, 0, 0, 0, 0, 0], [0, 0, 1, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 1], [0, 0, 0, 0, 1, 1, 0, 0], [0, 0, 0, 0, 0, 1, 1, 0], [0, 0, 0, 0, 0, 0, 1, 1] ] \$ \$ [ [0, 0, 0, 0, 0, 0, 1, 1], [0, 0, 0, 0, 1, 0, 0, 1], [0, 0, 0, 0, 1, 1, 0, 0], [0, 0, 0, 0, 0, 1, 1, 0], [0, 0, 1, 1, 0, 0, 0, 0], [1, 0, 0, 1, 0, 0, 0, 0], [1, 1, 0, 0, 0, 0, 0, 0], [0, 1, 1, 0, 0, 0, 0, 0] ] \$ \$ [ [12, 10, 8, 10, 8, 8, 8, 8], [10, 12, 10, 8, 8, 8, 8, 8], [8, 10, 12, 10, 8, 8, 8, 8], [10, 8, 10, 12, 8, 8, 8, 8], [8, 8, 8, 8, 12, 10, 8, 10], [8, 8, 8, 8, 10, 12, 10, 8], [8, 8, 8, 8, 8, 10, 12, 10], [8, 8, 8, 8, 10, 8, 10, 12] ] \$

CmmCk true, true, true

$$p' = s^2 \quad p' = s^3 \quad p' = s^4 \quad p' = s^5$$

$\Delta$ -Rank	A+(1/2) $\Delta$	A-(1/2) $\Delta$	R	B
1 vs 6	1 vs 6	1 vs 6	1 vs 4	1 vs 4

Omega Rank for R : cycles: {{1, 3}, {5, 7}}, net cycles: 2 . order: 2

\$ [ [2, 0, 2, 0, 2, 0, 2, 0], [2, 0, 2, 0, 2, 0, 2, 0], [2, 0, 2, 0, 2, 0, 2, 0], [2, 0, 2, 0, 2, 0, 2, 0] ] \$

$$[y_1, 0, y_1, 0, y_1, 0, y_1, 0]$$

$$p = s - s^2 \quad p' = -s + s^2 \quad p' = -s + s^3$$

Omega Rank for B : cycles: {{4, 6}, {2, 8}}, net cycles: 2 . order: 2

\$ [ [0, 2, 0, 2, 0, 2, 0, 2], [0, 2, 0, 2, 0, 2, 0, 2], [0, 2, 0, 2, 0, 2, 0, 2], [0, 2, 0, 2, 0, 2, 0, 2] ] \$

$$[0, y_1, 0, y_1, 0, y_1, 0, y_1]$$

$$p = -s + s^2 \quad p = -s + s^3 \quad p = -s + s^4$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[0, x_5, x_4, x_3, x_2, x_1]$$

For A+2 $\Delta$  :  $[y_1, -3y_1 - 3y_2 - y_3 - 3y_4 - y_5 - 3y_6 - y_7, y_2, y_3, y_4, y_5, y_6, y_7]$

For A-2 $\Delta$  :  $[y_6, y_5, y_4, y_3, y_2, y_1, -y_6 - 3y_5 - y_4 - 3y_3 - y_2 - 3y_1 - 3y_7, y_7]$

Range of  $\{\Omega\Delta^i\}$ :  $[-\mu_1, \mu_1, -\mu_1, \mu_1, -\mu_1, \mu_1, -\mu_1, \mu_1]$

rank of M is 8 , rank of N is 6

M \ ; N

$\$ [ [0, 0, 2, 0, 2, 0, 2, 0], [0, 0, 0, 2, 0, 2, 0, 2], [2, 0, 0, 0, 2, 0, 2, 0], [0, 2, 0, 0, 0, 2, 0, 2], [2, 0, 2, 0, 0, 0, 2, 0], [0, 2, 0, 2, 0, 0, 0, 2], [2, 0, 2, 0, 2, 0, 0, 0], [0, 2, 0, 2, 0, 2, 0, 0] ] \$$   $\$ [ [0, 1, 2, 1, 2, 2, 2, 2], [1, 0, 1, 2, 2, 2, 2, 2], [2, 1, 0, 1, 2, 2, 2, 2], [1, 2, 1, 0, 2, 2, 2, 2], [2, 2, 2, 2, 0, 1, 2, 1], [2, 2, 2, 2, 1, 0, 1, 2], [2, 2, 2, 2, 2, 1, 0, 1], [2, 2, 2, 2, 1, 2, 1, 0] ] \$$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [1, -1, 1, -1, 1, -1, 1, -1]$

ker M,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range M,  $[x_4, x_5, x_6, x_7, x_8, x_3, x_1, x_2]$

$\tau = 16$  ,  $r' = 3/4$

Ranges

Action of R on ranges,  $[[1], [1]]$

Action of B on ranges,  $[[2], [2]]$

$\beta(\{1, 3, 5, 7\}) = 1/2$

$\beta(\{2, 4, 6, 8\}) = 1/2$

ker N,  $[-\mu_1, \mu_1, -\mu_1, \mu_1, -\mu_2, \mu_2, -\mu_2, \mu_2]$

Range of N

$[y_1 - y_2 + y_3, y_1, y_2, y_3, y_4 - y_5 + y_6, y_4, y_5, y_6]$

Partitions

Action of R on partitions,  $[[4], [4], [4], [4]]$

Action of B on partitions,  $[[1], [1], [1], [1]]$

$\alpha(\{\{2, 3\}, \{1, 4\}, \{6, 7\}, \{5, 8\}\}) = 1/2$

$\alpha(\{\{1, 2\}, \{3, 4\}, \{6, 7\}, \{5, 8\}\}) = 0/1$

$\alpha(\{\{5, 6\}, \{7, 8\}, \{2, 3\}, \{1, 4\}\}) = 0/1$

$\alpha(\{\{1, 2\}, \{5, 6\}, \{7, 8\}, \{3, 4\}\}) = 1/2$

$b_1 = \{1, 2\}$  ,  $b_2 = \{5, 6\}$  ,  $b_3 = \{7, 8\}$  ,  $b_4 = \{3, 4\}$  ,  $b_5 = \{2, 3\}$  ,  $b_6 = \{1, 4\}$  ,  $b_7 = \{6, 7\}$  ,  $b_8 = \{5, 8\}$

Action of R and B on the blocks of the partitions:  $\$ [ [0, 0, 0, 1, 0, 0, 0, 1], [0, 0, 1, 0, 0, 1, 0, 0], [0, 1, 0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0, 1, 0], [0, 1, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 1, 0, 0, 0] ] \$ = \$ [ [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0] ] \$ + \$ [ [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$$



For  $\tau=1/2$ , [105, 57, 93, 47, 162, 38, 150, 66] . FixedPtCheck, [105, 57, 93, 47, 162, 38, 150, 66]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	3 vs 5	3 vs 5

Omega Rank for R : cycles:  $\{\{1, 3\}, \{5, 7\}\}$ , net cycles: 1 . order: 2

$\$ [ [2, 0, 1, 0, 2, 0, 2, 1], [1, 0, 2, 0, 3, 0, 2, 0], [2, 0, 1, 0, 2, 0, 3, 0], [1, 0, 2, 0, 3, 0, 2, 0], [2, 0, 1, 0, 2, 0, 3, 0] ] \$$

$$[-4y_1 + 3y_2, 0, y_1, 0, y_2, 0, -5y_1 + 4y_2 - y_3, y_3]$$

$$p = -s^2 + s^4 \quad p' = -s^2 + s^4$$

Omega Rank for B : cycles:  $\{\{2, 3, 8\}, \{4, 6\}\}$ , net cycles: 2 . order: 6

$\$ [ [0, 2, 1, 2, 0, 2, 0, 1], [0, 1, 2, 2, 0, 2, 0, 1], [0, 1, 1, 2, 0, 2, 0, 2], [0, 2, 1, 2, 0, 2, 0, 1], [0, 1, 2, 2, 0, 2, 0, 1] ] \$$

$$[0, -y_1 + 2y_2 - y_3, y_1, y_2, 0, y_2, 0, y_3]$$

$$p' = -s + s^4 \quad p = -s + s^4$$

Â» SYNC'D 15/2048 , 0.007324218750

3 . Coloring, {3}

**R**: [3, 3, 8, 1, 7, 7, 5, 5] **B**: [6, 8, 1, 6, 2, 4, 4, 2]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ '-4' ('1 + \tau')'' ('5 - \tau - \tau^2 + \tau^3')'' ('-1 + \tau')', 4' ('1 + \tau')'' ('-5 - \tau + \tau^2 + \tau^3')'' ('-1 + \tau')', \\ & 4' ('1 + \tau')''^2 ('-5 + \tau^2')'' ('-1 + \tau')', 4' ('-5 + 2\tau - 4\tau^2 - 2\tau^3 + \tau^4')'' ('-1 + \tau')', \\ & -4' ('1 + \tau')''^2 ('-5 + \tau - \tau^2 + \tau^3')', 4' ('5 + \tau + \tau^2 + \tau^3')'' ('-1 + \tau')''^2, -4' ('1 + \tau')''^2 \\ & ('1 + \tau^2')'' ('-5 + \tau^2')', 4' ('1 + \tau')'' ('-5 - 3\tau - \tau^2 + \tau^3')'' ('-1 + \tau')'' ]' \end{aligned}$$

For  $\tau=1/2$ , [105, 123, 171, 83, 333, 47, 285, 159] . FixedPtCheck, [105, 123, 171, 83, 333, 47, 285, 159]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	4 vs 5	3 vs 5

Omega Rank for R : cycles:  $\{\{5, 7\}\}$ , net cycles: 0 . order: 4

$\$ [ [1, 0, 2, 0, 2, 0, 2, 1], [0, 0, 1, 0, 3, 0, 2, 2], [0, 0, 0, 0, 4, 0, 3, 1], [0, 0, 0, 0, 4, 0, 4, 0], [0, 0, 0, 0, 4, 0, 4, 0] ] \$$

$$[y_4, 0, y_3, 0, y_2, 0, y_1, -y_4 + y_3 + y_2 - y_1]$$

$$p = s^4 - s^5$$

Omega Rank for B : cycles:  $\{\{2, 8\}, \{4, 6\}\}$ , net cycles: 1 . order: 2

$\$ [ [1, 2, 0, 2, 0, 2, 0, 1], [0, 1, 0, 2, 0, 3, 0, 2], [0, 2, 0, 3, 0, 2, 0, 1], [0, 1, 0, 2, 0, 3, 0, 2], [0, 2, 0, 3, 0, 2, 0, 1] ] \$$

$$[-y_1 + 4y_3 - 5y_2, 3y_3 - 4y_2, 0, y_1, 0, y_3, 0, y_2]$$

$$p = -s^2 + s^4 \quad p' = -s^2 + s^4$$

$\hat{A} \gg \text{SYNC'D } 5/512, 0.009765625000$

4 . Coloring,  $\{4\}$

**R:** [3, 3, 1, 6, 7, 7, 5, 5]    **B:** [6, 8, 8, 1, 2, 4, 4, 2]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$[ '4' ('5 - 3\tau + \tau^2 + \tau^3') ('1 + \tau')', -4' ('-1 + \tau') ('5 + \tau + \tau^2 + \tau^3')', 4' ('5 - \tau - \tau^2 + \tau^3') ('1 + \tau')', 4' ('-1 + \tau') ('1 + \tau') ('-5 + \tau^2')', 4' ('1 + \tau') ('5 + 2\tau + \tau^2')', -4' ('5 + \tau') ('-1 + \tau') ('1 + \tau')', -4' ('1 + \tau')^2 ('-5 + \tau')', 4' ('-1 + \tau') ('-5 + \tau^2') ]'$

For  $\tau=1/2$ , [93, 47, 105, 57, 150, 66, 162, 38] . FixedPtCheck, [93, 47, 105, 57, 150, 66, 162, 38]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	3 vs 5	3 vs 5

Omega Rank for R : cycles:  $\{\{1, 3\}, \{5, 7\}\}$ , net cycles: 1 . order: 2

$\$ [ [1, 0, 2, 0, 2, 1, 2, 0], [2, 0, 1, 0, 2, 0, 3, 0], [1, 0, 2, 0, 3, 0, 2, 0], [2, 0, 1, 0, 2, 0, 3, 0], [1, 0, 2, 0, 3, 0, 2, 0] ] \$$

$$[y_1, 0, -4y_1 + 3y_3, 0, -5y_1 - y_2 + 4y_3, y_2, y_3, 0]$$

$$p' = -s^2 + s^4 \quad p = -s^2 + s^4$$

Omega Rank for B : cycles:  $\{\{2, 8\}, \{1, 4, 6\}\}$ , net cycles: 2 . order: 6

$\$ [ [1, 2, 0, 2, 0, 1, 0, 2], [2, 2, 0, 1, 0, 1, 0, 2], [1, 2, 0, 1, 0, 2, 0, 2], [1, 2, 0, 2, 0, 1, 0, 2], [2, 2, 0, 1, 0, 1, 0, 2] ] \$$

$$[2y_3 - y_1 - y_2, y_3, 0, y_1, 0, y_2, 0, y_3]$$

$$p = -s + s^4 \quad p' = -s + s^4$$

Â» SYNC'D 15/2048 , 0.007324218750

5 . Coloring, {5}

**R:** [3, 3, 1, 1, 2, 7, 5, 5]   **B:** [6, 8, 8, 6, 7, 4, 4, 2]

' See graph

' ' See pair graph

Ω for  $A + \tau \Delta$  :

$$\begin{aligned} & [ '-4' ('1 + \tau')^{2'} ('1 + \tau^2')^{2'} ('-5 + \tau^2')^{2'} , 4' ('-1 + \tau')^{2'} ('1 + \tau')^{2'} ('-5 - 3\tau - \tau^2 + \tau^3')^{2'} \\ & ' , -4' ('1 + \tau')^{2'} ('-5 + \tau - \tau^2 + \tau^3')^{2'} , 4' ('-1 + \tau')^{2'} ('5 + \tau + \tau^2 + \tau^3')^{2'} , 4' ('-1 + \tau')^{2'} \\ & ' ('1 + \tau')^{2'} ('-5 + \tau^2')^{2'} , 4' ('-1 + \tau')^{2'} ('-5 + 2\tau - 4\tau^2 - 2\tau^3 + \tau^4')^{2'} , -4' ('-1 + \tau')^{2'} \\ & ('1 + \tau')^{2'} ('5 - \tau - \tau^2 + \tau^3')^{2'} , 4' ('-1 + \tau')^{2'} ('1 + \tau')^{2'} ('-5 - \tau + \tau^2 + \tau^3')^{2'} ]' \end{aligned}$$

For  $\tau=1/2$ , [285, 159, 333, 47, 171, 83, 105, 123] . FixedPtCheck, [285, 159, 333, 47, 171, 83, 105, 123]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	4 vs 5	3 vs 5

Omega Rank for R : cycles:  $\{\{1, 3\}\}$ , net cycles: 0 . order: 4

$\$ [ [2, 1, 2, 0, 2, 0, 1, 0], [2, 2, 3, 0, 1, 0, 0, 0], [3, 1, 4, 0, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0] ] \$$

$$[-y_1 + y_2 + y_3 - y_4, y_1, y_2, 0, y_3, 0, y_4, 0]$$

$$p = -s^4 + s^5$$

Omega Rank for B : cycles: {{2, 8}, {4, 6}}, net cycles: 1 . order: 2

$$\$ [ [0, 1, 0, 2, 0, 2, 1, 2], [0, 2, 0, 3, 0, 2, 0, 1], [0, 1, 0, 2, 0, 3, 0, 2], [0, 2, 0, 3, 0, 2, 0, 1], [0, 1, 0, 2, 0, 3, 0, 2] ] \$$$

$$[0, y_1, 0, y_2, 0, -5y_1 + 4y_2 - y_3, y_3, -4y_1 + 3y_2]$$

$$p = -s^2 + s^4 \quad p' = -s^2 + s^4$$

Â» SYNC'D 5/512 , 0.009765625000

6 . Coloring, {6}

**R:** [3, 3, 1, 1, 7, 4, 5, 5]    **B:** [6, 8, 8, 6, 2, 7, 4, 2]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ '-4' ('1 + \tau')^2 ('-5 + \tau')^4 ('-5 + \tau^2')^4 ('-1 + \tau')^4, 4' ('1 + \tau')^4 ('5 + 2\tau + \tau^2')^4, \\ & -4' ('5 + \tau')^4 ('1 + \tau')^4 ('-1 + \tau')^4, 4' ('1 + \tau')^4 ('5 - \tau - \tau^2 + \tau^3')^4, 4' ('1 + \tau')^4 ('-5 + \tau^2')^4 ('-1 + \tau')^4, \\ & 4' ('1 + \tau')^4 ('5 - 3\tau + \tau^2 + \tau^3')^4, -4' ('5 + \tau + \tau^2 + \tau^3')^4 ('-1 + \tau')^4 ] \end{aligned}$$

For τ=1/2, [162, 38, 150, 66, 105, 57, 93, 47] . FixedPtCheck, [162, 38, 150, 66, 105, 57, 93, 47]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	3 vs 5	3 vs 5

Omega Rank for R : cycles: {{1, 3}, {5, 7}}, net cycles: 1 . order: 2

$$\$ [ [2, 0, 2, 1, 2, 0, 1, 0], [3, 0, 2, 0, 1, 0, 2, 0], [2, 0, 3, 0, 2, 0, 1, 0], [3, 0, 2, 0, 1, 0, 2, 0], [2, 0, 3, 0, 2, 0, 1, 0] ] \$$$

$$[y_1, 0, 4y_1 - y_2 - 5y_3, y_2, 3y_1 - 4y_3, 0, y_3, 0]$$



$$p = -s^2 + s^4 \quad p' = -s^2 + s^4$$

Omega Rank for B : cycles: {{4, 6, 7}, {2, 8}}, net cycles: 2 . order: 6

$$\$ [ [0, 2, 0, 1, 0, 2, 1, 2], [0, 2, 0, 1, 0, 1, 2, 2], [0, 2, 0, 2, 0, 1, 1, 2], [0, 2, 0, 1, 0, 2, 1, 2], [0, 2, 0, 1, 0, 1, 2, 2] ] \$$$

$$[0, y_2, 0, 2y_2 - y_1 - y_3, 0, y_1, y_3, y_2]$$

$$p = -s + s^4 \quad p' = -s + s^4$$

Â» SYNC'D 15/2048 , 0.007324218750

7 . Coloring, {7}

**R**: [3, 3, 1, 1, 7, 7, 4, 5]    **B**: [6, 8, 8, 6, 2, 4, 5, 2]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$\begin{aligned} & [ '-4' ('1 + \tau')^2 ('-5 + \tau - \tau^2 + \tau^3')', 4' ('5 + \tau + \tau^2 + \tau^3')' ('-1 + \tau')^2, -4' ('1 + \tau') \\ & ('1 + \tau^2')' ('-5 + \tau^2')', 4' ('1 + \tau')' ('-5 - 3\tau - \tau^2 + \tau^3')' ('-1 + \tau')', -4' ('1 + \tau') \\ & ('5 - \tau - \tau^2 + \tau^3')' ('-1 + \tau')', 4' ('1 + \tau')' ('-5 - \tau + \tau^2 + \tau^3')' ('-1 + \tau')', 4' ('1 + \tau') \\ & (')^2 ('-5 + \tau^2')' ('-1 + \tau')', 4' ('-5 + 2\tau - 4\tau^2 - 2\tau^3 + \tau^4')' ('-1 + \tau')' ]' \end{aligned}$$

For τ=1/2, [333, 47, 285, 159, 105, 123, 171, 83] . FixedPtCheck, [333, 47, 285, 159, 105, 123, 171, 83]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	4 vs 5	3 vs 5

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 4

$$\$ [ [2, 0, 2, 1, 1, 0, 2, 0], [3, 0, 2, 2, 0, 0, 1, 0], [4, 0, 3, 1, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0] ] \$$$

$$[y_4, 0, y_4 - y_1 - y_2 + y_3, y_1, y_2, 0, y_3, 0]$$

$$p = -s^4 + s^5$$

Omega Rank for B : cycles: {{4, 6}, {2, 8}}, net cycles: 1 . order: 2

\$ [ [0, 2, 0, 1, 1, 2, 0, 2] , [0, 3, 0, 2, 0, 1, 0, 2] , [0, 2, 0, 1, 0, 2, 0, 3] , [0, 3, 0, 2, 0, 1, 0, 2] , [0, 2, 0, 1, 0, 2, 0, 3] ] \$

$$[0, y_1, 0, y_2, 4y_1 - 5y_2 - y_3, 3y_1 - 4y_2, 0, y_3]$$

$$p = -s^2 + s^4 \quad p' = s^2 - s^4$$

Â» SYNC'D 5/512 , 0.009765625000

8 . Coloring, {8}

**R:** [3, 3, 1, 1, 7, 7, 5, 2]    **B:** [6, 8, 8, 6, 2, 4, 4, 5]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$\begin{bmatrix} 4(1+\tau)^2(5+2\tau+\tau^2), & -4(5+\tau)(-1+\tau)(1+\tau), & -4(1+\tau)^2(-5+\tau), & 4(-1+\tau)(-5+\tau^2), & 4(1+\tau)(5-3\tau+\tau^2+\tau^3), & -4(-1+\tau)(5+\tau+\tau^2+\tau^3), & 4(1+\tau)(5-\tau-\tau^2+\tau^3), & 4(-1+\tau)(1+\tau)(-5+\tau^2) \end{bmatrix}$$

For τ=1/2, [150, 66, 162, 38, 93, 47, 105, 57] . FixedPtCheck, [150, 66, 162, 38, 93, 47, 105, 57]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	3 vs 5	3 vs 5

Omega Rank for R : cycles: {{1, 3}, {5, 7}}, net cycles: 1 . order: 2

\$ [ [2, 1, 2, 0, 1, 0, 2, 0] , [2, 0, 3, 0, 2, 0, 1, 0] , [3, 0, 2, 0, 1, 0, 2, 0] , [2, 0, 3, 0, 2, 0, 1, 0] , [3, 0, 2, 0, 1, 0, 2, 0] ] \$

$$[-y_1 + 4y_2 - 5y_3, y_1, y_2, 0, y_3, 0, 3y_2 - 4y_3, 0]$$

$$p = -s^2 + s^4 \quad p' = -s^2 + s^4$$

Omega Rank for B : cycles: {{2, 5, 8}, {4, 6}}, net cycles: 2 . order: 6

\$ [ [0, 1, 0, 2, 1, 2, 0, 2] , [0, 1, 0, 2, 2, 2, 0, 1] , [0, 2, 0, 2, 1, 2, 0, 1] , [0, 1, 0, 2, 1, 2, 0, 2] , [0, 1, 0, 2, 2, 2, 0, 1] ] \$

$$[0, y_3, 0, y_2, y_1, y_2, 0, -y_3 + 2y_2 - y_1]$$

$$p = -s + s^4 \quad p' = -s + s^4$$

Â» SYNC'D 15/2048 , 0.007324218750

9. Coloring, {2, 3}

**R:** [3, 8, 8, 1, 7, 7, 5, 5]   **B:** [6, 3, 1, 6, 2, 4, 4, 2]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ '-2' (' - 1 + \tau ')'' (' - 5 + 2\tau - 4\tau^2 - 2\tau^3 + \tau^4 ')'' (' 1 + \tau ')', -2' (' - 1 + \tau ')'' (' 1 + \tau ')'^2 ' \\ & (' - 5 - \tau + \tau^2 + \tau^3 ')', 2' (' 5 - 3\tau + \tau^2 + \tau^3 ')'' (' - 1 + \tau ')'' (' 1 + \tau ')'^2, 2' (' 5 - \tau + 12\tau^2 - \tau \\ & 4 + \tau^5 ')'' (' - 1 + \tau ')', 6' (' - 5 - 3\tau - 3\tau^2 + 3\tau^3 ')'' (' 1 + \tau ')'^2, -6' (' 5 + \tau + 7\tau^2 + 3\tau^3 ')'' (' \\ & - 1 + \tau ')'^2, 2' (' - 5 - 2\tau - 12\tau^2 + 2\tau^3 + \tau^4 ')'' (' 1 + \tau ')', -2' (' - 1 + \tau ')'' (' 1 + \tau ')'^3 ' (' - 5 + \\ & \tau ')'' ]' \end{aligned}$$

For  $\tau=1/2$ , [-249, -369, -279, -239, -990, -122, -834, -486] . FixedPtCheck, [249, 369, 279, 239, 990, 122, 834, 486]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	6 vs 6	6 vs 6	5 vs 5	5 vs 5

Omega Rank for R : cycles: {{5, 7}}, net cycles: 0 . order: 4

$$[y_3, 0, y_4, 0, y_5, 0, y_1, y_2]$$

$$\begin{aligned} R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, \\ 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0] \\ , [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, \\ 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 1, -1, -11/16, 13/16], [0, 0, 1, -3/16, -11/16] \\ , [0, 0, 1, -3/16, -11/16], [1, -1, -1, 13/16, 5/16], [0, 0, 0, -3/16, 5/16], [0, 0, 0, -3/16, 5/16], [0, 0, 0, 5/16, \\ -3/16], [0, 0, 0, 5/16, -3/16] ] \times \$ [ [1, 0, 1, 0, 2, 0, 2, 2], [0, 0, 1, 0, 4, 0, 2, 1], [0, 0, 0, 0, 3, 0, 4, 1], [0, \\ 0, 0, 0, 5, 0, 3, 0], [0, 0, 0, 0, 3, 0, 5, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$[y_3, y_2, y_1, y_5, 0, y_4, 0, 0]$$

$$\mathbf{B} = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] = \$ [ [0, 0, 0, 5/16, -3/16], [0, 1/2, -1/4, -3/16, 1/16], [0, 0, 1/2, -3/16, -3/16], [0, 0, 0, 5/16, -3/16], [1/2, -1/4, -1/8, 1/16, -1/16], [0, 0, 0, -3/16, 5/16], [0, 0, 0, -3/16, 5/16], [1/2, -1/4, -1/8, 1/16, -1/16] ] \times \$ [ [1, 2, 1, 2, 0, 2, 0, 0], [1, 0, 2, 2, 0, 3, 0, 0], [2, 0, 0, 3, 0, 3, 0, 0], [0, 0, 0, 3, 0, 5, 0, 0], [0, 0, 0, 5, 0, 3, 0, 0] ] \$$$

Â» SYNC'D 87/2048 , 0.04248046875

10 . Coloring, {2, 4}

$$\mathbf{R}: [3, 8, 1, 6, 7, 7, 5, 5] \quad \mathbf{B}: [6, 3, 8, 1, 2, 4, 4, 2]$$

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[ '2' (' - 1 + \tau ')', 2' (' - 1 + \tau ')', 2' (' - 1 + \tau ')', 2' (' - 1 + \tau ')', -2' (' 1 + \tau ')', 2' (' - 1 + \tau ')', -2' (' 1 + \tau ')', 2' (' - 1 + \tau ')']$$

For τ=1/2, [-1, -1, -1, -1, -3, -1, -3, -1] . FixedPtCheck, [1, 1, 1, 1, 3, 1, 3, 1]

$$\det(\mathbf{A} + \tau \Delta) = 0$$

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

$$\begin{aligned} & \$ [ [1, 0, 1, 0, 2, 1, 2, 1], [3, 1, 3, 1, 3, 1, 3, 1], [3, 2, 3, 2, 2, 1, 2, 1], [5, 5, 5, 5, 3, 3, 3, 3], [4, 5, 4, 5, 3, \\ & 4, 3, 4], [7, 9, 7, 9, 7, 9, 7, 9] ] \$ \$ [ [1, 2, 1, 2, 0, 1, 0, 1], [1, 3, 1, 3, 1, 3, 1, 3], [1, 2, 1, 2, 2, 3, 2, 3], \\ & [3, 3, 3, 3, 5, 5, 5, 5], [4, 3, 4, 3, 5, 4, 5, 4], [9, 7, 9, 7, 9, 7, 9, 7] ] \$ \$ [ [0, -1, 0, -1, 1, 0, 1, 0], [1, -1, 1, \\ & -1, 1, -1, 1, -1], [1, 0, 1, 0, 0, -1, 0, -1], [1, 1, 1, 1, -1, -1, -1, -1], [0, 1, 0, 1, -1, 0, -1, 0], [-1, 1, -1, 1, -1, \\ & 1, -1, 1] ] \$ \end{aligned}$$

$$[y_2, -y_1, y_2, -y_1, y_1, -y_2, y_1, -y_2]$$

$$p = s + 4s^5$$

$$\begin{aligned} & \mathbf{S}^+ \quad \backslash ; \quad \mathbf{S}^- \quad \backslash ; \quad \mathbf{NM} \\ & \$ [ [1, 1, 5, 3, 2, 2, 2, 4], [4, 4, 2, 0, 3, 1, 1, 5], [5, 3, 1, 1, 2, 4, 2, 2], [2, 0, 4, 4, 1, 5, 3, 1], [4, 2, 0, 4, 1, \\ & 1, 5, 3], [3, 3, 1, 3, 0, 4, 6, 0], [0, 4, 4, 2, 5, 3, 1, 1], [1, 3, 3, 3, 6, 0, 0, 4] ] \$ \$ [ [3, 1, 3, 3, 4, 6, 0, 0] ] \$ \end{aligned}$$

[0, 2, 6, 2, 3, 3, 1, 3], [3, 3, 3, 1, 0, 0, 4, 6], [6, 2, 0, 2, 1, 3, 3, 3], [2, 6, 2, 0, 3, 1, 3, 3], [3, 1, 1, 5, 4, 2, 2, 2], [2, 0, 2, 6, 3, 3, 3, 1], [1, 5, 3, 1, 2, 2, 4, 2]] \$ \$ [ [2, 1, 0, 1, 1, 1, 1, 1], [1, 2, 1, 0, 1, 1, 1, 1], [0, 1, 2, 1, 1, 1, 1, 1], [1, 0, 1, 2, 1, 1, 1, 1], [1, 1, 1, 1, 2, 1, 0, 1], [1, 1, 1, 1, 1, 2, 1, 0], [1, 1, 1, 1, 0, 1, 2, 1], [1, 1, 1, 1, 1, 0, 1, 2]] \$

CmmCk true, true, true

$$p' = s + 4s^5 \quad p' = s^2 - 2s^4 + 4s^5 \quad p' = s^3 - 2s^4 + 2s^5$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
2 vs 6	3 vs 7	3 vs 7	2 vs 6	3 vs 6

Omega Rank for R : cycles: {{1, 3}, {5, 7}}, net cycles: 0 . order: 2

\$ [ [1, 0, 1, 0, 2, 1, 2, 1], [1, 0, 1, 0, 3, 0, 3, 0], [1, 0, 1, 0, 3, 0, 3, 0], [1, 0, 1, 0, 3, 0, 3, 0], [1, 0, 1, 0, 3, 0, 3, 0], [1, 0, 1, 0, 3, 0, 3, 0]] \$

$$[y_1, 0, y_1, 0, 3y_1 - y_2, y_2, 3y_1 - y_2, y_2]$$

$$p = -s^2 + s^3 \quad p = -s^2 + s^4 \quad p = -s^2 + s^5 \quad p = -s^2 + s^6$$

Omega Rank for B : cycles: {{1, 4, 6}, {2, 3, 8}}, net cycles: 2 . order: 3

\$ [ [1, 2, 1, 2, 0, 1, 0, 1], [2, 1, 2, 1, 0, 1, 0, 1], [1, 1, 1, 1, 0, 2, 0, 2], [1, 2, 1, 2, 0, 1, 0, 1], [2, 1, 2, 1, 0, 1, 0, 1], [1, 1, 1, 1, 0, 2, 0, 2]] \$

$$[y_1, y_2, y_1, y_2, 0, y_3, 0, y_3]$$

$$p' = -s^2 + s^5 \quad p' = -s + s^4 \quad p = -s + s^4$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[x_1, x_4, x_2, x_3, 4x_1 - 2x_3 - 2x_2, 4x_4 + 4x_2 + 2x_3]$$

$$\text{For } A+2\Delta: [-y_1, -y_2 - 3y_3 - 3y_4, y_1, y_2, y_3, -y_5, y_4, y_5]$$

$$\text{For } A-2\Delta: [-y_2, y_1, y_2, y_3, -3y_1 - 3y_3 - y_4, -y_5, y_4, y_5]$$

Range of  $\{\Omega\Delta^i\}$  :  $[-\mu_2, -\mu_1, -\mu_2, -\mu_1, \mu_1, \mu_2, \mu_1, \mu_2]$

rank of M is 8 , rank of N is 5

M \ ; N

\$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0]] \$ \$ [ [0, 1, 2, 1, 1, 1, 1, 1], [1, 0, 1, 2, 1, 1, 1, 1], [2, 1, 0, 1, 1, 1, 1, 1], [1, 2, 1, 0, 1, 1, 1, 1], [1, 1, 1, 1, 0, 1, 2, 1], [1, 1, 1, 1, 1, 0, 1, 2], [1, 1, 1, 1, 2, 1, 0, 1], [1, 1, 1, 1, 1, 2, 1, 0]] \$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [0, -1, 0, -1, 1, 0, 1, 0]$

$\ker M$ ,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range  $M$ ,  $[x_6, x_7, x_8, x_5, x_1, x_2, x_3, x_4]$

$\tau = 32$ ,  $r' = 1/2$

Ranges

Action of  $R$  on ranges,  $[[1], [4], [3], [3]]$

Action of  $B$  on ranges,  $[[4], [1], [2], [2]]$

$\beta(\{1, 3\}) = 1/4$

$\beta(\{2, 4\}) = 1/4$

$\beta(\{5, 7\}) = 1/4$

$\beta(\{6, 8\}) = 1/4$

$\ker N$ ,  $[-\mu_1 - \mu_3 - \mu_2, \mu_1, -\mu_1 - \mu_3 - \mu_2, \mu_1, \mu_2, \mu_3, \mu_2, \mu_3]$

Range of  $N$

$[y_1 - y_2 + y_3, y_1, y_2, y_3, y_4, y_5, y_1 + y_3 - y_4, y_1 + y_3 - y_5]$

Partitions

Action of  $R$  on partitions,  $[[5], [6], [2], [2], [5], [6], [1], [1]]$

Action of  $B$  on partitions,  $[[4], [3], [7], [4], [7], [8], [8], [3]]$

$\alpha(\{\{1, 4, 5, 6\}, \{2, 3, 7, 8\}\}) = 1/8$

$\alpha(\{\{3, 4, 5, 6\}, \{1, 2, 7, 8\}\}) = 1/8$

$\alpha(\{\{1, 2, 6, 7\}, \{3, 4, 5, 8\}\}) = 1/8$

$\alpha(\{\{1, 4, 6, 7\}, \{2, 3, 5, 8\}\}) = 1/8$

$\alpha(\{\{1, 2, 5, 6\}, \{3, 4, 7, 8\}\}) = 1/8$

$\alpha(\{\{2, 3, 5, 6\}, \{1, 4, 7, 8\}\}) = 1/8$

$\alpha(\{\{1, 4, 5, 8\}, \{2, 3, 6, 7\}\}) = 1/8$

$\alpha(\{\{3, 4, 6, 7\}, \{1, 2, 5, 8\}\}) = 1/8$

$b_1 = \{1, 4, 5, 8\}$  ' , '  $b_2 = \{3, 4, 6, 7\}$  ' , '  $b_3 = \{1, 2, 5, 6\}$  ' , '  $b_4 = \{3, 4, 7, 8\}$  ' , '  $b_5 = \{2, 3, 5, 6\}$  ' , '  $b_6 = \{3, 4, 5, 6\}$  ' , '  $b_7 = \{1, 4, 5, 6\}$  ' , '  $b_8 = \{1, 2, 7, 8\}$  ' , '  $b_9 = \{1, 4, 7, 8\}$  ' , '  $b_{10} = \{1, 2, 5, 8\}$  ' , '  $b_{11} = \{2, 3, 6, 7\}$  ' , '  $b_{12} = \{1, 2, 6, 7\}$  ' , '  $b_{13} = \{3, 4, 5, 8\}$  ' , '  $b_{14} = \{1, 4, 6, 7\}$  ' , '  $b_{15} = \{2, 3, 7, 8\}$  ' , '  $b_{16} = \{2, 3, 5, 8\}$

Action of  $R$  and  $B$  on the blocks of the partitions:  $\$ [ [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0], [1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$



$$-2\tau + \tau^2 \text{ ' (' - 1 + \tau ' ) ' , 4 ' ( ' 5 + \tau ' ) ' ( ' 1 + \tau ' ) ' ( ' - 1 + \tau ' ) ' , -4 ' ( ' 1 + \tau ' ) ' ( ' 5 + 2\tau + \tau^2 ' ) ' ] ' ' }$$

For  $\tau=1/2$ , [-111, -159, -123, -25, -162, -34, -66, -150] . FixedPtCheck, [111, 159, 123, 25, 162, 34, 66, 150]

$$\det(A + \tau \Delta) = 1 \text{ ' ( ' \tau ' ) ' }^2 \text{ ' ( ' 1 + \tau ' ) ' }^2 \text{ ' ( ' - 1 + \tau ' ) ' }^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	5 vs 6	3 vs 6

Omega Rank for R : cycles: {{1, 3}, {2, 5, 8}}, net cycles: 1 . order: 6

$$\$ [ [2, 1, 1, 0, 2, 0, 1, 1], [1, 2, 2, 0, 2, 0, 0, 1], [2, 2, 1, 0, 1, 0, 0, 2], [1, 1, 2, 0, 2, 0, 0, 2], [2, 2, 1, 0, 2, 0, 0, 1], [1, 2, 2, 0, 1, 0, 0, 2] ] \$$$

$$[3 y_1, 5 y_1 + 5 y_2 - 3 y_3 - 3 y_4 - 3 y_5, 3 y_2, 0, 3 y_3, 0, 3 y_4, 3 y_5]$$

$$p = -s^2 - s^3 + s^5 + s^6$$

Omega Rank for B : cycles: {{4, 6}, {2, 3, 8}}, net cycles: 1 . order: 6

$$\$ [ [0, 1, 1, 2, 0, 2, 1, 1], [0, 1, 1, 3, 0, 2, 0, 1], [0, 1, 1, 2, 0, 3, 0, 1], [0, 1, 1, 3, 0, 2, 0, 1], [0, 1, 1, 2, 0, 3, 0, 1], [0, 1, 1, 3, 0, 2, 0, 1] ] \$$$

$$[0, y_3, y_3, 5 y_3 - y_1 - y_2, 0, y_1, y_2, y_3]$$

$$p = -s^2 + s^4 \quad p = -s^2 + s^6 \quad p' = -s^2 + s^4$$

$\hat{A}$ » SYNC'D 2641/131072 , 0.02014923096

12 . Coloring, {2, 6}

**R**: [3, 8, 1, 1, 7, 4, 5, 5]    **B**: [6, 3, 8, 6, 2, 7, 4, 2]

' See graph

' ' See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

$$[ ' 2 \text{ ' ( ' 1 + \tau ' ) ' ( ' 5 - 2\tau + \tau^2 ' ) ' , 2 \text{ ' ( ' - 1 + \tau ' ) ' ( ' - 5 + \tau^2 ' ) ' , 2 \text{ ' ( ' 5 - \tau + 3\tau^2 + \tau^3 ' ) ' , } \\ -2 \text{ ' ( ' - 1 + \tau ' ) ' ( ' 5 + 2\tau + \tau^2 ' ) ' , 2 \text{ ' ( ' 1 + \tau ' ) ' ( ' 5 - 2\tau + \tau^2 ' ) ' , 2 \text{ ' ( ' - 1 + \tau ' ) ' ( ' - 5 + \tau^2 ' ) ' , } \\ 2 \text{ ' ( ' 5 - \tau + 3\tau^2 + \tau^3 ' ) ' , -2 \text{ ' ( ' - 1 + \tau ' ) ' ( ' 5 + 2\tau + \tau^2 ' ) ' } ] ' ' }$$



For  $\tau=1/2$ , [51, 19, 43, 25, 51, 19, 43, 25] . FixedPtCheck, [51, 19, 43, 25, 51, 19, 43, 25]

$$\det(A + \tau \Delta) = 1 \cdot (-1 + \tau)^2 \cdot (\tau)^2 \cdot (1 + \tau)^2$$

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

\$ [ [2, 0, 1, 1, 2, 0, 1, 1] , [2, 1, 4, 1, 2, 1, 4, 1] , [5, 5, 5, 1, 5, 5, 5, 1] , [3, 5, 4, 4, 3, 5, 4, 4] , [8, 9, 6, 9, 8, 9, 6, 9] , [15, 15, 15, 19, 15, 15, 15, 19] ] \$ \$ [ [0, 2, 1, 1, 0, 2, 1, 1] , [2, 3, 0, 3, 2, 3, 0, 3] , [3, 3, 3, 7, 3, 3, 3, 7] , [5, 3, 4, 4, 5, 3, 4, 4] , [8, 7, 10, 7, 8, 7, 10, 7] , [17, 17, 17, 13, 17, 17, 17, 13] ] \$ \$ [ [1, -1, 0, 0, 1, -1, 0, 0] , [0, -1, 2, -1, 0, -1, 2, -1] , [1, 1, 1, -3, 1, 1, 1, -3] , [-1, 1, 0, 0, -1, 1, 0, 0] , [0, 1, -2, 1, 0, 1, -2, 1] , [-1, -1, -1, 3, -1, -1, -1, 3] ] \$

$$[-y_2 - y_3 - y_1, y_1, y_2, y_3, -y_2 - y_3 - y_1, y_1, y_2, y_3]$$

$$p = s + 4s^4$$

$$S+ \quad \backslash ; \quad S- \quad \backslash ; \quad NM$$

\$ [ [1, 1, 2, 1, 0, 2, 2, 1] , [1, 1, 2, 1, 1, 0, 1, 3] , [2, 1, 1, 1, 2, 1, 0, 2] , [2, 1, 1, 1, 1, 3, 1, 0] , [0, 2, 2, 1, 1, 1, 2, 1] , [1, 0, 1, 3, 1, 1, 2, 1] , [2, 1, 0, 2, 2, 1, 1, 1] , [1, 3, 1, 0, 2, 1, 1, 1] ] \$ \$ [ [1, 1, 2, 1, 0, 2, 2, 1] , [1, 1, 2, 1, 1, 0, 1, 3] , [2, 1, 1, 1, 2, 1, 0, 2] , [2, 1, 1, 1, 1, 3, 1, 0] , [0, 2, 2, 1, 1, 1, 2, 1] , [1, 0, 1, 3, 1, 1, 2, 1] , [2, 1, 0, 2, 2, 1, 1, 1] , [1, 3, 1, 0, 2, 1, 1, 1] ] \$ \$ [ [4, 2, 2, 3, 0, 2, 2, 1] , [2, 4, 1, 2, 2, 0, 3, 2] , [2, 1, 4, 3, 2, 3, 0, 1] , [3, 2, 3, 4, 1, 2, 1, 0] , [0, 2, 2, 1, 4, 2, 2, 3] , [2, 0, 3, 2, 2, 4, 1, 2] , [2, 3, 0, 1, 2, 1, 4, 3] , [1, 2, 1, 0, 3, 2, 3, 4] ] \$

CmmCk true, true, true

$$p' = s + 4s^4 \quad p' = s^2 + 4s^5$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
3 vs 6	4 vs 8	4 vs 8	2 vs 6	3 vs 6

Omega Rank for R : cycles: {{1, 3}, {5, 7}}, net cycles: 0 . order: 2

\$ [ [2, 0, 1, 1, 2, 0, 1, 1] , [2, 0, 2, 0, 2, 0, 2, 0] , [2, 0, 2, 0, 2, 0, 2, 0] , [2, 0, 2, 0, 2, 0, 2, 0] , [2, 0, 2, 0, 2, 0, 2, 0] , [2, 0, 2, 0, 2, 0, 2, 0] ] \$

$$[y_1, 0, y_2, y_1 - y_2, y_1, 0, y_2, y_1 - y_2]$$

$$p' = s^2 - s^4 \quad p = s^2 - s^5 \quad p' = s^3 - s^4 \quad p' = -s^4 + s^5$$

Omega Rank for B : cycles: {{2, 3, 8}, {4, 6, 7}}, net cycles: 2 . order: 3

\$ [ [0, 2, 1, 1, 0, 2, 1, 1] , [0, 1, 2, 1, 0, 1, 2, 1] , [0, 1, 1, 2, 0, 1, 1, 2] , [0, 2, 1, 1, 0, 2, 1, 1] , [0, 1, 2, 1, 0, 1, 2, 1] , [0, 1, 1, 2, 0, 1, 1, 2] ] \$

$$[0, y_2, y_1, y_3, 0, y_2, y_1, y_3]$$

$$p = -s + s^4 \quad p' = -s^2 + s^5 \quad p'' = -s + s^4$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[x_1, x_2, x_3, 4x_1, 4x_2, 4x_3]$$

$$\text{For } A+2\Delta: [y_1, y_2, y_3, -y_4, -y_1, -y_2, -y_3, y_4]$$

$$\text{For } A-2\Delta: [-y_1, -y_2, -y_3, -y_4, y_1, y_2, y_3, y_4]$$

$$\text{Range of } \{\Omega\Delta^i\}: [-\mu_1 - \mu_2 - \mu_3, \mu_1, \mu_2, \mu_3, -\mu_1 - \mu_2 - \mu_3, \mu_1, \mu_2, \mu_3]$$

rank of M is 8 , rank of N is 5

$$M \quad \setminus ; \quad N$$

$$\begin{aligned} & \$ [ [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], \\ & [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0] ] \$ \quad \$ [ [0, 2, 2, 1, 4, 2, 2, 3], \\ & [2, 0, 3, 2, 2, 4, 1, 2], [2, 3, 0, 1, 2, 1, 4, 3], [1, 2, 1, 0, 3, 2, 3, 4], [4, 2, 2, 3, 0, 2, 2, 1], [2, 4, 1, 2, 2, 0, 3, \\ & 2], [2, 1, 4, 3, 2, 3, 0, 1], [3, 2, 3, 4, 1, 2, 1, 0] ] \$ \end{aligned}$$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [1, -1, 0, 0, 1, -1, 0, 0]$

ker M,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range M,  $[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$

$$\tau = 32, r' = 1/2$$

Ranges

Action of R on ranges,  $[[3], [4], [1], [1]]$

Action of B on ranges,  $[[2], [3], [4], [2]]$

$$\beta(\{1, 5\}) = 1/4$$

$$\beta(\{2, 6\}) = 1/4$$

$$\beta(\{3, 7\}) = 1/4$$

$$\beta(\{4, 8\}) = 1/4$$

ker N,  $[-\mu_2 - \mu_3 - \mu_1, \mu_1, \mu_2, \mu_3, -\mu_2 - \mu_3 - \mu_1, \mu_1, \mu_2, \mu_3]$

Range of N

$$[y_1 + y_5 - y_2, y_1 + y_5 - y_3, y_1 - y_4 + y_5, y_1, y_2, y_3, y_4, y_5]$$

Partitions

Action of R on partitions, [[5], [3], [4], [4], [5]]

Action of B on partitions, [[5], [1], [2], [5], [2]]

$$\alpha(\{1, 2, 3, 4\}, \{5, 6, 7, 8\}) = 1/8$$

$$\alpha(\{3, 5, 6, 8\}, \{1, 2, 4, 7\}) = 1/4$$

$$\alpha(\{3, 4, 5, 6\}, \{1, 2, 7, 8\}) = 1/8$$

$$\alpha(\{2, 3, 4, 5\}, \{1, 6, 7, 8\}) = 1/8$$

$$\alpha(\{1, 3, 4, 6\}, \{2, 5, 7, 8\}) = 3/8$$

b1 = {3, 5, 6, 8} ‘ , ‘ b2 = {1, 2, 4, 7} ‘ , ‘ b3 = {3, 4, 5, 6} ‘ , ‘ b4 = {2, 3, 4, 5} ‘ , ‘ b5 = {1, 2, 7, 8} ‘ , ‘  
 b6 = {1, 6, 7, 8} ‘ , ‘ b7 = {1, 3, 4, 6} ‘ , ‘ b8 = {1, 2, 3, 4} ‘ , ‘ b9 = {2, 5, 7, 8} ‘ , ‘ b10 = {5, 6, 7, 8}

Action of R and B on the blocks of the partitions: \$ [ [0, 0, 0, 0, 1, 0, 0, 1, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0, 0, 1] , [0, 1, 0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0, 1, 0] , [1, 0, 0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 1, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0, 1, 0] , [1, 0, 0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 1, 0, 1, 0] ] \$ = \$ [ [0, 0, 0, 0, 1, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 1, 0] ] \$ + \$ [ [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 1] , [0, 1, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 1, 0] , [1, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 1, 0] , [1, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0, 0, 0] ] \$  
 [‘5‘, ‘3‘, ‘6‘, ‘6‘, ‘4‘, ‘4‘, ‘7‘, ‘7‘, ‘9‘, ‘9‘], [‘8‘, A, ‘2‘, ‘9‘, ‘1‘, ‘7‘, ‘2‘, ‘9‘, ‘1‘, ‘7‘] with invariant measure [2, 2, 1, 1, 1, 1, 3, 1, 3, 1]

N by blocks, check: true . ‘ See partition graph.

‘ ‘ See level-2 partition graph.

‘

Sandwich	
<b>Coloring</b>	{2, 6}
<b>Rank</b>	2
<b>R,B</b>	[3, 8, 1, 1, 7, 4, 5, 5], [6, 3, 8, 6, 2, 7, 4, 2]
$\pi_2$	[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]
$u_2$	[2, 2, 1, 4, 2, 2, 3, 3, 2, 2, 4, 1, 2, 1, 2, 1, 4, 3, 3, 2, 3, 4, 2, 2, 1, 3, 2, 1] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

13 . Coloring, {2, 7}

**R:** [3, 8, 1, 1, 7, 7, 4, 5]    **B:** [6, 3, 8, 6, 2, 4, 5, 2]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$[ 4(1+\tau)^2(1+\tau^2)(5-2\tau+\tau^2), 4(-1+\tau)^2(5+\tau+\tau^2+\tau^3), 4(5-3\tau+10\tau^2+2\tau^3+\tau^4+\tau^5), -4(-1+\tau)(5+4\tau+6\tau^2+\tau^4), -4(-1+\tau)(1+\tau)(5-2\tau+\tau^2), 4(-1+\tau)(-5-\tau-3\tau^2+\tau^3), -4(-1+\tau)(1+\tau)(5+2\tau+\tau^2), -4(-1+\tau)(5-\tau+3\tau^2+\tau^3) ]$$

For τ=1/2, [255, 47, 203, 137, 102, 98, 150, 86] . FixedPtCheck, [255, 47, 203, 137, 102, 98, 150, 86]

$$\det(A + \tau \Delta) = 1(-1+\tau)^2(1+\tau)^2(\tau)^2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	6 vs 6	5 vs 6

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 6

$$[y_1, 0, y_2, y_3, y_4, 0, y_6, y_5]$$

$$R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, 0, -3/16, 5/16], [1, -1, -1, 2, 5/16, -19/16], [0, 0, 0, 0, 5/16, -3/16], [0, 0, 0, 0, 5/16, -3/16], [0, 0, 1, -1, -11/16, 13/16], [0, 0, 1, -1, -11/16, 13/16], [0, 0, 0, 1, -3/16, -11/16], [0, 1, -1, -1, 13/16, 5/16] ] \$ x \$ [ [2, 0, 1, 1, 1, 0, 2, 1], [2, 0, 2, 2, 1, 0, 1, 0], [4, 0, 2, 1, 0, 0, 1, 0], [3, 0, 4, 1, 0, 0, 0, 0], [5, 0, 3, 0, 0, 0, 0, 0], [3, 0, 5, 0, 0, 0, 0, 0] ] \$$$

Omega Rank for B : cycles: {{4, 6}, {2, 3, 8}}, net cycles: 1 . order: 6

$$\$ [ [0, 2, 1, 1, 1, 2, 0, 1], [0, 2, 2, 2, 0, 1, 0, 1], [0, 1, 2, 1, 0, 2, 0, 2], [0, 2, 1, 2, 0, 1, 0, 2], [0, 2, 2, 1, 0, 2, 0, 1], [0, 1, 2, 2, 0, 1, 0, 2] ] \$$$

$$[0, 3y_2, 3y_3, 3y_4, 3y_1, 3y_5, 0, -3y_2 - 3y_3 + 5y_4 - 3y_1 + 5y_5]$$

$$p = -s^2 - s^3 + s^5 + s^6$$

Â» SYNC'D 3891/65536 , 0.05937194824

14 . Coloring, {2, 8}

**R:** [3, 8, 1, 1, 7, 7, 5, 2]    **B:** [6, 3, 8, 6, 2, 4, 4, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

' [ ' 2' ( ' 1 + τ ' ) ' , 2' ( ' 1 + τ ' ) ' , 2' ( ' 1 + τ ' ) ' , -2' ( ' - 1 + τ ' ) ' , 2' ( ' 1 + τ ' ) ' , -2' ( ' - 1 + τ ' ) ' ,  
2' ( ' 1 + τ ' ) ' , 2' ( ' 1 + τ ' ) ' ] '

For τ=1/2, [3, 3, 3, 1, 3, 1, 3, 3] . FixedPtCheck, [3, 3, 3, 1, 3, 1, 3, 3]

$$\det(A + \tau \Delta) = 1' ( ' \tau ' ) ' ^2 ( ' 1 + \tau ' ) ' ^2 ( ' - 1 + \tau ' ) ' ^2$$

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

[1, 1, 1, 1, 1, 1, 1, 1]

+ \ ; - \ ; Δ

\$ [ [2, 1, 1, 0, 1, 0, 2, 1] , [1, 2, 3, 2, 3, 2, 1, 2] , [5, 3, 3, 5, 3, 5, 5, 3] , [4, 4, 5, 3, 5, 3, 4, 4] , [8, 7, 8, 9, 8, 9, 8, 7] , [17, 15, 17, 15, 17, 15, 17, 15] ] \$ \$ [ [0, 1, 1, 2, 1, 2, 0, 1] , [3, 2, 1, 2, 1, 2, 3, 2] , [3, 5, 5, 3, 5, 3, 3, 5] , [4, 4, 3, 5, 3, 5, 4, 4] , [8, 9, 8, 7, 8, 7, 8, 9] , [15, 17, 15, 17, 15, 17, 15, 17] ] \$ \$ [ [1, 0, 0, -1, 0, -1, 1, 0] , [-1, 0, 1, 0, 1, 0, -1, 0] , [1, -1, -1, 1, -1, 1, 1, -1] , [0, 0, 1, -1, 1, -1, 0, 0] , [0, -1, 0, 1, 0, 1, 0, -1] , [1, -1, 1, -1, 1, -1, 1, -1] ] \$

[y<sub>2</sub>, y<sub>3</sub>, -y<sub>2</sub> - y<sub>3</sub> - y<sub>1</sub>, y<sub>1</sub>, -y<sub>2</sub> - y<sub>3</sub> - y<sub>1</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>]

$$p = s + 4s^4 + 8s^5 - 16s^6$$

S+ \ ; S- \ ; NM

\$ [ [8, 7, 7, 7, 6, 9, 8, 6] , [6, 9, 9, 5, 7, 5, 7, 10] , [7, 7, 8, 7, 8, 6, 6, 9] , [9, 5, 6, 9, 7, 10, 7, 5] , [6, 9, 8, 6, 8, 7, 7, 7] , [7, 5, 7, 10, 6, 9, 9, 5] , [8, 6, 6, 9, 7, 7, 8, 7] , [7, 10, 7, 5, 9, 5, 6, 9] ] \$ \$ [ [8, 7, 7, 7, 6, 9, 8, 6] , [6, 9, 9, 5, 7, 5, 7, 10] , [7, 7, 8, 7, 8, 6, 6, 9] , [9, 5, 6, 9, 7, 10, 7, 5] , [6, 9, 8, 6, 8, 7, 7, 7] , [7, 5, 7, 10, 6, 9, 9, 5] , [8, 6, 6, 9, 7, 7, 8, 7] , [7, 10, 7, 5, 9, 5, 6, 9] ] \$ \$ [ [11, 4, 5, 8, 6, 3, 0, 7] , [4, 11, 6, 5, 5, 6, 7, 0] , [5, 6, 11, 8, 0, 3, 6, 5] , [8, 5, 8, 11, 3, 0, 3, 6] , [6, 5, 0, 3, 11, 8, 5, 6] , [3, 6, 3, 0, 8, 11, 8, 5] , [0, 7, 6, 3, 5, 8, 11, 4] , [7, 0, 5, 6, 6, 5, 4, 11] ] \$

CmmCk true, true, true

$$p' = s - 4s^3 - 4s^4 + 8s^5 \quad p'' = s^2 + 2s^3 - 4s^5$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
3 vs 6	4 vs 8	4 vs 8	2 vs 6	1 vs 6

Omega Rank for R : cycles: {{1, 3}, {2, 8}, {5, 7}}, net cycles: 3 . order: 2

\$ [ [2, 1, 1, 0, 1, 0, 2, 1] , [1, 1, 2, 0, 2, 0, 1, 1] , [2, 1, 1, 0, 1, 0, 2, 1] , [1, 1, 2, 0, 2, 0, 1, 1] , [2, 1, 1, 0, 1, 0, 2, 1] , [1, 1, 2, 0, 2, 0, 1, 1] ] \$

$$[y_2, y_1, -y_2 + 3y_1, 0, -y_2 + 3y_1, 0, y_2, y_1]$$

$$p = -s + s^3 \quad p' = -s + s^3 \quad p = -s + s^5 \quad p' = -s + s^5$$

Omega Rank for B : cycles: {{4, 6}, {2, 3, 5, 8}}, net cycles: 2 . order: 4

\$ [ [0, 1, 1, 2, 1, 2, 0, 1] , [0, 1, 1, 2, 1, 2, 0, 1] , [0, 1, 1, 2, 1, 2, 0, 1] , [0, 1, 1, 2, 1, 2, 0, 1] , [0, 1, 1, 2, 1, 2, 0, 1] , [0, 1, 1, 2, 1, 2, 0, 1] ] \$

$$[0, y_1, y_1, 2y_1, y_1, 2y_1, 0, y_1]$$

$$p' = s^4 - s^5 \quad p' = s^3 - s^5 \quad p' = s^2 - s^5 \quad p' = s - s^5 \quad p = s - s^6$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[x_1, x_2, x_3, -4x_2 + 2x_3 + 4x_1, 8x_1 - 4x_2, -4x_3 - 16x_1 + 8x_2]$$

$$\text{For } A+2\Delta : [-y_3, y_4, -y_1, -y_2, y_1, y_2, y_3, -y_4]$$

$$\text{For } A-2\Delta : [-y_3, -y_4, -y_1, -y_2, y_1, y_2, y_3, y_4]$$

Range of  $\{\Omega\Delta^i\}$  :  $[\mu_2, \mu_3, -\mu_2 - \mu_3 - \mu_1, \mu_1, -\mu_2 - \mu_3 - \mu_1, \mu_1, \mu_2, \mu_3]$

rank of M is 8 , rank of N is 5

M \ ; N

\$ [ [0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \$ [ [0, 7, 6, 3, 5, 8, 11, 4] , [7, 0, 5, 6, 6, 5, 4, 11] , [6, 5, 0, 3, 11, 8, 5, 6] , [3, 6, 3, 0, 8, 11, 8, 5] , [5, 6, 11, 8, 0, 3, 6, 5] , [8, 5, 8, 11, 3, 0, 3, 6] , [11, 4, 5, 8, 6, 3, 0, 7] , [4, 11, 6, 5, 5, 6, 7, 0] ] \$

Check is  $\Omega\Delta N$  zero? true,  $\pi\Delta = [1, 0, 0, -1, 0, -1, 1, 0]$

ker M,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range M,  $[x_8, x_7, x_6, x_5, x_4, x_3, x_2, x_1]$

$\tau = 32$  ,  $r' = 1/2$

Ranges

Action of R on ranges,  $[[3], [2], [1], [1]]$

Action of B on ranges,  $[[4], [3], [2], [4]]$

$\beta(\{1, 7\}) = 1/4$

$\beta(\{2, 8\}) = 1/4$

$$\beta(\{3, 5\}) = 1/4$$

$$\beta(\{4, 6\}) = 1/4$$

ker N,  $[-\mu_1 - \mu_3 - \mu_2, \mu_3, \mu_1, \mu_2, \mu_1, \mu_2, -\mu_1 - \mu_3 - \mu_2, \mu_3]$

Range of N

$$[y_1 + y_3 - y_5, y_1, y_2, -y_4 + y_1 + y_3, -y_2 + y_1 + y_3, y_4, y_5, y_3]$$

Partitions

Action of R on partitions,  $[[4], [2], [6], [1], [2], [6]]$

Action of B on partitions,  $[[4], [4], [1], [5], [3], [5]]$

$$\alpha(\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}) = 2/11$$

$$\alpha(\{\{1, 5, 6, 8\}, \{2, 3, 4, 7\}\}) = 2/11$$

$$\alpha(\{\{3, 6, 7, 8\}, \{1, 2, 4, 5\}\}) = 1/11$$

$$\alpha(\{\{2, 5, 6, 7\}, \{1, 3, 4, 8\}\}) = 3/11$$

$$\alpha(\{\{2, 3, 6, 7\}, \{1, 4, 5, 8\}\}) = 2/11$$

$$\alpha(\{\{1, 2, 5, 6\}, \{3, 4, 7, 8\}\}) = 1/11$$

$$b1 = \{2, 3, 6, 7\} \text{ ' , ' } b2 = \{1, 5, 6, 8\} \text{ ' , ' } b3 = \{2, 5, 6, 7\} \text{ ' , ' } b4 = \{3, 6, 7, 8\} \text{ ' , ' } b5 = \{2, 3, 4, 7\} \text{ ' , ' }$$

$$b6 = \{1, 2, 4, 5\} \text{ ' , ' } b7 = \{1, 3, 4, 8\} \text{ ' , ' } b8 = \{1, 4, 5, 8\} \text{ ' , ' } b9 = \{1, 2, 5, 6\} \text{ ' , ' } b10 = \{3, 4, 7, 8\} \text{ ' , ' }$$

$$b11 = \{1, 2, 3, 4\} \text{ ' , ' } b12 = \{5, 6, 7, 8\}$$

Action of R and B on the blocks of the partitions: \$ [ [0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0], [0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] ] \$

[ '2', '5', C, '9', '2', A, B, '5', A, '9', '7', '3', [ '6', '7', '8', B, '3', C, '1', '4', '8', '1', '3', '7' ] with invariant measure [2, 2, 3, 1, 2, 1, 3, 2, 1, 1, 2, 2]

N by blocks, check: true . ' See partition graph.

' ' See level-2 partition graph.

'

<b>Sandwich</b>	
<b>Coloring</b>	{2, 8}
<b>Rank</b>	2
<b>R,B</b>	[3, 8, 1, 1, 7, 7, 5, 2], [6, 3, 8, 6, 2, 4, 4, 5]
$\pi_2$	[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0]
$u_2$	[7, 6, 3, 5, 8, 11, 4, 5, 6, 6, 5, 4, 11, 3, 11, 8, 5, 6, 8, 11, 8, 5, 3, 6, 5, 3, 6, 7] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

15 . Coloring, {3, 4}

**R:** [3, 3, 8, 6, 7, 7, 5, 5]    **B:** [6, 8, 1, 1, 2, 4, 4, 2]

‘ See graph

‘ ‘ See pair graph

$\Omega$  for  $A+\tau\Delta$  :

$[ -2'(-1+\tau)'^2, (5+\tau+\tau^2+\tau^3)', 2'(5-3\tau+\tau^2+\tau^3)'(-1+\tau)'(1+\tau)', 2'(-1+\tau)'(5-\tau-\tau^2+\tau^3)'(1+\tau)', -2'(-1+\tau)'(-5+2\tau-4\tau^2-2\tau^3+\tau^4)', -6'(5-4\tau+3\tau^2)'(1+\tau)^2, 6'(-1+\tau)'(5-3\tau+3\tau^2+3\tau^3)', -2'(5-\tau+3\tau^2+\tau^3)'(1+\tau)', 2'(-1+\tau)'(5-2\tau+\tau^2)'(1+\tau)']$

For  $\tau=1/2$ , [-47, -93, -105, -83, -270, -74, -258, -102] . FixedPtCheck, [47, 93, 105, 83, 270, 74, 258, 102]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	6 vs 6	6 vs 6	4 vs 5	4 vs 5

Omega Rank for R : cycles: {{5, 7}}, net cycles: -1 . order: 4

$\$ [ [0, 0, 2, 0, 2, 1, 2, 1], [0, 0, 0, 0, 3, 0, 3, 2], [0, 0, 0, 0, 5, 0, 3, 0], [0, 0, 0, 0, 3, 0, 5, 0], [0, 0, 0, 0, 5, 0, 3, 0] ] \$$

$[0, 0, 2 y_2, 0, y_1, y_2, y_3, y_4]$



$$p = s^3 - s^5$$

Omega Rank for B : cycles: {{2, 8}, {1, 4, 6}}, net cycles: 2 . order: 6

$$\$ [ [2, 2, 0, 2, 0, 1, 0, 1], [2, 1, 0, 1, 0, 2, 0, 2], [1, 2, 0, 2, 0, 2, 0, 1], [2, 1, 0, 2, 0, 1, 0, 2], [2, 2, 0, 1, 0, 2, 0, 1] ] \$$$

$$[5y_1 - 3y_2 - 3y_3 + 5y_4, 3y_1, 0, 3y_2, 0, 3y_3, 0, 3y_4]$$

$$p = -s - s^2 + s^4 + s^5$$

Â» SYNC'D 1/16 , 0.06250000000

16 . Coloring, {3, 5}

**R:** [3, 3, 8, 1, 2, 7, 5, 5] **B:** [6, 8, 1, 6, 7, 4, 4, 2]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$[ '-2' (' - 1 + \tau ')^{2} (' 1 + \tau ')^{2}, 2' (' 1 + \tau ')^{2}, 2' (' 1 + \tau ')^{2}, 2' (' - 1 + \tau ')^{2}, 2' (' 1 + \tau ')^{2}, 2' (' - 1 + \tau ')^{2}, -2' (' - 1 + \tau ')^{2} (' 1 + \tau ')^{2}, 2' (' 1 + \tau ')^{2} ' ]$$

For τ=1/2, [3, 9, 9, 1, 9, 1, 3, 9] . FixedPtCheck, [3, 9, 9, 1, 9, 1, 3, 9]

$$\det(A + \tau \Delta) = 1' (' \tau ')^{2} (' - 1 + \tau ')^{2} (' 1 + \tau ')^{2}$$

Delta Range : [y2, -y1 - y3 - y5, -y2 - y4 - y6, y1, y3, y4, y5, y6]

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

$$\$ [ [1, 1, 2, 0, 2, 0, 1, 1], [0, 3, 2, 3, 2, 3, 0, 3], [5, 3, 3, 5, 3, 5, 5, 3], [5, 4, 4, 3, 4, 3, 5, 4], [7, 8, 9, 8, 9, 8, 7, 8], [15, 17, 15, 17, 15, 17, 15, 17] ] \$ \$ [ [1, 1, 0, 2, 0, 2, 1, 1], [4, 1, 2, 1, 2, 1, 4, 1], [3, 5, 5, 3, 5, 3, 3, 5], [3, 4, 4, 5, 4, 5, 3, 4], [9, 8, 7, 8, 7, 8, 9, 8], [17, 15, 17, 15, 17, 15, 17, 15] ] \$ \$ [ [0, 0, 1, -1, 1, -1, 0, 0], [-2, 1, 0, 1, 0, 1, -2, 1], [1, -1, -1, 1, -1, 1, 1, -1], [1, 0, 0, -1, 0, -1, 1, 0], [-1, 0, 1, 0, 1, 0, -1, 0], [-1, 1, -1, 1, -1, 1, -1, 1] ] \$$$

$$[y_3, y_2, y_1, -y_3 - y_2 - y_1, y_1, -y_3 - y_2 - y_1, y_3, y_2]$$

$$p' = s - 4s^4 - 8s^5 \quad p = s - 4s^4 - 8s^5$$

S+ \ ; S- \ ; NM

$$\$ [ [2, 2, 4, 3, 3, 5, 4, 3], [3, 4, 5, 3, 3, 1, 2, 5], [4, 3, 2, 2, 4, 3, 3, 5], [3, 3, 3, 2, 2, 7, 5, 1], [3, 5, 4, 3, 2,$$

2, 4, 3] , [5, 1, 2, 7, 3, 2, 3, 3] , [4, 3, 3, 5, 4, 3, 2, 2] , [2, 5, 3, 1, 5, 3, 3, 4] ] \$ \$ [ [4, 2, 4, 5, 3, 3, 2, 3] ,  
 [3, 4, 3, 1, 5, 3, 2, 5] , [4, 5, 4, 2, 2, 3, 3, 3] , [5, 1, 3, 6, 2, 3, 3, 3] , [3, 3, 2, 3, 4, 2, 4, 5] , [3, 3, 2, 3, 3, 6, 5,  
 1] , [2, 3, 3, 3, 4, 5, 4, 2] , [2, 5, 5, 3, 3, 1, 3, 4] ] \$ \$ [ [7, 5, 3, 5, 4, 2, 0, 2] , [5, 7, 3, 3, 4, 4, 2, 0] , [3, 3,  
 7, 2, 0, 5, 4, 4] , [5, 3, 2, 7, 5, 0, 2, 4] , [4, 4, 0, 5, 7, 2, 3, 3] , [2, 4, 5, 0, 2, 7, 5, 3] , [0, 2, 4, 2, 3, 5, 7, 5] ,  
 [2, 0, 4, 4, 3, 3, 5, 7] ] \$

CmmCk true, true, true

$$p' = s^2 + 2s^3 + 4s^4 + 4s^5$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
3 vs 6	4 vs 8	4 vs 8	2 vs 6	2 vs 6

Omega Rank for R : cycles: {{2, 3, 5, 8}}, net cycles: -1 . order: 4

\$ [ [1, 1, 2, 0, 2, 0, 1, 1] , [0, 2, 2, 0, 2, 0, 0, 2] , [0, 2, 2, 0, 2, 0, 0, 2] , [0, 2, 2, 0, 2, 0, 0, 2] , [0, 2, 2, 0, 2,  
 0, 0, 2] , [0, 2, 2, 0, 2, 0, 0, 2] ] \$

$$[y_1, y_2, y_1 + y_2, 0, y_1 + y_2, 0, y_1, y_2]$$

$$p = -s^2 + s^6 \quad p = -s^2 + s^4 \quad p = -s^2 + s^5 \quad p = -s^2 + s^3$$

Omega Rank for B : cycles: {{2, 8}, {4, 6}}, net cycles: 0 . order: 2

\$ [ [1, 1, 0, 2, 0, 2, 1, 1] , [0, 1, 0, 3, 0, 3, 0, 1] , [0, 1, 0, 3, 0, 3, 0, 1] , [0, 1, 0, 3, 0, 3, 0, 1] , [0, 1, 0, 3, 0,  
 3, 0, 1] , [0, 1, 0, 3, 0, 3, 0, 1] ] \$

$$[y_1, y_2, 0, 3y_2 - y_1, 0, 3y_2 - y_1, y_1, y_2]$$

$$p = -s^2 + s^3 \quad p = -s^2 + s^4 \quad p = -s^2 + s^5 \quad p = -s^2 + s^6$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[x_1, x_2, x_3, -4x_1 + 2x_3, -4x_2 - 8x_1 + 4x_3, -8x_2 + 4x_3]$$

$$\text{For } A+2\Delta : [y_1, -y_4, -y_2, -y_3, y_2, y_3, -y_1, y_4]$$

$$\text{For } A-2\Delta : [-y_3, -y_4, -y_1, -y_2, y_1, y_2, y_3, y_4]$$

Range of  $\{\Omega\Delta^i\}$ :  $[\mu_2, \mu_3, -\mu_2 - \mu_3 - \mu_1, \mu_1, -\mu_2 - \mu_3 - \mu_1, \mu_1, \mu_2, \mu_3]$

rank of M is 8 , rank of N is 5

M \ ; N

\$ [ [0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 1, 0, 0,  
 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \$ [ [0, 2, 4, 2, 3, 5, 7, 5] ,  
 [2, 0, 4, 4, 3, 3, 5, 7] , [4, 4, 0, 5, 7, 2, 3, 3] , [2, 4, 5, 0, 2, 7, 5, 3] , [3, 3, 7, 2, 0, 5, 4, 4] , [5, 3, 2, 7, 5, 0, 2,  
 4] , [7, 5, 3, 5, 4, 2, 0, 2] , [5, 7, 3, 3, 4, 4, 2, 0] ] \$



invariant measure [2, 3, 2, 5, 5, 2, 2, 3, 1, 1, 1, 1]

N by blocks, check: true . ‘ See partition graph.

‘ ‘ See level-2 partition graph.

‘

<b>Sandwich</b>	
<b>Coloring</b>	{3, 5}
<b>Rank</b>	2
<b>R,B</b>	[3, 3, 8, 1, 2, 7, 5, 5], [6, 8, 1, 6, 7, 4, 4, 2]
$\pi_2$	[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0]
$u_2$	[2, 4, 2, 3, 5, 7, 5, 4, 4, 3, 3, 5, 7, 5, 7, 2, 3, 3, 2, 7, 5, 3, 5, 4, 4, 2, 4, 2] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

17 . Coloring, {3, 6}

**R:** [3, 3, 8, 1, 7, 4, 5, 5]    **B:** [6, 8, 1, 6, 2, 7, 4, 2]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ -4(1+\tau)'(5-2\tau+\tau^2)'(-1+\tau)', 4(-5-\tau-3\tau^2+\tau^3)'(-1+\tau)', -4(1+\tau)'(5+2\tau+\tau^2)'(-1+\tau)', \\ & -4(5-\tau+3\tau^2+\tau^3)'(-1+\tau)', 4(1+\tau^2)'(1+\tau)'(5-2\tau+\tau^2)', \\ & 4(5+\tau+\tau^2+\tau^3)'(-1+\tau)'^2, 4(5-3\tau+10\tau^2+2\tau^3+\tau^4+\tau^5)', \\ & -4(5+4\tau+6\tau^2+\tau^4)'(-1+\tau)']' \end{aligned}$$

For  $\tau=1/2$ , [102, 98, 150, 86, 255, 47, 203, 137] . FixedPtCheck, [102, 98, 150, 86, 255, 47, 203, 137]

$$\det(A + \tau \Delta) = 1'(-1 + \tau)'^2 (\tau)'^2 (1 + \tau)'^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	6 vs 6	5 vs 6

Omega Rank for R : cycles: {{5, 7}}, net cycles: 0 . order: 6

$$[y_3, 0, y_1, y_2, y_4, 0, y_5, y_6]$$

$$R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 1, -1, -11/16, 13/16], [0, 0, 1, -1, -11/16, 13/16], [0, 0, 0, 1, -3/16, -11/16], [0, 1, -1, -1, 13/16, 5/16], [0, 0, 0, 0, -3/16, 5/16], [1, -1, -1, 2, 5/16, -19/16], [0, 0, 0, 0, 5/16, -3/16], [0, 0, 0, 0, 5/16, -3/16] ] \$ x \$ [ [1, 0, 2, 1, 2, 0, 1, 1], [1, 0, 1, 0, 2, 0, 2, 2], [0, 0, 1, 0, 4, 0, 2, 1], [0, 0, 0, 0, 3, 0, 4, 1], [0, 0, 0, 0, 5, 0, 3, 0], [0, 0, 0, 0, 3, 0, 5, 0] ] \$$$

Omega Rank for B : cycles: {{2, 8}, {4, 6, 7}}, net cycles: 1 . order: 6

$$\$ [ [1, 2, 0, 1, 0, 2, 1, 1], [0, 1, 0, 1, 0, 2, 2, 2], [0, 2, 0, 2, 0, 1, 2, 1], [0, 1, 0, 2, 0, 2, 1, 2], [0, 2, 0, 1, 0, 2, 2, 1], [0, 1, 0, 2, 0, 1, 2, 2] ] \$$$

$$[3 y_5, 3 y_4, 0, 3 y_3, 0, 3 y_2, -3 y_5 + 5 y_4 - 3 y_3 - 3 y_2 + 5 y_1, 3 y_1]$$

$$p = -s^2 - s^3 + s^5 + s^6$$

Â» SYNC'D 3891/65536 , 0.05937194824

18 . Coloring, {3, 7}

**R:** [3, 3, 8, 1, 7, 7, 4, 5] **B:** [6, 8, 1, 6, 2, 4, 5, 2]

‘ See graph

‘ ‘ See pair graph

‘

Ω for A+τΔ :

$$[ \text{' -2' ( ' 1 + \tau \text{' )'' ( ' 5 - 2\tau + \tau^2 \text{' )' , 2' ( ' - 1 + \tau \text{' )'' ( ' 5 + 2\tau + \tau^2 \text{' )' , 2' ( ' - 5 + \tau^2 \text{' )'' ( ' 1 + \tau \text{' )' , 2' ( ' - 5 - \tau - 3\tau^2 + \tau^3 \text{' )' , -2' ( ' 1 + \tau \text{' )'' ( ' 5 - 2\tau + \tau^2 \text{' )' , 2' ( ' - 1 + \tau \text{' )'' ( ' 5 + 2\tau + \tau^2 \text{' )' , 2' ( ' - 5 + \tau^2 \text{' )'' ( ' 1 + \tau \text{' )' , 2' ( ' - 5 - \tau - 3\tau^2 + \tau^3 \text{' )' } ] \text{'$$

For τ=1/2, [-51, -25, -57, -49, -51, -25, -57, -49] . FixedPtCheck, [51, 25, 57, 49, 51, 25, 57, 49]

$$\det(A + \tau \Delta) = 1 \text{' ( ' \tau \text{' )' }^2 \text{' ( ' 1 + \tau \text{' )' }^2 \text{' ( ' - 1 + \tau \text{' )' }^2$$

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

\$ [ [1, 0, 2, 1, 1, 0, 2, 1], [1, 2, 1, 4, 1, 2, 1, 4], [7, 3, 3, 3, 7, 3, 3, 3], [4, 3, 5, 4, 4, 3, 5, 4], [7, 8, 7, 10, 7, 8, 7, 10], [19, 15, 15, 15, 19, 15, 15, 15] ] \$ \$ [ [1, 2, 0, 1, 1, 2, 0, 1], [3, 2, 3, 0, 3, 2, 3, 0], [1, 5, 5, 5, 1, 5, 5, 5], [4, 5, 3, 4, 4, 5, 3, 4], [9, 8, 9, 6, 9, 8, 9, 6], [13, 17, 17, 17, 13, 17, 17, 17] ] \$ \$ [ [0, -1, 1, 0, 0, -1, 1, 0], [-1, 0, -1, 2, -1, 0, -1, 2], [3, -1, -1, -1, 3, -1, -1, -1], [0, -1, 1, 0, 0, -1, 1, 0], [-1, 0, -1, 2, -1, 0, -1, 2], [3, -1, -1, -1, 3, -1, -1, -1] ] \$

$$[-y_1 - y_2 - y_3, y_1, y_2, y_3, -y_1 - y_2 - y_3, y_1, y_2, y_3]$$

$$p' = s - 4s^4 \quad p' = s^2 - 4s^5 \quad p = s - 4s^4$$

S+ \ ; S- \ ; NM

\$ [ [1, 2, 3, 1, 0, 1, 1, 1], [1, 1, 3, 2, 1, 0, 0, 2], [1, 1, 1, 0, 3, 1, 0, 3], [3, 2, 1, 1, 0, 2, 1, 0], [0, 1, 1, 1, 1, 2, 3, 1], [1, 0, 0, 2, 1, 1, 3, 2], [3, 1, 0, 3, 1, 1, 1, 0], [0, 2, 1, 0, 3, 2, 1, 1] ] \$ \$ [ [1, 0, 1, 1, 0, 3, 3, 1], [1, 1, 1, 0, 1, 0, 2, 4], [3, 1, 1, 2, 1, 1, 0, 1], [1, 0, 1, 1, 2, 4, 1, 0], [0, 3, 3, 1, 1, 0, 1, 1], [1, 0, 2, 4, 1, 1, 1, 0], [1, 1, 0, 1, 3, 1, 1, 2], [2, 4, 1, 0, 1, 0, 1, 1] ] \$ \$ [ [4, 3, 2, 3, 0, 1, 2, 1], [3, 4, 2, 2, 1, 0, 2, 2], [2, 2, 4, 1, 2, 2, 0, 3], [3, 2, 1, 4, 1, 2, 3, 0], [0, 1, 2, 1, 4, 3, 2, 3], [1, 0, 2, 2, 3, 4, 2, 2], [2, 2, 0, 3, 2, 2, 4, 1], [1, 2, 3, 0, 3, 2, 1, 4] ] \$

CmmCk true, true, true

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
3 vs 6	4 vs 8	4 vs 8	3 vs 6	2 vs 6

Omega Rank for R : cycles: {{1, 3, 4, 5, 7, 8}}, net cycles: 1 . order: 6

\$ [ [1, 0, 2, 1, 1, 0, 2, 1], [1, 0, 1, 2, 1, 0, 1, 2], [2, 0, 1, 1, 2, 0, 1, 1], [1, 0, 2, 1, 1, 0, 2, 1], [1, 0, 1, 2, 1, 0, 1, 2], [2, 0, 1, 1, 2, 0, 1, 1] ] \$

$$[y_1, 0, y_2, y_3, y_1, 0, y_2, y_3]$$

$$p' = -s^2 + s^5 \quad p = -s + s^4 \quad p' = -s + s^4$$

Omega Rank for B : cycles: {{2, 8}, {4, 6}}, net cycles: 0 . order: 2

\$ [ [1, 2, 0, 1, 1, 2, 0, 1], [0, 2, 0, 2, 0, 2, 0, 2], [0, 2, 0, 2, 0, 2, 0, 2], [0, 2, 0, 2, 0, 2, 0, 2], [0, 2, 0, 2, 0, 2, 0, 2], [0, 2, 0, 2, 0, 2, 0, 2] ] \$

$$[y_1 - y_2, y_1, 0, y_2, y_1 - y_2, y_1, 0, y_2]$$

$$p' = s^2 - s^4 \quad p' = -s^4 + s^5 \quad p = s^2 - s^5 \quad p' = s^3 - s^4$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[x_3, x_2, x_1, -4x_3, -4x_2, -4x_1]$$

$$\text{For } A+2\Delta: [y_4, y_3, y_2, y_1, -y_4, -y_3, -y_2, -y_1]$$

$$\text{For } A-2\Delta: [y_4, y_3, y_2, y_1, -y_4, -y_3, -y_2, -y_1]$$

Range of  $\{\Omega\Delta^i\}$ :  $[-\mu_2 - \mu_1 - \mu_3, \mu_2, \mu_3, \mu_1, -\mu_2 - \mu_1 - \mu_3, \mu_2, \mu_3, \mu_1]$

rank of M is 8 , rank of N is 5

M \ ; N

\$ [ [0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 0, 1] , [1, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0] ] \$ \$ [ [0, 1, 2, 1, 4, 3, 2, 3] , [1, 0, 2, 2, 3, 4, 2, 2] , [2, 2, 0, 3, 2, 2, 4, 1] , [1, 2, 3, 0, 3, 2, 1, 4] , [4, 3, 2, 3, 0, 1, 2, 1] , [3, 4, 2, 2, 1, 0, 2, 2] , [2, 2, 4, 1, 2, 2, 0, 3] , [3, 2, 1, 4, 1, 2, 3, 0] ] \$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [0, -1, 1, 0, 0, -1, 1, 0]$

ker M,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range M,  $[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$

$\tau = 32$  ,  $r' = 1/2$

Ranges

Action of R on ranges,  $[[3], [3], [4], [1]]$

Action of B on ranges,  $[[2], [4], [1], [2]]$

$\beta(\{1, 5\}) = 1/4$

$\beta(\{2, 6\}) = 1/4$

$\beta(\{3, 7\}) = 1/4$

$\beta(\{4, 8\}) = 1/4$

ker N,  $[\mu_1, \mu_2, -\mu_1 - \mu_2 - \mu_3, \mu_3, \mu_1, \mu_2, -\mu_1 - \mu_2 - \mu_3, \mu_3]$

Range of N

$[y_1 - y_2 + y_5, y_1 + y_5 - y_3, y_1 + y_5 - y_4, y_1, y_2, y_3, y_4, y_5]$

Partitions

Action of R on partitions,  $[[5], [3], [1], [5], [3]]$

Action of B on partitions,  $[[5], [4], [2], [4], [5]]$

$\alpha(\{1, 2, 3, 4\}, \{5, 6, 7, 8\}) = 1/8$

$\alpha(\{2, 3, 5, 8\}, \{1, 4, 6, 7\}) = 1/8$

$\alpha(\{1, 2, 3, 8\}, \{4, 5, 6, 7\}) = 1/4$

$\alpha(\{2, 5, 7, 8\}, \{1, 3, 4, 6\}) = 1/8$

$\alpha(\{1, 2, 4, 7\}, \{3, 5, 6, 8\}) = 3/8$

$b_1 = \{1, 2, 3, 8\}$  ,  $b_2 = \{2, 3, 5, 8\}$  ,  $b_3 = \{2, 5, 7, 8\}$  ,  $b_4 = \{1, 4, 6, 7\}$  ,  $b_5 = \{1, 3, 4, 6\}$  ,

$b_6 = \{1, 2, 4, 7\}$  ,  $b_7 = \{3, 5, 6, 8\}$  ,  $b_8 = \{1, 2, 3, 4\}$  ,  $b_9 = \{5, 6, 7, 8\}$  ,  $b_{10} = \{4, 5, 6, 7\}$

Action of R and B on the blocks of the partitions: \$ [ [0, 1, 0, 0, 0, 0, 1, 0, 0] , [1, 0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 1, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0, 0] , [1, 0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 1, 0, 0] , [0, 0, 0, 0, 0, 1, 1, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0] ,

$1, 0] ] \$ = \$ [ [0, 0, 0, 0, 0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0] ] \$ + \$ [ [0, 1, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0, 0, 0] ] \$$   
 $[ '8', '1', '7', A, '6', A, '1', '6', '7', '9' ], [ '2', '3', '3', '5', '5', '7', '6', '7', '6', '4' ]$  with invariant measure  $[2, 1, 1, 1, 1, 3, 3, 1, 1, 2]$

N by blocks, check: true . ‘ See partition graph.

‘ ‘ See level-2 partition graph.

‘

<b>Sandwich</b>	
<b>Coloring</b>	{3, 7}
<b>Rank</b>	2
<b>R,B</b>	[3, 3, 8, 1, 7, 7, 4, 5], [6, 8, 1, 6, 2, 4, 5, 2]
$\pi_2$	[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]
$u_2$	[1, 2, 1, 4, 3, 2, 3, 2, 2, 3, 4, 2, 2, 3, 2, 2, 4, 1, 3, 2, 1, 4, 1, 2, 1, 2, 2, 3] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

19 . Coloring, {3, 8}

**R:** [3, 3, 8, 1, 7, 7, 5, 2]    **B:** [6, 8, 1, 6, 2, 4, 4, 5]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

$[ '4' ( '5 + \tau ' ) ' ( '1 + \tau ' ) ' ( ' - 1 + \tau ' ) ' , -4' ( '1 + \tau ' ) ' ( '5 + 2\tau + \tau^2 ' ) ' , 4' ( '1 + \tau ' ) ' ^2 ' ( ' - 5 + \tau ' ) ' , 4' ( ' - 1 + \tau ' ) ' ( '5 - 2\tau + \tau^2 ' ) ' , 4' ( '1 + \tau ' ) ' ( ' - 5 - \tau + \tau^2 + \tau^3 ' ) ' , -4' ( ' - 1 + \tau ' ) ' ^2 ' ( '5 + 2\tau + \tau^2 ' ) ' , 4' ( '1 + \tau ' ) ' ( ' - 5 + \tau - \tau^2 + \tau^3 ' ) ' , 4' ( '1 + \tau ' ) ' ( ' - 5 - 3\tau - \tau^2 + \tau^3 ' ) ' ] ' '$



For  $\tau=1/2$ , [-66, -150, -162, -34, -123, -25, -111, -159] . FixedPtCheck, [66, 150, 162, 34, 123, 25, 111, 159]

$$\det(A + \tau \Delta) = 1 \cdot (1 + \tau)^2 \cdot (\tau)^2 \cdot (-1 + \tau)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	5 vs 6	3 vs 6

Omega Rank for R : cycles: {{5, 7}, {2, 3, 8}}, net cycles: 1 . order: 6

\$ [ [1, 1, 2, 0, 1, 0, 2, 1], [0, 1, 2, 0, 2, 0, 1, 2], [0, 2, 1, 0, 1, 0, 2, 2], [0, 2, 2, 0, 2, 0, 1, 1], [0, 1, 2, 0, 1, 0, 2, 2], [0, 2, 1, 0, 2, 0, 1, 2] ] \$

$$[3 y_5, 3 y_4, 3 y_3, 0, 3 y_2, 0, 3 y_1, -3 y_5 - 3 y_4 - 3 y_3 + 5 y_2 + 5 y_1]$$

$$p = s^2 + s^3 - s^5 - s^6$$

Omega Rank for B : cycles: {{4, 6}, {2, 5, 8}}, net cycles: 1 . order: 6

\$ [ [1, 1, 0, 2, 1, 2, 0, 1], [0, 1, 0, 2, 1, 3, 0, 1], [0, 1, 0, 3, 1, 2, 0, 1], [0, 1, 0, 2, 1, 3, 0, 1], [0, 1, 0, 3, 1, 2, 0, 1], [0, 1, 0, 2, 1, 3, 0, 1] ] \$

$$[5 y_3 - y_1 - y_2, y_3, 0, y_1, y_3, y_2, 0, y_3]$$

$$p = -s^2 + s^4 \quad p' = -s^2 + s^4 \quad p = -s^2 + s^6$$

Â» SYNC'D 2641/131072 , 0.02014923096

20 . Coloring, {4, 5}

**R**: [3, 3, 1, 6, 2, 7, 5, 5]    **B**: [6, 8, 8, 1, 7, 4, 4, 2]

' See graph

' ' See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ ' 4 ( ' 5 - 3\tau + 10\tau^2 + 2\tau^3 + \tau^4 + \tau^5 ) ', -4 ( ' 5 + 4\tau + 6\tau^2 + \tau^4 ) ' ( ' -1 + \tau ' ) ', 4 ( ' 1 + \tau^2 ' ) ' ( ' 1 + \tau ' ) ' ( ' 5 - 2\tau + \tau^2 ' ) ', 4 ( ' -1 + \tau ' )^2 ( ' 5 + \tau + \tau^2 + \tau^3 ' ) ', -4 ( ' -1 + \tau ' ) ' ( ' 1 + \tau ' ) ' ( ' 5 + 2\tau + \tau^2 ' ) ', -4 ( ' -1 + \tau ' ) ' ( ' 5 - \tau + 3\tau^2 + \tau^3 ' ) ', -4 ( ' -1 + \tau ' ) ' ( ' 1 + \tau ' ) ' ( ' 5 - 2\tau + \tau^2 ' ) ', 4 ( ' -1 + \tau ' ) ' ( ' -5 - \tau - 3\tau^2 + \tau^3 ' ) ' ] ' \end{aligned}$$

For  $\tau=1/2$ , [203, 137, 255, 47, 150, 86, 102, 98] . FixedPtCheck, [203, 137, 255, 47, 150, 86, 102, 98]

$$\det(A + \tau \Delta) = 1^{\epsilon} (-1 + \tau^{\epsilon})^{\epsilon^2} (\tau^{\epsilon})^{\epsilon^2} (\epsilon + 1 + \tau^{\epsilon})^{\epsilon^2}$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	6 vs 6	5 vs 6

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 6

$$[y_1, y_2, y_3, 0, y_4, y_5, y_6, 0]$$

$$\begin{aligned} R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, \\ 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0] \\ , [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, \\ 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] = \$ [ [0, 0, 0, 0, 5/16, -3/16], [0, 0, 0, 0, 5/16, -3/16], \\ [0, 0, 0, 0, -3/16, 5/16], [1, -1, -1, 2, 5/16, -19/16], [0, 0, 0, 1, -3/16, -11/16], [0, 1, -1, -1, 13/16, 5/16], \\ [0, 0, 1, -1, -11/16, 13/16], [0, 0, 1, -1, -11/16, 13/16] ] \times \$ [ [1, 1, 2, 0, 2, 1, 1, 0], [2, 2, 2, 0, 1, 0, 1, 0], \\ [2, 1, 4, 0, 1, 0, 0, 0], [4, 1, 3, 0, 0, 0, 0, 0], [3, 0, 5, 0, 0, 0, 0, 0], [5, 0, 3, 0, 0, 0, 0, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{2, 8}, {1, 4, 6}}, net cycles: 1 . order: 6

$$\begin{aligned} \$ [ [1, 1, 0, 2, 0, 1, 1, 2], [2, 2, 0, 2, 0, 1, 0, 1], [2, 1, 0, 1, 0, 2, 0, 2], [1, 2, 0, 2, 0, 2, 0, 1], [2, 1, 0, 2, 0, \\ 1, 0, 2], [2, 2, 0, 1, 0, 2, 0, 1] ] \$ \end{aligned}$$

$$[3 y_3, 3 y_4, 0, 3 y_5, 0, -3 y_3 + 5 y_4 - 3 y_5 - 3 y_2 + 5 y_1, 3 y_2, 3 y_1]$$

$$p = -s^2 - s^3 + s^5 + s^6$$

Â» SYNC'D 3891/65536 , 0.05937194824

21 . Coloring, {4, 6}

**R:** [3, 3, 1, 6, 7, 4, 5, 5] **B:** [6, 8, 8, 1, 2, 7, 4, 2]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} [ '2^{\epsilon} (\epsilon + 1 + \tau^{\epsilon})^{\epsilon}, -2^{\epsilon} (\epsilon - 1 + \tau^{\epsilon})^{\epsilon}, 2^{\epsilon} (\epsilon + 1 + \tau^{\epsilon})^{\epsilon}, 2^{\epsilon} (\epsilon + 1 + \tau^{\epsilon})^{\epsilon}, 2^{\epsilon} (\epsilon + 1 + \tau^{\epsilon})^{\epsilon}, 2^{\epsilon} (\epsilon + 1 + \tau^{\epsilon})^{\epsilon}, 2^{\epsilon} \\ (\epsilon + 1 + \tau^{\epsilon})^{\epsilon}, -2^{\epsilon} (\epsilon - 1 + \tau^{\epsilon})^{\epsilon} ]^{\epsilon} \end{aligned}$$

For  $\tau=1/2$ , [3, 1, 3, 3, 3, 3, 3, 1] . FixedPtCheck, [3, 1, 3, 3, 3, 3, 3, 1]

$$\det(A + \tau \Delta) = 1^{\epsilon} (\tau^{\epsilon})^{\epsilon^2} (\epsilon - 1 + \tau^{\epsilon})^{\epsilon^2} (\epsilon + 1 + \tau^{\epsilon})^{\epsilon^2}$$

Delta Range :  $[y_2, -y_1 - y_3 - y_5, -y_2 - y_4 - y_6, y_1, y_3, y_4, y_5, y_6]$

$[1, 1, 1, 1, 1, 1, 1]$

+ \; - \; \Delta

$\$ [ [1, 0, 2, 1, 2, 1, 1, 0], [3, 2, 1, 2, 1, 2, 3, 2], [3, 5, 5, 3, 5, 3, 3, 5], [5, 3, 4, 4, 4, 4, 5, 3], [8, 9, 8, 7, 8, 7, 8, 9], [17, 15, 17, 15, 17, 15, 17, 15] ] \$ [ [1, 2, 0, 1, 0, 1, 1, 2], [1, 2, 3, 2, 3, 2, 1, 2], [5, 3, 3, 5, 3, 5, 5, 3], [3, 5, 4, 4, 4, 4, 3, 5], [8, 7, 8, 9, 8, 9, 8, 7], [15, 17, 15, 17, 15, 17, 15, 17] ] \$ [ [0, -1, 1, 0, 1, 0, 0, -1], [1, 0, -1, 0, -1, 0, 1, 0], [-1, 1, 1, -1, 1, -1, -1, 1], [1, -1, 0, 0, 0, 0, 1, -1], [0, 1, 0, -1, 0, -1, 0, 1], [1, -1, 1, -1, 1, -1, 1, -1] ] \$$

$[-y_1 - y_2 - y_3, y_1, y_3, y_2, y_3, y_2, -y_1 - y_2 - y_3, y_1]$

$p' = s - 4s^3 - 4s^4 + 8s^5$   $p' = s^2 + 2s^3 - 4s^5$   $p = s + 4s^4 + 8s^5 - 16s^6$   
 S+ \; S- \; NM

$\$ [ [8, 7, 7, 7, 6, 9, 8, 6], [6, 9, 9, 5, 7, 5, 7, 10], [7, 7, 8, 7, 8, 6, 6, 9], [9, 5, 6, 9, 7, 10, 7, 5], [6, 9, 8, 6, 8, 7, 7, 7], [7, 5, 7, 10, 6, 9, 9, 5], [8, 6, 6, 9, 7, 7, 8, 7], [7, 10, 7, 5, 9, 5, 6, 9] ] \$ [ [8, 7, 7, 7, 6, 9, 8, 6], [6, 9, 9, 5, 7, 5, 7, 10], [7, 7, 8, 7, 8, 6, 6, 9], [9, 5, 6, 9, 7, 10, 7, 5], [6, 9, 8, 6, 8, 7, 7, 7], [7, 5, 7, 10, 6, 9, 9, 5], [8, 6, 6, 9, 7, 7, 8, 7], [7, 10, 7, 5, 9, 5, 6, 9] ] \$ [ [11, 8, 5, 6, 6, 5, 0, 3], [8, 11, 8, 5, 3, 6, 3, 0], [5, 8, 11, 4, 0, 7, 6, 3], [6, 5, 4, 11, 7, 0, 5, 6], [6, 3, 0, 7, 11, 4, 5, 8], [5, 6, 7, 0, 4, 11, 6, 5], [0, 3, 6, 5, 5, 6, 11, 8], [3, 0, 3, 6, 8, 5, 8, 11] ] \$$

CmmCk true, true, true

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
3 vs 6	4 vs 8	4 vs 8	2 vs 6	1 vs 6

Omega Rank for R : cycles:  $\{\{1, 3\}, \{4, 6\}, \{5, 7\}\}$ , net cycles: 3 . order: 2

$\$ [ [1, 0, 2, 1, 2, 1, 1, 0], [2, 0, 1, 1, 1, 1, 2, 0], [1, 0, 2, 1, 2, 1, 1, 0], [2, 0, 1, 1, 1, 1, 2, 0], [1, 0, 2, 1, 2, 1, 1, 0], [2, 0, 1, 1, 1, 1, 2, 0] ] \$$

$[y_2, 0, -y_2 + 3y_1, y_1, -y_2 + 3y_1, y_1, y_2, 0]$

$p = -s + s^3$   $p = -s + s^5$   $p' = -s + s^5$   $p' = -s + s^3$

Omega Rank for B : cycles:  $\{\{2, 8\}, \{1, 4, 6, 7\}\}$ , net cycles: 2 . order: 4

$\$ [ [1, 2, 0, 1, 0, 1, 1, 2], [1, 2, 0, 1, 0, 1, 1, 2], [1, 2, 0, 1, 0, 1, 1, 2], [1, 2, 0, 1, 0, 1, 1, 2], [1, 2, 0, 1, 0, 1, 1, 2], [1, 2, 0, 1, 0, 1, 1, 2] ] \$$

$[y_1, 2y_1, 0, y_1, 0, y_1, y_1, 2y_1]$

$p = -s + s^2$   $p = -s + s^3$   $p = -s + s^4$   $p = -s + s^5$   $p = -s + s^6$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[x_1, x_2, x_3, -4x_2 + 2x_3 + 4x_1, 8x_1 - 4x_2, -4x_3 - 16x_1 + 8x_2]$$

$$\text{For } A+2\Delta: [-y_3, -y_4, -y_1, -y_2, y_1, y_2, y_3, y_4]$$

$$\text{For } A-2\Delta: [-y_3, -y_4, -y_1, -y_2, y_1, y_2, y_3, y_4]$$

$$\text{Range of } \{\Omega\Delta^i\}: [\mu_3, \mu_2, \mu_1, -\mu_3 - \mu_2 - \mu_1, \mu_1, -\mu_3 - \mu_2 - \mu_1, \mu_3, \mu_2]$$

rank of M is 8 , rank of N is 5

M \ ; N

$$\begin{aligned} & \$ [ [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], \\ & [0, 0, 0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \quad \$ [ [0, 3, 6, 5, 5, 6, 11, 8], \\ & [3, 0, 3, 6, 8, 5, 8, 11], [6, 3, 0, 7, 11, 4, 5, 8], [5, 6, 7, 0, 4, 11, 6, 5], [5, 8, 11, 4, 0, 7, 6, 3], [6, 5, 4, 11, \\ & 7, 0, 5, 6], [11, 8, 5, 6, 6, 5, 0, 3], [8, 11, 8, 5, 3, 6, 3, 0] ] \$ \end{aligned}$$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [0, -1, 1, 0, 1, 0, 0, -1]$

ker M,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range M,  $[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$

$$\tau = 32, r' = 1/2$$

Ranges

Action of R on ranges,  $[[3], [3], [1], [4]]$

Action of B on ranges,  $[[4], [2], [2], [1]]$

$$\beta(\{1, 7\}) = 1/4$$

$$\beta(\{2, 8\}) = 1/4$$

$$\beta(\{3, 5\}) = 1/4$$

$$\beta(\{4, 6\}) = 1/4$$

ker N,  $[\mu_2, \mu_3, -\mu_1 - \mu_2 - \mu_3, \mu_1, -\mu_1 - \mu_2 - \mu_3, \mu_1, \mu_2, \mu_3]$

Range of N

$$[y_2 + y_4 - y_5, -y_1 + y_2 + y_4, -y_3 + y_2 + y_4, y_2, y_3, y_4, y_5, y_1]$$

Partitions

Action of R on partitions,  $[[1], [2], [4], [3], [2], [1]]$

Action of B on partitions,  $[[3], [6], [6], [3], [4], [5]]$

$$\alpha(\{\{3, 6, 7, 8\}, \{1, 2, 4, 5\}\}) = 2/11$$

$$\alpha(\{\{3, 4, 7, 8\}, \{1, 2, 5, 6\}\}) = 1/11$$

$$\alpha(\{\{1, 2, 3, 6\}, \{4, 5, 7, 8\}\}) = 3/11$$

$$\alpha(\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}) = 2/11$$

$$\alpha(\{\{2, 3, 4, 7\}, \{1, 5, 6, 8\}\}) = 1/11$$

$$\alpha(\{\{2, 3, 6, 7\}, \{1, 4, 5, 8\}\}) = 2/11$$

$b_1 = \{1, 2, 3, 6\}$  ,  $b_2 = \{4, 5, 7, 8\}$  ,  $b_3 = \{2, 3, 6, 7\}$  ,  $b_4 = \{3, 6, 7, 8\}$  ,  $b_5 = \{2, 3, 4, 7\}$  ,  $b_6 = \{3, 4, 7, 8\}$  ,  $b_7 = \{1, 5, 6, 8\}$  ,  $b_8 = \{1, 2, 3, 4\}$  ,  $b_9 = \{1, 2, 4, 5\}$  ,  $b_{10} = \{5, 6, 7, 8\}$  ,  $b_{11} = \{1, 4, 5, 8\}$  ,  $b_{12} = \{1, 2, 5, 6\}$

Action of R and B on the blocks of the partitions: \$ [ [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0] , [0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1] , [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0] , [1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] , [1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0] ] \$ = \$ [ [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0] ] \$ + \$ [ [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] ] \$

['8', A, '9', '9', C, C, '6', '1', '4', '2', '4', '6'], [B, '3', '7', '1', A, '3', '8', '2', '2', '1', '5', B] with invariant measure [3, 3, 2, 2, 1, 1, 1, 2, 2, 2, 2, 1]

N by blocks, check: true . ' See partition graph.

' ' See level-2 partition graph.

'

<b>Sandwich</b>	
<b>Coloring</b>	{4, 6}
<b>Rank</b>	2
<b>R,B</b>	[3, 3, 1, 6, 7, 4, 5, 5], [6, 8, 8, 1, 2, 7, 4, 2]
$\pi_2$	[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0]
$u_2$	[3, 6, 5, 5, 6, 11, 8, 3, 6, 8, 5, 8, 11, 7, 11, 4, 5, 8, 4, 11, 6, 5, 7, 6, 3, 5, 6, 3] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

22 . Coloring, {4, 7}

**R:** [3, 3, 1, 6, 7, 7, 4, 5] **B:** [6, 8, 8, 1, 2, 4, 5, 2]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & \left[ 4(1+\tau)^2(-5-\tau+\tau^2+\tau^3), -4(-1+\tau)^2(5+2\tau+\tau^2), 4(-5+\tau-\tau^2+\tau^3)(1+\tau), \right. \\ & \left. 4(1+\tau)^2(-5-3\tau-\tau^2+\tau^3), 4(5+\tau)(1+\tau)(-1+\tau), -4(1+\tau)(5+2\tau+\tau^2), \right. \\ & \left. 4(-1+\tau)^2(-5+\tau), 4(-1+\tau)(5-2\tau+\tau^2) \right] \end{aligned}$$

For  $\tau=1/2$ , [-123, -25, -111, -159, -66, -150, -162, -34] . FixedPtCheck, [123, 25, 111, 159, 66, 150, 162, 34]

$$\det(A + \tau \Delta) = 1^2 (\tau)^2 (1 + \tau)^2 (-1 + \tau)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	5 vs 6	3 vs 6

Omega Rank for R : cycles: {{1, 3}, {4, 6, 7}}, net cycles: 1 . order: 6

$$\$ [ [1, 0, 2, 1, 1, 1, 2, 0], [2, 0, 1, 2, 0, 1, 2, 0], [1, 0, 2, 2, 0, 2, 1, 0], [2, 0, 1, 1, 0, 2, 2, 0], [1, 0, 2, 2, 0, 1, 2, 0], [2, 0, 1, 2, 0, 2, 1, 0] ] \$$$

$$[3y_1, 0, 3y_2, 5y_1 + 5y_2 - 3y_3 - 3y_4 - 3y_5, 3y_3, 3y_4, 3y_5, 0]$$

$$p = -s^2 - s^3 + s^5 + s^6$$

Omega Rank for B : cycles: {{2, 8}, {1, 4, 6}}, net cycles: 1 . order: 6

$$\$ [ [1, 2, 0, 1, 1, 1, 0, 2], [1, 3, 0, 1, 0, 1, 0, 2], [1, 2, 0, 1, 0, 1, 0, 3], [1, 3, 0, 1, 0, 1, 0, 2], [1, 2, 0, 1, 0, 1, 0, 3], [1, 3, 0, 1, 0, 1, 0, 2] ] \$$$

$$[y_2, 5y_2 - y_1 - y_3, 0, y_2, y_1, y_2, 0, y_3]$$

$$p = s^2 - s^4 \quad p' = s^2 - s^4 \quad p'' = -s^3 + s^5$$

Â» SYNC'D 2641/131072 , 0.02014923096

23 . Coloring, {4, 8}

**R:** [3, 3, 1, 6, 7, 7, 5, 2] **B:** [6, 8, 8, 1, 2, 4, 4, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$[ '2' ( '5 - \tau + 3\tau^2 + \tau^3 ' ) , -2' ( ' - 1 + \tau ' ) ( '5 + 2\tau + \tau^2 ' ) , 2' ( '1 + \tau ' ) ( '5 - 2\tau + \tau^2 ' ) , 2' ( ' - 1 + \tau ' ) ( ' - 5 + \tau^2 ' ) , 2' ( '5 - \tau + 3\tau^2 + \tau^3 ' ) , -2' ( ' - 1 + \tau ' ) ( '5 + 2\tau + \tau^2 ' ) , 2' ( '1 + \tau ' ) ( '5 - 2\tau + \tau^2 ' ) , 2' ( ' - 1 + \tau ' ) ( ' - 5 + \tau^2 ' ) ]$$

For τ=1/2, [43, 25, 51, 19, 43, 25, 51, 19] . FixedPtCheck, [43, 25, 51, 19, 43, 25, 51, 19]

$$\det(A + \tau \Delta) = 1' ( ' - 1 + \tau ' )^2 ( '1 + \tau ' )^2 ( ' \tau ' )^2$$

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

\$ [ [1, 1, 2, 0, 1, 1, 2, 0] , [4, 1, 2, 1, 4, 1, 2, 1] , [5, 1, 5, 5, 5, 1, 5, 5] , [4, 4, 3, 5, 4, 4, 3, 5] , [6, 9, 8, 9, 6, 9, 8, 9] , [15, 19, 15, 15, 15, 19, 15, 15] ] \$ \$ [ [1, 1, 0, 2, 1, 1, 0, 2] , [0, 3, 2, 3, 0, 3, 2, 3] , [3, 7, 3, 3, 3, 7, 3, 3] , [4, 4, 5, 3, 4, 4, 5, 3] , [10, 7, 8, 7, 10, 7, 8, 7] , [17, 13, 17, 17, 17, 13, 17, 17] ] \$ \$ [ [0, 0, 1, -1, 0, 0, 1, -1] , [2, -1, 0, -1, 2, -1, 0, -1] , [1, -3, 1, 1, 1, -3, 1, 1] , [0, 0, -1, 1, 0, 0, -1, 1] , [-2, 1, 0, 1, -2, 1, 0, 1] , [-1, 3, -1, -1, -1, 3, -1, -1] ] \$

$$[-y_2 - y_1 - y_3, y_1, y_2, y_3, -y_2 - y_1 - y_3, y_1, y_2, y_3]$$

$$p' = s^2 + 4s^5 \quad p' = s + 4s^4 \quad p = s + 4s^4$$

S+ \ ; S- \ ; NM  
 \$ [ [1, 1, 2, 1, 0, 2, 2, 1] , [1, 1, 2, 1, 1, 0, 1, 3] , [2, 1, 1, 1, 2, 1, 0, 2] , [2, 1, 1, 1, 1, 3, 1, 0] , [0, 2, 2, 1, 1, 1, 2, 1] , [1, 0, 1, 3, 1, 1, 2, 1] , [2, 1, 0, 2, 2, 1, 1, 1] , [1, 3, 1, 0, 2, 1, 1, 1] ] \$ \$ [ [1, 1, 2, 1, 0, 2, 2, 1] , [1, 1, 2, 1, 1, 0, 1, 3] , [2, 1, 1, 1, 2, 1, 0, 2] , [2, 1, 1, 1, 1, 3, 1, 0] , [0, 2, 2, 1, 1, 1, 2, 1] , [1, 0, 1, 3, 1, 1, 2, 1] , [2, 1, 0, 2, 2, 1, 1, 1] , [1, 3, 1, 0, 2, 1, 1, 1] ] \$ \$ [ [4, 3, 2, 1, 0, 1, 2, 3] , [3, 4, 3, 2, 1, 0, 1, 2] , [2, 3, 4, 2, 2, 1, 0, 2] , [1, 2, 2, 4, 3, 2, 2, 0] , [0, 1, 2, 3, 4, 3, 2, 1] , [1, 0, 1, 2, 3, 4, 3, 2] , [2, 1, 0, 2, 2, 3, 4, 2] , [3, 2, 2, 0, 1, 2, 2, 4] ] \$

CmmCk true, true, true

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
3 vs 6	4 vs 8	4 vs 8	2 vs 6	3 vs 6

Omega Rank for R : cycles: {{1, 3}, {5, 7}}, net cycles: 0 . order: 2

\$ [ [1, 1, 2, 0, 1, 1, 2, 0] , [2, 0, 2, 0, 2, 0, 2, 0] , [2, 0, 2, 0, 2, 0, 2, 0] , [2, 0, 2, 0, 2, 0, 2, 0] , [2, 0, 2, 0, 2, 0, 2, 0] , [2, 0, 2, 0, 2, 0, 2, 0] ] \$

$$[-y_1 + y_2, y_1, y_2, 0, -y_1 + y_2, y_1, y_2, 0]$$

$$p' = s^4 - s^5 \quad p = s^2 - s^6 \quad p' = s^2 - s^5 \quad p' = s^3 - s^5$$

Omega Rank for B : cycles: {{1, 4, 6}, {2, 5, 8}}, net cycles: 2 . order: 3

$$\$ [ [1, 1, 0, 2, 1, 1, 0, 2], [2, 1, 0, 1, 2, 1, 0, 1], [1, 2, 0, 1, 1, 2, 0, 1], [1, 1, 0, 2, 1, 1, 0, 2], [2, 1, 0, 1, 2, 1, 0, 1], [1, 2, 0, 1, 1, 2, 0, 1] ] \$$$

$$[y_1, y_2, 0, y_3, y_1, y_2, 0, y_3]$$

$$p' = -s + s^4 \quad p' = -s^2 + s^5 \quad p = -s + s^4$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[x_2, x_1, x_3, 4x_2, 4x_1, 4x_3]$$

$$\text{For } A+2\Delta : [-y_1, -y_2, -y_3, -y_4, y_1, y_2, y_3, y_4]$$

$$\text{For } A-2\Delta : [-y_1, -y_2, -y_3, -y_4, y_1, y_2, y_3, y_4]$$

$$\text{Range of } \{\Omega\Delta^i\} : [-\mu_3 - \mu_1 - \mu_2, \mu_3, \mu_1, \mu_2, -\mu_3 - \mu_1 - \mu_2, \mu_3, \mu_1, \mu_2]$$

rank of M is 8 , rank of N is 5

$$M \quad \setminus ; \quad N$$

$$\$ [ [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0] ] \$ \quad \$ [ [0, 1, 2, 3, 4, 3, 2, 1], [1, 0, 1, 2, 3, 4, 3, 2], [2, 1, 0, 2, 2, 3, 4, 2], [3, 2, 2, 0, 1, 2, 2, 4], [4, 3, 2, 1, 0, 1, 2, 3], [3, 4, 3, 2, 1, 0, 1, 2], [2, 3, 4, 2, 2, 1, 0, 2], [1, 2, 2, 4, 3, 2, 2, 0] ] \$$$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [0, 0, 1, -1, 0, 0, 1, -1]$

$$\ker M, [0, 0, 0, 0, 0, 0, 0, 0]$$

$$\text{Range } M, [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$$

$$\tau = 32, r' = 1/2$$

Ranges

$$\text{Action of } R \text{ on ranges, } [[3], [3], [1], [2]]$$

$$\text{Action of } B \text{ on ranges, } [[2], [4], [4], [1]]$$

$$\beta(\{1, 5\}) = 1/4$$

$$\beta(\{2, 6\}) = 1/4$$

$$\beta(\{3, 7\}) = 1/4$$

$$\beta(\{4, 8\}) = 1/4$$



ker N,  $[\mu_1, \mu_2, \mu_3, -\mu_1 - \mu_2 - \mu_3, \mu_1, \mu_2, \mu_3, -\mu_1 - \mu_2 - \mu_3]$

Range of N

$$[y_1 - y_2 + y_5, y_1 + y_5 - y_3, y_1 + y_5 - y_4, y_1, y_2, y_3, y_4, y_5]$$

Partitions

Action of R on partitions,  $[[1], [3], [1], [5], [5]]$

Action of B on partitions,  $[[5], [4], [2], [5], [2]]$

$$\alpha(\{\{3, 5, 6, 8\}, \{1, 2, 4, 7\}\}) = 1/8$$

$$\alpha(\{\{2, 3, 4, 5\}, \{1, 6, 7, 8\}\}) = 1/4$$

$$\alpha(\{\{3, 4, 5, 6\}, \{1, 2, 7, 8\}\}) = 1/8$$

$$\alpha(\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}) = 1/8$$

$$\alpha(\{\{4, 5, 6, 7\}, \{1, 2, 3, 8\}\}) = 3/8$$

$b_1 = \{3, 5, 6, 8\}$  ,  $b_2 = \{4, 5, 6, 7\}$  ,  $b_3 = \{2, 3, 4, 5\}$  ,  $b_4 = \{1, 2, 3, 8\}$  ,  $b_5 = \{3, 4, 5, 6\}$  ,  $b_6 = \{1, 6, 7, 8\}$  ,  $b_7 = \{1, 2, 7, 8\}$  ,  $b_8 = \{1, 2, 3, 4\}$  ,  $b_9 = \{5, 6, 7, 8\}$  ,  $b_{10} = \{1, 2, 4, 7\}$

Action of R and B on the blocks of the partitions:  $\$ [ [0, 0, 0, 1, 0, 0, 0, 0, 0, 1] , [0, 1, 0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 1, 0] , [0, 0, 1, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0, 0, 1] , [0, 0, 0, 0, 1, 0, 0, 1, 0, 0] , [1, 0, 1, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 1, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 1, 0, 0, 0, 0, 0, 0] , [1, 1, 0, 0, 0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 0, 0, 0, 0, 0, 0, 0, 0, 1] , [0, 1, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 1, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0, 0, 0] ] \$ + \$ [ [0, 0, 0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 1, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0, 0, 0] ] \$$

$[A, '2', '7', '4', A, '5', '1', '4', '2', '1']$ ,  $['4', '6', '9', '3', '6', '8', '3', '2', '4', '2']$  with invariant measure  $[1, 3, 2, 3, 1, 2, 1, 1, 1, 1]$

N by blocks, check: true . ' See partition graph.

' ' See level-2 partition graph.

Sandwich	
<b>Coloring</b>	{4, 8}
<b>Rank</b>	2
<b>R,B</b>	[3, 3, 1, 6, 7, 7, 5, 2], [6, 8, 8, 1, 2, 4, 4, 5]
$\pi_2$	[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]
$u_2$	[1, 2, 3, 4, 3, 2, 1, 1, 2, 3, 4, 3, 2, 2, 2, 3, 4, 2, 1, 2, 2, 4, 1, 2, 3, 1, 2, 2] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

24 . Coloring, {5, 6}

**R:** [3, 3, 1, 1, 2, 4, 5, 5]    **B:** [6, 8, 8, 6, 7, 7, 4, 2]

' See graph

' ' See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

' [ '-2' (' 1 + \tau ' )'' (' 5 - \tau + 3\tau ^ 2 + \tau ^ 3 ' )' , 2' (' - 1 + \tau ' )'' (' 1 + \tau ' )'' (' 5 - 2\tau + \tau ^ 2 ' )' , -6' (' 1 + \tau ' )' ^ 2 ' (' 5 - 4\tau + 3\tau ^ 2 ' )' , 6' (' - 1 + \tau ' )'' (' 5 - 3\tau + 3\tau ^ 2 + 3\tau ^ 3 ' )' , 2' (' - 1 + \tau ' )'' (' 5 - \tau - \tau ^ 2 + \tau ^ 3 ' )'' (' 1 + \tau ' )' , -2' (' - 1 + \tau ' )'' (' - 5 + 2\tau - 4\tau ^ 2 - 2\tau ^ 3 + \tau ^ 4 ' )' , -2' (' - 1 + \tau ' )' ^ 2 ' (' 5 + \tau + \tau ^ 2 + \tau ^ 3 ' )' , 2' (' - 1 + \tau ' )'' (' 5 - 3\tau + \tau ^ 2 + \tau ^ 3 ' )'' (' 1 + \tau ' )'' ]'

For  $\tau=1/2$ , [-258, -102, -270, -74, -105, -83, -47, -93] . FixedPtCheck, [258, 102, 270, 74, 105, 83, 47, 93]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	6 vs 6	6 vs 6	4 vs 5	4 vs 5

Omega Rank for R : cycles: {{1, 3}}, net cycles: -1 . order: 4

\$ [ [2, 1, 2, 1, 2, 0, 0, 0] , [3, 2, 3, 0, 0, 0, 0, 0] , [3, 0, 5, 0, 0, 0, 0, 0] , [5, 0, 3, 0, 0, 0, 0, 0] , [3, 0, 5, 0, 0, 0, 0, 0] ] \$

$[y_1, y_2, y_3, y_4, 2y_4, 0, 0, 0]$

$$p = -s^3 + s^5$$

Omega Rank for B : cycles: {{2, 8}, {4, 6, 7}}, net cycles: 2 . order: 6

\$ [ [0, 1, 0, 1, 0, 2, 2, 2] , [0, 2, 0, 2, 0, 1, 2, 1] , [0, 1, 0, 2, 0, 2, 1, 2] , [0, 2, 0, 1, 0, 2, 2, 1] , [0, 1, 0, 2, 0, 1, 2, 2] ] \$

$[0, 3y_1, 0, 5y_1 - 3y_2 - 3y_3 + 5y_4, 0, 3y_2, 3y_3, 3y_4]$

$$p = -s - s^2 + s^4 + s^5$$

Â» SYNC'D 1/16 , 0.06250000000

25 . Coloring, {5, 7}

**R:** [3, 3, 1, 1, 2, 7, 4, 5]    **B:** [6, 8, 8, 6, 7, 4, 5, 2]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

‘ [ ‘ -2‘ ( ‘ 1 +  $\tau$  ‘ ) ‘ , 2‘ ( ‘ - 1 +  $\tau$  ‘ ) ‘ , -2‘ ( ‘ 1 +  $\tau$  ‘ ) ‘ , 2‘ ( ‘ - 1 +  $\tau$  ‘ ) ‘ , 2‘ ( ‘ - 1 +  $\tau$  ‘ ) ‘ , 2‘ ( ‘ - 1 +  $\tau$  ‘ ) ‘ , 2‘ ( ‘ - 1 +  $\tau$  ‘ ) ‘ , 2‘ ( ‘ - 1 +  $\tau$  ‘ ) ‘ ] ‘

For  $\tau=1/2$ , [-3, -1, -3, -1, -1, -1, -1, -1] . FixedPtCheck, [3, 1, 3, 1, 1, 1, 1, 1]

$\det(A + \tau \Delta) = 0$

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

[1, 1, 1, 1, 1, 1, 1, 1]

+ \ ; - \ ;  $\Delta$

\$ [ [2, 1, 2, 1, 1, 0, 1, 0] , [3, 3, 3, 3, 1, 1, 1, 1] , [3, 2, 3, 2, 2, 1, 2, 1] , [5, 5, 5, 5, 3, 3, 3, 3] , [5, 4, 5, 4, 4, 3, 4, 3] , [9, 9, 9, 9, 7, 7, 7, 7] ] \$ \$ [ [0, 1, 0, 1, 1, 2, 1, 2] , [1, 1, 1, 1, 3, 3, 3, 3] , [1, 2, 1, 2, 2, 3, 2, 3] , [3, 3, 3, 3, 5, 5, 5, 5] , [3, 4, 3, 4, 4, 5, 4, 5] , [7, 7, 7, 7, 9, 9, 9, 9] ] \$ \$ [ [1, 0, 1, 0, 0, -1, 0, -1] , [1, 1, 1, 1, -1, -1, -1, -1] , [1, 0, 1, 0, 0, -1, 0, -1] , [1, 1, 1, 1, -1, -1, -1, -1] ] \$

[-y<sub>2</sub>, -y<sub>1</sub>, -y<sub>2</sub>, -y<sub>1</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>1</sub>, y<sub>2</sub>]

$p' = s - 4s^5$      $p' = s^3 - 2s^5$      $p = s - 4s^5$   
 S+ \ ; S- \ ; NM

\$ [ [0, 0, 6, 4, 3, 3, 1, 3] , [1, 3, 5, 1, 4, 4, 0, 2] , [6, 4, 0, 0, 1, 3, 3, 3] , [5, 1, 1, 3, 0, 2, 4, 4] , [3, 5, 1, 1, 4, 0, 2, 4] , [2, 4, 2, 2, 1, 3, 5, 1] , [1, 1, 3, 5, 2, 4, 4, 0] , [2, 2, 2, 4, 5, 1, 1, 3] ] \$ \$ [ [4, 2, 2, 2, 3, 5, 1, 1] , [3, 3, 3, 1, 2, 0, 2, 6] , [2, 2, 4, 2, 1, 1, 3, 5] , [3, 1, 3, 3, 2, 6, 2, 0] , [3, 3, 1, 3, 0, 2, 6, 2] , [4, 0, 0, 6, 3, 3, 3, 1] , [1, 3, 3, 3, 6, 2, 0, 2] , [0, 6, 4, 0, 3, 1, 3, 3] ] \$ \$ [ [2, 1, 0, 1, 1, 1, 1, 1] , [1, 2, 1, 0, 1, 1, 1, 1] , [0, 1, 2, 1, 1, 1, 1, 1] , [1, 0, 1, 2, 1, 1, 1, 1] , [1, 1, 1, 1, 2, 1, 0, 1] , [1, 1, 1, 1, 1, 2, 1, 0] , [1, 1, 1, 1, 0, 1, 2, 1] , [1, 1, 1, 1, 1, 0, 1, 2] ] \$

CmmCk true, true, true

$p' = s^2 - 2s^4$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
2 vs 6	3 vs 7	3 vs 7	3 vs 6	2 vs 6

Omega Rank for R : cycles:  $\{\{1, 3\}\}$ , net cycles: -1 . order: 4

$\$ [ [2, 1, 2, 1, 1, 0, 1, 0], [3, 1, 3, 1, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0] ] \$$

$[y_1, y_2, y_1, y_2, y_3, 0, y_3, 0]$

$p = -s^3 + s^5 \quad p = -s^3 + s^6 \quad p = -s^3 + s^4$

Omega Rank for B : cycles:  $\{\{2, 8\}, \{4, 6\}, \{5, 7\}\}$ , net cycles: 3 . order: 2

$\$ [ [0, 1, 0, 1, 1, 2, 1, 2], [0, 2, 0, 2, 1, 1, 1, 1], [0, 1, 0, 1, 1, 2, 1, 2], [0, 2, 0, 2, 1, 1, 1, 1], [0, 1, 0, 1, 1, 2, 1, 2], [0, 2, 0, 2, 1, 1, 1, 1] ] \$$

$[0, 3y_1 - y_2, 0, 3y_1 - y_2, y_1, y_2, y_1, y_2]$

$p' = -s + s^3 \quad p = -s + s^5 \quad p' = -s + s^5 \quad p = -s + s^3$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$[x_1, x_2, x_3, x_4, -4x_1 - 2x_3, -4x_2 - 2x_4]$

For  $A+2\Delta$  :  $[y_1, -y_3, y_2, y_3, -y_4, -3y_1 - 3y_2 - y_5, y_4, y_5]$

For  $A-2\Delta$  :  $[-y_1 - 3y_3 - 3y_5, -y_2, y_1, y_2, -y_4, y_3, y_4, y_5]$

Range of  $\{\Omega\Delta^i\}$  :  $[-\mu_1, -\mu_2, -\mu_1, -\mu_2, \mu_2, \mu_1, \mu_2, \mu_1]$

rank of M is 8 , rank of N is 5

M \ ; N

$\$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0] ] \$ \quad \$ [ [0, 1, 2, 1, 1, 1, 1, 1], [1, 0, 1, 2, 1, 1, 1, 1], [2, 1, 0, 1, 1, 1, 1, 1], [1, 2, 1, 0, 1, 1, 1, 1], [1, 1, 1, 1, 0, 1, 2, 1], [1, 1, 1, 1, 1, 0, 1, 2], [1, 1, 1, 1, 2, 1, 0, 1], [1, 1, 1, 1, 1, 2, 1, 0] ] \$$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [1, 0, 1, 0, 0, -1, 0, -1]$

ker M,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range M,  $[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$

$\tau = 32$  ,  $r' = 1/2$

Ranges



0], [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] ] \$  
 ['3', A, C, E, D, A, '8', '7', E, B, '8', '7', C, B, D, '3'], ['5', '4', '5', '2', '1', '10', '1', '2', F, '4', '9', F, '6', '10', '9', '6'] with invariant measure [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]

N by blocks, check: true . ' See partition graph.

' ' See level-2 partition graph.

'

<b>Sandwich</b>	
<b>Coloring</b>	{5, 7}
<b>Rank</b>	2
<b>R,B</b>	[3, 3, 1, 1, 2, 7, 4, 5], [6, 8, 8, 6, 7, 4, 5, 2]
$\pi_2$	[0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0]
$u_2$	[1, 2, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 2, 1] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

26 . Coloring, {5, 8}

**R:** [3, 3, 1, 1, 2, 7, 5, 2]    **B:** [6, 8, 8, 6, 7, 4, 4, 5]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

' [ '2' (' - 5 - 2 $\tau$  - 12 $\tau^2$  + 2 $\tau^3$  +  $\tau^4$  )'' (' 1 +  $\tau$  ' )' , -2' (' - 1 +  $\tau$  ' )'' (' - 5 +  $\tau$  ' )'' (' 1 +  $\tau$  ' )' 3 ,  
 6' (' - 5 - 3 $\tau$  - 3 $\tau^2$  + 3 $\tau^3$  ' )'' (' 1 +  $\tau$  ' )' 2 , -6' (' - 1 +  $\tau$  ' )' 2 ' (' 5 +  $\tau$  + 7 $\tau^2$  + 3 $\tau^3$  ' )' , 2' (' - 1 +  $\tau$  ' )'' (' 5 - 3 $\tau$  +  $\tau^2$  +  $\tau^3$  ' )'' (' 1 +  $\tau$  ' )' 2 , 2' (' - 1 +  $\tau$  ' )'' (' 5 -  $\tau$  + 12 $\tau^2$  -  $\tau^4$  +  $\tau^5$  ' )' , -2' (' - 1 +  $\tau$  ' )'' (' - 5 + 2 $\tau$  - 4 $\tau^2$  - 2 $\tau^3$  +  $\tau^4$  ' )'' (' 1 +  $\tau$  ' )' , -2' (' - 1 +  $\tau$  ' )'' (' - 5 -  $\tau$  +  $\tau^2$  +  $\tau^3$  ' )'' (' 1 +  $\tau$  ' )' 2 ' ]'

For  $\tau=1/2$ , [-834, -486, -990, -122, -279, -239, -249, -369] . FixedPtCheck, [834, 486, 990, 122, 279, 239, 249, 369]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	6 vs 6	6 vs 6	5 vs 5	5 vs 5

Omega Rank for R : cycles:  $\{\{1, 3\}\}$ , net cycles: 0 . order: 4

$$[y_1, y_2, y_3, 0, y_4, 0, y_5, 0]$$

$$\begin{aligned} R = & \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, \\ & 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0] \\ & , [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, \\ & 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 0, 0, 5/16, -3/16], [0, 0, 0, 5/16, -3/16], [0, \\ & 0, 0, -3/16, 5/16], [0, 0, 0, -3/16, 5/16], [0, 0, 1, -3/16, -11/16], [1, -1, -1, 13/16, 5/16], [0, 1, -1, -11/16, \\ & 13/16], [0, 0, 1, -3/16, -11/16] ] \times \$ [ [2, 2, 2, 0, 1, 0, 1, 0], [2, 1, 4, 0, 1, 0, 0, 0], [4, 1, 3, 0, 0, 0, 0, 0], \\ & [3, 0, 5, 0, 0, 0, 0, 0], [5, 0, 3, 0, 0, 0, 0, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles:  $\{\{4, 6\}\}$ , net cycles: 0 . order: 4

$$[0, 0, 0, y_2, y_1, y_3, y_4, y_5]$$

$$\begin{aligned} B = & \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, \\ & 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \times \$ [ [0, 0, 0, 0, 0, 0, 0, 0] \\ & , [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, \\ & 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, -3/16, 5/16], [1/2, -1/4, -1/8, 1/16, \\ & -1/16], [1/2, -1/4, -1/8, 1/16, -1/16], [0, 0, 0, -3/16, 5/16], [0, 0, 1/2, -3/16, -3/16], [0, 0, 0, 5/16, -3/16], \\ & [0, 0, 0, 5/16, -3/16], [0, 1/2, -1/4, -3/16, 1/16] ] \times \$ [ [0, 0, 0, 2, 1, 2, 1, 2], [0, 0, 0, 3, 2, 2, 1, 0], [0, 0, \\ & 0, 3, 0, 3, 2, 0], [0, 0, 0, 5, 0, 3, 0, 0], [0, 0, 0, 3, 0, 5, 0, 0] ] \$ \end{aligned}$$

Â» SYNC'D 87/2048 , 0.04248046875

27 . Coloring,  $\{6, 7\}$

**R:** [3, 3, 1, 1, 7, 4, 4, 5] **B:** [6, 8, 8, 6, 2, 7, 5, 2]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ '6' ('1 + \tau')^2 ('-5 - 3\tau - 3\tau^2 + 3\tau^3')^2, -6' ('-1 + \tau')^2 ('5 + \tau + 7\tau^2 + 3\tau^3')^2, 2' \\ & ('-5 - 2\tau - 12\tau^2 + 2\tau^3 + \tau^4') ('1 + \tau')^2, -2' ('-1 + \tau') ('-5 + \tau') ('1 + \tau')^3, -2' ('-1 + \\ & \tau') ('1 + \tau') ('-5 + 2\tau - 4\tau^2 - 2\tau^3 + \tau^4')^2, -2' ('-1 + \tau') ('-5 - \tau + \tau^2 + \tau^3') ('1 + \tau')^2, \\ & 2' ('-1 + \tau') ('1 + \tau')^2 ('5 - 3\tau + \tau^2 + \tau^3')^2, 2' ('-1 + \tau') ('5 - \tau + 12\tau^2 - \tau^4 + \tau \end{aligned}$$

5 ‘,’‘’‘

For  $\tau=1/2$ , [-990, -122, -834, -486, -249, -369, -279, -239] . FixedPtCheck, [990, 122, 834, 486, 249, 369, 279, 239]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	6 vs 6	6 vs 6	5 vs 5	5 vs 5

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 4

$$[y_1, 0, y_2, y_3, y_4, 0, y_5, 0]$$

$$\begin{aligned} \mathbf{R} = \$ & [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, \\ & 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], \\ & [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, \\ & 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 0, 0, -3/16, 5/16], [0, 0, 0, -3/16, 5/16], [0, \\ & 0, 0, 5/16, -3/16], [0, 0, 0, 5/16, -3/16], [0, 1, -1, -11/16, 13/16], [0, 0, 1, -3/16, -11/16], [0, 0, 1, -3/16, \\ & -11/16], [1, -1, -1, 13/16, 5/16] ] \$ \times \$ [ [2, 0, 2, 2, 1, 0, 1, 0], [4, 0, 2, 1, 0, 0, 1, 0], [3, 0, 4, 1, 0, 0, 0, 0], \\ & [5, 0, 3, 0, 0, 0, 0, 0], [3, 0, 5, 0, 0, 0, 0, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{2, 8}}, net cycles: 0 . order: 4

$$[0, y_1, 0, 0, y_2, y_3, y_4, y_5]$$

$$\begin{aligned} \mathbf{B} = \$ & [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, \\ & 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [0, 0, 0, 0, 0, 0, 0, 0], \\ & [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, \\ & 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [1/2, -1/4, -1/8, 1/16, -1/16], [0, 0, 0, -3/16, \\ & 5/16], [0, 0, 0, -3/16, 5/16], [1/2, -1/4, -1/8, 1/16, -1/16], [0, 0, 0, 5/16, -3/16], [0, 1/2, -1/4, -3/16, 1/16], \\ & [0, 0, 1/2, -3/16, -3/16], [0, 0, 0, 5/16, -3/16] ] \$ \times \$ [ [0, 2, 0, 0, 1, 2, 1, 2], [0, 3, 0, 0, 1, 0, 2, 2], [0, 3, 0, \\ & 0, 2, 0, 0, 3], [0, 5, 0, 0, 0, 0, 0, 3], [0, 3, 0, 0, 0, 0, 0, 5] ] \$ \end{aligned}$$

Â» SYNC'D 87/2048 , 0.04248046875

28 . Coloring, {6, 8}

**R**: [3, 3, 1, 1, 7, 4, 5, 2] **B**: [6, 8, 8, 6, 2, 7, 4, 5]

‘ See graph

‘ ‘ See pair graph



Ω for A+τΔ :

$$[-2^{(-1+\tau)}, 2^{(-1+\tau)}, -2^{(-1+\tau)}, 2^{(-1+\tau)}, 2^{(-1+\tau)}, 2^{(-1+\tau)}, 2^{(-1+\tau)}, 2^{(-1+\tau)}]$$

For τ=1/2, [-3, -1, -3, -1, -1, -1, -1, -1] . FixedPtCheck, [3, 1, 3, 1, 1, 1, 1, 1]

$$\det(A + \tau \Delta) = 0$$

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

$$\begin{aligned} & \$ [ [2, 1, 2, 1, 1, 0, 1, 0], [3, 1, 3, 1, 3, 1, 3, 1], [2, 1, 2, 1, 3, 2, 3, 2], [3, 3, 3, 3, 5, 5, 5, 5], [3, 4, 3, 4, 4, \\ & 5, 4, 5], [7, 9, 7, 9, 7, 9, 7, 9] ] \$ [ [0, 1, 0, 1, 1, 2, 1, 2], [1, 3, 1, 3, 1, 3, 1, 3], [2, 3, 2, 3, 1, 2, 1, 2], \\ & [5, 5, 5, 5, 3, 3, 3, 3], [5, 4, 5, 4, 4, 3, 4, 3], [9, 7, 9, 7, 9, 7, 9, 7] ] \$ [ [1, 0, 1, 0, 0, -1, 0, -1], [1, -1, 1, \\ & -1, 1, -1, 1, -1], [0, -1, 0, -1, 1, 0, 1, 0], [-1, -1, -1, -1, 1, 1, 1, 1], [-1, 0, -1, 0, 0, 1, 0, 1], [-1, 1, -1, 1, -1, \\ & 1, -1, 1] ] \$ \end{aligned}$$

$$[-y_2, -y_1, -y_2, -y_1, y_1, y_2, y_1, y_2]$$

$$p = s + 4s^5$$

$$S+ \quad \backslash ; \quad S- \quad \backslash ; \quad NM$$

$$\begin{aligned} & \$ [ [1, 1, 5, 3, 4, 2, 0, 4], [0, 4, 6, 0, 3, 3, 1, 3], [5, 3, 1, 1, 0, 4, 4, 2], [6, 0, 0, 4, 1, 3, 3, 3], [2, 2, 2, 4, 1, \\ & 1, 5, 3], [3, 1, 1, 5, 4, 4, 2, 0], [2, 4, 2, 2, 5, 3, 1, 1], [1, 5, 3, 1, 2, 0, 4, 4] ] \$ [ [3, 1, 3, 3, 2, 6, 2, 0], \\ & [4, 2, 2, 2, 3, 1, 1, 5], [3, 3, 3, 1, 2, 0, 2, 6], [2, 2, 4, 2, 1, 5, 3, 1], [4, 6, 0, 0, 3, 1, 3, 3], [3, 3, 1, 3, 0, 2, 6, \\ & 2], [0, 0, 4, 6, 3, 3, 3, 1], [1, 3, 3, 3, 6, 2, 0, 2] ] \$ [ [2, 1, 0, 1, 1, 1, 1, 1], [1, 2, 1, 0, 1, 1, 1, 1], [0, 1, \\ & 2, 1, 1, 1, 1, 1], [1, 0, 1, 2, 1, 1, 1, 1], [1, 1, 1, 1, 2, 1, 0, 1], [1, 1, 1, 1, 2, 1, 0], [1, 1, 1, 1, 0, 1, 2, 1], \\ & [1, 1, 1, 1, 1, 0, 1, 2] ] \$ \end{aligned}$$

CmmCk true, true, true

$$p' = s + 4s^5 \quad p' = s^2 - 2s^4 + 4s^5 \quad p' = s^3 - 2s^4 + 2s^5$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
2 vs 6	3 vs 7	3 vs 7	2 vs 6	3 vs 6

Omega Rank for R : cycles: {{1, 3}, {5, 7}}, net cycles: 0 . order: 2

$$\begin{aligned} & \$ [ [2, 1, 2, 1, 1, 0, 1, 0], [3, 0, 3, 0, 1, 0, 1, 0], [3, 0, 3, 0, 1, 0, 1, 0], [3, 0, 3, 0, 1, 0, 1, 0], [3, 0, 3, 0, 1, \\ & 0, 1, 0], [3, 0, 3, 0, 1, 0, 1, 0] ] \$ \end{aligned}$$

$$[y_1, -y_1 + 3y_2, y_1, -y_1 + 3y_2, y_2, 0, y_2, 0]$$

$$p = -s^2 + s^3 \quad p = -s^2 + s^4 \quad p = -s^2 + s^5 \quad p = -s^2 + s^6$$

Omega Rank for B : cycles: {{4, 6, 7}, {2, 5, 8}}, net cycles: 2 . order: 3

$$\$ [ [0, 1, 0, 1, 1, 2, 1, 2], [0, 1, 0, 1, 2, 1, 2, 1], [0, 2, 0, 2, 1, 1, 1, 1], [0, 1, 0, 1, 1, 2, 1, 2], [0, 1, 0, 1, 2, 1, 2, 1], [0, 2, 0, 2, 1, 1, 1, 1] ] \$$$

$$[0, y_3, 0, y_3, y_2, y_1, y_2, y_1]$$

$$p' = -s^2 + s^5 \quad p' = -s + s^4 \quad p = -s + s^4$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[x_2, x_3, x_1, x_4, 4x_2 - 2x_4 - 2x_1, 4x_3 + 4x_1 + 2x_4]$$

$$\text{For } A+2\Delta : [y_1, y_2, y_3, -y_2, -y_4, -3y_1 - 3y_3 - y_5, y_4, y_5]$$

$$\text{For } A-2\Delta : [-y_4 - 3y_1 - 3y_5, y_3, y_4, -y_3, -y_2, y_1, y_2, y_5]$$

$$\text{Range of } \{\Omega\Delta^i\} : [-\mu_1, -\mu_2, -\mu_1, -\mu_2, \mu_2, \mu_1, \mu_2, \mu_1]$$

rank of M is 8 , rank of N is 5

$$M \quad \setminus ; \quad N$$

$$\$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0] ] \$ \quad \$ [ [0, 1, 2, 1, 1, 1, 1, 1], [1, 0, 1, 2, 1, 1, 1, 1], [2, 1, 0, 1, 1, 1, 1, 1], [1, 2, 1, 0, 1, 1, 1, 1], [1, 1, 1, 1, 0, 1, 2, 1], [1, 1, 1, 1, 1, 0, 1, 2], [1, 1, 1, 1, 2, 1, 0, 1], [1, 1, 1, 1, 1, 2, 1, 0] ] \$$$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [1, 0, 1, 0, 0, -1, 0, -1]$

$$\ker M, [0, 0, 0, 0, 0, 0, 0, 0]$$

$$\text{Range } M, [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$$

$$\tau = 32, r' = 1/2$$

Ranges

Action of R on ranges, [[1], [1], [3], [2]]

Action of B on ranges, [[4], [4], [2], [3]]

$$\beta(\{1, 3\}) = 1/4$$

$$\beta(\{2, 4\}) = 1/4$$

$$\beta(\{5, 7\}) = 1/4$$

$$\beta(\{6, 8\}) = 1/4$$

ker N,  $[-\mu_1 - \mu_2 - \mu_3, \mu_1, -\mu_1 - \mu_2 - \mu_3, \mu_1, \mu_2, \mu_3, \mu_2, \mu_3]$

Range of N

$$[y_3 + y_5 - y_1, -y_2 + y_3 + y_5, y_1, y_2, -y_4 + y_3 + y_5, y_3, y_4, y_5]$$

Partitions

Action of R on partitions,  $[[3], [7], [6], [2], [2], [6], [7], [3]]$

Action of B on partitions,  $[[8], [4], [5], [1], [5], [1], [8], [4]]$

$$\alpha(\{\{2, 3, 6, 7\}, \{1, 4, 5, 8\}\}) = 1/8$$

$$\alpha(\{\{3, 4, 5, 6\}, \{1, 2, 7, 8\}\}) = 1/8$$

$$\alpha(\{\{3, 4, 6, 7\}, \{1, 2, 5, 8\}\}) = 1/8$$

$$\alpha(\{\{2, 3, 5, 6\}, \{1, 4, 7, 8\}\}) = 1/8$$

$$\alpha(\{\{1, 4, 6, 7\}, \{2, 3, 5, 8\}\}) = 1/8$$

$$\alpha(\{\{3, 4, 7, 8\}, \{1, 2, 5, 6\}\}) = 1/8$$

$$\alpha(\{\{3, 4, 5, 8\}, \{1, 2, 6, 7\}\}) = 1/8$$

$$\alpha(\{\{2, 3, 7, 8\}, \{1, 4, 5, 6\}\}) = 1/8$$

$b_1 = \{1, 4, 6, 7\}$  ,  $b_2 = \{2, 3, 5, 8\}$  ,  $b_3 = \{3, 4, 6, 7\}$  ,  $b_4 = \{2, 3, 7, 8\}$  ,  $b_5 = \{1, 4, 5, 6\}$  ,  $b_6 = \{2, 3, 5, 6\}$  ,  $b_7 = \{3, 4, 5, 6\}$  ,  $b_8 = \{1, 2, 7, 8\}$  ,  $b_9 = \{2, 3, 6, 7\}$  ,  $b_{10} = \{1, 2, 5, 8\}$  ,  $b_{11} = \{3, 4, 5, 8\}$  ,  $b_{12} = \{1, 2, 6, 7\}$  ,  $b_{13} = \{3, 4, 7, 8\}$  ,  $b_{14} = \{1, 2, 5, 6\}$  ,  $b_{15} = \{1, 4, 5, 8\}$  ,  $b_{16} = \{1, 4, 7, 8\}$

Action of R and B on the blocks of the partitions: \$ [ [1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1] , [0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] ] \$

[ '7' , '8' , E , A , '3' , '8' , C , B , A , D , C , B , E , D , '3' , '7' ] , [ '1' , '2' , '1' , '6' , '10' , F , '10' , '6' , '5' , '2' , '4' , '5' , '9' , F , '4' , '9' ] with invariant measure [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]

N by blocks, check: true . ‘ See partition graph.

‘ ‘ See level-2 partition graph.

‘

<b>Sandwich</b>	
<b>Coloring</b>	{6, 8}
<b>Rank</b>	2
<b>R,B</b>	[3, 3, 1, 1, 7, 4, 5, 2], [6, 8, 8, 6, 2, 7, 4, 5]
$\pi_2$	[0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0]
$u_2$	[1, 2, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 2, 1] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

29 . Coloring, {7, 8}

**R:** [3, 3, 1, 1, 7, 7, 4, 2]    **B:** [6, 8, 8, 6, 2, 4, 5, 5]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ \text{‘} -6 \text{‘} (\text{‘} 1 + \tau \text{‘})^2 \text{‘} (\text{‘} 5 - 4\tau + 3\tau^2 \text{‘}) \text{‘}, 6 \text{‘} (\text{‘} -1 + \tau \text{‘}) \text{‘} (\text{‘} 5 - 3\tau + 3\tau^2 + 3\tau^3 \text{‘}) \text{‘}, -2 \text{‘} (\text{‘} 1 + \tau \text{‘} \\ & ) \text{‘} (\text{‘} 5 - \tau + 3\tau^2 + \tau^3 \text{‘}) \text{‘}, 2 \text{‘} (\text{‘} 1 + \tau \text{‘}) \text{‘} (\text{‘} 5 - 2\tau + \tau^2 \text{‘}) \text{‘} (\text{‘} -1 + \tau \text{‘}) \text{‘}, -2 \text{‘} (\text{‘} 5 + \tau + \tau^2 + \tau^3 \text{‘}) \text{‘} \\ & (\text{‘} -1 + \tau \text{‘})^2 \text{‘}, 2 \text{‘} (\text{‘} 5 - 3\tau + \tau^2 + \tau^3 \text{‘}) \text{‘} (\text{‘} 1 + \tau \text{‘}) \text{‘} (\text{‘} -1 + \tau \text{‘}) \text{‘}, 2 \text{‘} (\text{‘} 1 + \tau \text{‘}) \text{‘} (\text{‘} 5 - \tau - \tau^2 + \tau^3 \\ & \text{‘}) \text{‘} (\text{‘} -1 + \tau \text{‘}) \text{‘}, -2 \text{‘} (\text{‘} -1 + \tau \text{‘}) \text{‘} (\text{‘} -5 + 2\tau - 4\tau^2 - 2\tau^3 + \tau^4 \text{‘}) \text{‘} ] \text{‘} \end{aligned}$$

For  $\tau=1/2$ , [-270, -74, -258, -102, -47, -93, -105, -83] . FixedPtCheck, [270, 74, 258, 102, 47, 93, 105, 83]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	6 vs 6	6 vs 6	4 vs 5	4 vs 5

Omega Rank for R : cycles: {{1, 3}}, net cycles: -1 . order: 4

\$ [ [2, 1, 2, 1, 0, 0, 2, 0] , [3, 0, 3, 2, 0, 0, 0, 0] , [5, 0, 3, 0, 0, 0, 0, 0] , [3, 0, 5, 0, 0, 0, 0, 0] , [5, 0, 3, 0, 0, 0, 0, 0] ] \$

$[y_1, y_2, y_3, y_4, 0, 0, 2y_2, 0]$

$$p = -s^3 + s^5$$

Omega Rank for B : cycles: {{4, 6}, {2, 5, 8}}, net cycles: 2 . order: 6

\$ [ [0, 1, 0, 1, 2, 2, 0, 2] , [0, 2, 0, 2, 2, 1, 0, 1] , [0, 2, 0, 1, 1, 2, 0, 2] , [0, 1, 0, 2, 2, 1, 0, 2] , [0, 2, 0, 1, 2, 2, 0, 1] ] \$

$[0, 5y_1 - 3y_2 + 5y_4 - 3y_3, 0, 3y_1, 3y_2, 3y_4, 0, 3y_3]$

$$p = -s - s^2 + s^4 + s^5$$

Â» SYNC'D 1/16 , 0.06250000000

30 . Coloring, {2, 3, 4}

**R:** [3, 8, 8, 6, 7, 7, 5, 5] **B:** [6, 3, 1, 1, 2, 4, 4, 2]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$[ '4' ('1 + \tau^2')'' ('-5 + \tau^2')'' ('-1 + \tau')'^2 , 4' ('5 - 3\tau + \tau^2 + \tau^3')'' ('1 + \tau')'^2 ('-1 + \tau')' , -4' ('5 + \tau + \tau^2 + \tau^3')'' ('1 + \tau')'' ('-1 + \tau')'^2 , 4' ('5 - \tau + 12\tau^2 - \tau^4 + \tau^5')'' ('-1 + \tau')' , -4' ('1 + \tau')'^2 ('5 + 2\tau^2 + \tau^4')' , 4' ('5 - 3\tau + 10\tau^2 + 2\tau^3 + \tau^4 + \tau^5')'' ('-1 + \tau')' , -4' ('1 + \tau')'' ('5 + \tau + 10\tau^2 - 2\tau^3 + \tau^4 + \tau^5')' , 4' ('1 + \tau')'^2 ('-1 + \tau')'' ('5 - \tau - \tau^2 + \tau^3')'' ]'$

For  $\tau=1/2$ , [-95, -279, -141, -239, -801, -203, -753, -315] . FixedPtCheck, [95, 279, 141, 239, 801, 203, 753, 315]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	3 vs 5	5 vs 5

Omega Rank for R : cycles: {{5, 7}}, net cycles: -1 . order: 4

\$ [ [0, 0, 1, 0, 2, 1, 2, 2] , [0, 0, 0, 0, 4, 0, 3, 1] , [0, 0, 0, 0, 4, 0, 4, 0] , [0, 0, 0, 0, 4, 0, 4, 0] , [0, 0, 0, 0, 4, 0, 4, 0] ] \$

[0, 0, y<sub>2</sub>, 0, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, 2y<sub>2</sub> + y<sub>1</sub> - y<sub>3</sub>]

$$p = -s^3 + s^4 \quad p = -s^3 + s^5$$

Omega Rank for B : cycles: {{1, 4, 6}}, net cycles: 0 . order: 3

[y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>, 0, y<sub>5</sub>, 0, 0]

B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 0, 3/8, 3/8, -5/8] , [0, 1/2, -5/8, 3/8, -1/8] , [0, 0, 3/8, -5/8, 3/8] , [0, 0, 3/8, -5/8, 3/8] , [1/2, -1/4, 3/8, -1/8, -3/8] , [0, 0, -5/8, 3/8, 3/8] , [0, 0, -5/8, 3/8, 3/8] , [1/2, -1/4, 3/8, -1/8, -3/8] ] \$ x \$ [ [2, 2, 1, 2, 0, 1, 0, 0] , [3, 0, 2, 1, 0, 2, 0, 0] , [3, 0, 0, 2, 0, 3, 0, 0] , [2, 0, 0, 3, 0, 3, 0, 0] , [3, 0, 0, 3, 0, 2, 0, 0] ] \$

Â» SYNC'D 3/64 , 0.04687500000

31 . Coloring, {2, 3, 5}

**R:** [3, 8, 8, 1, 2, 7, 5, 5] **B:** [6, 3, 1, 6, 7, 4, 4, 2]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

[ '4' (' - 1 + τ ' ) ' 2 ' (' - 5 - 3τ - τ <sup>2</sup> + τ <sup>3</sup> ' ) ' (' 1 + τ ' ) ' , 4' (' - 5 + τ - τ <sup>2</sup> + τ <sup>3</sup> ' ) ' (' 1 + τ ' ) ' <sup>3</sup> , -4' (' - 1 + τ ' ) ' (' 1 + τ ' ) ' <sup>2</sup> ' (' - 5 - τ + τ <sup>2</sup> + τ <sup>3</sup> ' ) ' , -4' (' - 1 + τ ' ) ' <sup>2</sup> ' (' 5 + 2τ <sup>2</sup> + τ <sup>4</sup> ' ) ' , 12' (' 1 + τ ' ) ' <sup>2</sup> ' (' - 5 - 3τ - 3τ <sup>2</sup> + 3τ <sup>3</sup> ' ) ' , 12' (' - 1 + τ ' ) ' <sup>3</sup> ' (' 5 + 4τ + 3τ <sup>2</sup> ' ) ' , -4' (' - 1 + τ ' ) ' (' - 5 - τ - 3τ <sup>2</sup> + τ <sup>3</sup> ' ) ' (' 1 + τ ' ) ' , 4' (' - 5 + τ <sup>2</sup> ' ) ' (' 1 + τ ' ) ' <sup>3</sup> ' ]

For τ=1/2, [-159, -999, -369, -89, -990, -62, -294, -1026] . FixedPtCheck, [159, 999, 369, 89, 990, 62, 294, 1026]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	5 vs 6	4 vs 6

Omega Rank for R : cycles: {{2, 5, 8}}, net cycles: -1 . order: 3

\$ [ [1, 1, 1, 0, 2, 0, 1, 2] , [0, 2, 1, 0, 3, 0, 0, 2] , [0, 3, 0, 0, 2, 0, 0, 3] , [0, 2, 0, 0, 3, 0, 0, 3] , [0, 3, 0, 0, 3, 0, 0, 2] , [0, 3, 0, 0, 2, 0, 0, 3] ] \$

[y<sub>4</sub>, y<sub>1</sub>, y<sub>2</sub>, 0, y<sub>3</sub>, 0, y<sub>4</sub>, y<sub>5</sub>]

$$p = s^3 - s^6$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: -1 . order: 4

\$ [ [1, 1, 1, 2, 0, 2, 1, 0] , [1, 0, 1, 3, 0, 3, 0, 0] , [1, 0, 0, 3, 0, 4, 0, 0] , [0, 0, 0, 4, 0, 4, 0, 0] , [0, 0, 0, 4, 0, 4, 0, 0] , [0, 0, 0, 4, 0, 4, 0, 0] ] \$

[y<sub>1</sub> - y<sub>2</sub> + y<sub>3</sub>, y<sub>4</sub>, y<sub>1</sub>, y<sub>2</sub>, 0, y<sub>3</sub>, y<sub>4</sub>, 0]

$$p = -s^4 + s^6 \quad p = -s^4 + s^5$$

Â» SYNC'D 855/65536 , 0.01304626465

32 . Coloring, {2, 3, 6}

**R:** [3, 8, 8, 1, 7, 4, 5, 5] **B:** [6, 3, 1, 6, 2, 7, 4, 2]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

' [ '4' ('1 + τ')' ('-1 + τ')' ('-5 + 3τ - 7τ<sup>2</sup> + τ<sup>3</sup>')' , 4' ('1 + τ')' ('-1 + τ')' ('-5 - τ - 3τ<sup>2</sup> + τ<sup>3</sup>')' , -4' ('1 + τ')' ('-1 + τ')' ('5 - τ + 3τ<sup>2</sup> + τ<sup>3</sup>')' , -4' ('-1 + τ')' ('5 + 10τ<sup>2</sup> + τ<sup>4</sup>')' , 12' ('1 + τ')' ('5 + 2τ + 8τ<sup>2</sup> - 2τ<sup>3</sup> + 3τ<sup>4</sup>')' , 12' ('-1 + τ')'<sup>2</sup> ('5 + τ + 7τ<sup>2</sup> + 3τ<sup>3</sup>')' , 12' ('5 - τ + 3τ<sup>2</sup> + τ<sup>3</sup>')' ('1 + 3τ<sup>2</sup>')' , -12' ('1 + τ')'<sup>2</sup> ('-1 + τ')' ('5 + 3τ<sup>2</sup>')' ]'

For τ=1/2, [123, 147, 129, 121, 381, 61, 301, 207] . FixedPtCheck, [123, 147, 129, 121, 381, 61, 301, 207]

det(A + τ Δ) = 0

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	5 vs 6	6 vs 6

Omega Rank for R : cycles: {{5, 7}}, net cycles: 0 . order: 6

\$ [ [1, 0, 1, 1, 2, 0, 1, 2] , [1, 0, 1, 0, 3, 0, 2, 1] , [0, 0, 1, 0, 3, 0, 3, 1] , [0, 0, 0, 0, 4, 0, 3, 1] , [0, 0, 0, 0, 4, 0, 4, 0] , [0, 0, 0, 0, 4, 0, 4, 0] ] \$

$$[y_3, 0, y_4, y_5, y_3 - y_4 - y_5 + y_1 + y_2, 0, y_1, y_2]$$

$$p = -s^5 + s^6$$

Omega Rank for B : cycles: {{4, 6, 7}}, net cycles: 0 . order: 6

$$[y_1, y_2, y_3, y_6, 0, y_4, y_5, 0]$$

$$\mathbf{B} = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, 3/8, -1/8, -1/8], [0, 1/2, -1/4, -1/8, -1/8, 1/8], [0, 0, 1/2, -1/8, -1/8, -1/8], [0, 0, 0, 3/8, -1/8, -1/8], [1/2, -1/4, -1/8, -1/8, 1/8, 0], [0, 0, 0, -1/8, 3/8, -1/8], [0, 0, 0, -1/8, -1/8, 3/8], [1/2, -1/4, -1/8, -1/8, 1/8, 0] ] \times \$ [ [1, 2, 1, 1, 0, 2, 1, 0], [1, 0, 2, 1, 0, 2, 2, 0], [2, 0, 0, 2, 0, 2, 2, 0], [0, 0, 0, 2, 0, 4, 2, 0], [0, 0, 0, 2, 0, 2, 4, 0], [0, 0, 0, 4, 0, 2, 2, 0] ] \$$$

Â» SYNC'D 555/8192 , 0.06774902344

33 . Coloring, {2, 3, 7}

**R:** [3, 8, 8, 1, 7, 7, 4, 5] **B:** [6, 3, 1, 6, 2, 4, 5, 2]

‘ See graph

‘ ‘ See pair graph

‘

Ω for A+τΔ :

$$\begin{aligned} & [ ' 4 ' ( ' - 5 + 3\tau - 16\tau^2 + 4\tau^3 - 3\tau^4 + \tau^5 ' ) ' ( ' 1 + \tau ' ) ' , 4 ' ( ' - 1 + \tau ' ) ' ( ' 1 + \tau^2 ' ) ' ( ' 1 + \tau ' ) ' \\ & ( ' 5 + 2\tau + \tau^2 ' ) ' , -4 ' ( ' 1 + \tau ' ) ' ( ' 5 - 3\tau + 10\tau^2 + 2\tau^3 + \tau^4 + \tau^5 ' ) ' , -4 ' ( ' 5 + 2\tau + 19\tau^2 + \\ & 7\tau^4 - 2\tau^5 + \tau^6 ' ) ' , -12 ' ( ' 5 + 2\tau + 8\tau^2 - 2\tau^3 + 3\tau^4 ' ) ' ( ' 1 + \tau ' ) ' , 12 ' ( ' - 1 + \tau ' ) ' ( ' 1 + 3\tau^2 ' ) ' \\ & ) ' ( ' 5 + 2\tau + \tau^2 ' ) ' , 4 ' ( ' - 5 - 2\tau - 12\tau^2 + 2\tau^3 + \tau^4 ' ) ' ( ' 1 + \tau ' ) ' , 4 ' ( ' - 5 + 3\tau - 7\tau^2 + \tau^3 ' ) ' \\ & ( ' 1 + \tau ' ) ' ^2 ' ] ' \end{aligned}$$

For τ=1/2, [-687, -375, -609, -713, -762, -350, -834, -738] . FixedPtCheck, [687, 375, 609, 713, 762, 350, 834, 738]

$$\det(\mathbf{A} + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	6 vs 6	5 vs 6

Omega Rank for R : cycles: {{1, 3, 4, 5, 7, 8}}, net cycles: 1 . order: 6



$$[y_1, 0, y_2, y_3, y_4, 0, y_5, y_6]$$

$$\mathbf{R} = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [-1/16, -9/16, -1/16, 7/16, -1/16, 7/16], [7/16, -1/16, -9/16, -1/16, 7/16, -1/16], [-9/16, -1/16, 7/16, -1/16, 7/16, -1/16], [7/16, -1/16, 7/16, -1/16, -9/16, -1/16], [7/16, -1/16, 7/16, -1/16, -9/16, -1/16], [-1/16, 7/16, -1/16, 7/16, -1/16, -9/16], [-1/16, 7/16, -1/16, -9/16, -1/16, 7/16] ] \$ \times \$ [ [1, 0, 1, 1, 1, 0, 2, 2], [1, 0, 1, 2, 2, 0, 1, 1], [2, 0, 1, 1, 1, 0, 2, 1], [1, 0, 2, 2, 1, 0, 1, 1], [2, 0, 1, 1, 1, 0, 1, 2], [1, 0, 2, 1, 2, 0, 1, 1] ] \$$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: 0 . order: 6

$$\$ [ [1, 2, 1, 1, 1, 2, 0, 0], [1, 1, 2, 2, 0, 2, 0, 0], [2, 0, 1, 2, 0, 3, 0, 0], [1, 0, 0, 3, 0, 4, 0, 0], [0, 0, 0, 4, 0, 4, 0, 0], [0, 0, 0, 4, 0, 4, 0, 0] ] \$$$

$$[y_3, y_4, y_2, -y_3 - y_4 + y_2 + y_1 + y_5, y_1, y_5, 0, 0]$$

$$p = -s^5 + s^6$$

Â» SYNC'D 2665/65536 , 0.04066467285

34 . Coloring, {2, 3, 8}

**R**: [3, 8, 8, 1, 7, 7, 5, 2] **B**: [6, 3, 1, 6, 2, 4, 4, 5]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$[ '-4' (' - 5 + \tau^2 ')'' (' - 1 + \tau ')'' (' 1 + \tau ')', -4' (' 5 + 2\tau + \tau^2 ')'' (' 1 + \tau ')'^2, 4' (' 5 + \tau ')'' (' - 1 + \tau ')'' (' 1 + \tau ')'^2, 4' (' 5 - \tau + 3\tau^2 + \tau^3 ')'' (' - 1 + \tau ')', 4' (' - 5 + \tau^2 ')'' (' 1 + \tau ')'^2, -4' (' 5 + 2\tau + \tau^2 ')'' (' - 1 + \tau ')'^2, 4' (' - 5 - \tau - 3\tau^2 + \tau^3 ')'' (' 1 + \tau ')', 4' (' - 5 + \tau ')'' (' 1 + \tau ')'^3 ]'$$

For τ=1/2, [-57, -225, -99, -43, -171, -25, -147, -243] . FixedPtCheck, [57, 225, 99, 43, 171, 25, 147, 243]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	4 vs 6	5 vs 6

Omega Rank for R : cycles: {{2, 8}, {5, 7}}, net cycles: 1 . order: 4

\$ [ [1, 1, 1, 0, 1, 0, 2, 2] , [0, 2, 1, 0, 2, 0, 1, 2] , [0, 2, 0, 0, 1, 0, 2, 3] , [0, 3, 0, 0, 2, 0, 1, 2] , [0, 2, 0, 0, 1, 0, 2, 3] , [0, 3, 0, 0, 2, 0, 1, 2] ] \$

$$[4y_1 + 4y_2 - 5y_3 - y_4, y_1, y_2, 0, y_3, 0, 3y_1 + 3y_2 - 4y_3, y_4]$$

$$p = -s^3 + s^5 \quad p' = -s^3 + s^5$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: 0 . order: 6

\$ [ [1, 1, 1, 2, 1, 2, 0, 0] , [1, 1, 1, 2, 0, 3, 0, 0] , [1, 0, 1, 3, 0, 3, 0, 0] , [1, 0, 0, 3, 0, 4, 0, 0] , [0, 0, 0, 4, 0, 4, 0, 0] , [0, 0, 0, 4, 0, 4, 0, 0] ] \$

$$[-y_1 + y_2 - y_3 + y_4 + y_5, y_1, y_2, y_3, y_4, y_5, 0, 0]$$

$$p = -s^5 + s^6$$

Â» SYNC'D 463/65536 , 0.007064819336

35 . Coloring, {2, 4, 5}

**R:** [3, 8, 1, 6, 2, 7, 5, 5]    **B:** [6, 3, 8, 1, 7, 4, 4, 2]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$[ '4' ( '5 - 4\tau + 6\tau^2 + \tau^4 ' ) ' , 4' ( '5 + 4\tau + 6\tau^2 + \tau^4 ' ) ' , 4' ( '5 + 2\tau^2 + \tau^4 ' ) ' , 4' ( ' - 1 + \tau ' ) ' ^2 ( '5 + 2\tau + \tau^2 ' ) ' , 4' ( '5 + \tau + \tau^2 + \tau^3 ' ) ' ( '1 + \tau ' ) ' , 4' ( ' - 1 + \tau ' ) ' ( ' - 5 + \tau - \tau^2 + \tau^3 ' ) ' , -4' ( ' - 1 + \tau ' ) ' ( '1 + \tau ' ) ' ( '5 + \tau^2 ' ) ' , 4' ( '1 + \tau^2 ' ) ' ( '5 + 2\tau + \tau^2 ' ) ' ] '$$

For τ=1/2, [73, 137, 89, 25, 141, 37, 63, 125] . FixedPtCheck, [73, 137, 89, 25, 141, 37, 63, 125]

$$\det(A + \tau \Delta) = 1' ( ' \tau ' ) ' ^2 ( ' 1 + \tau ' ) ' ( ' 1 + \tau^2 ' ) ' ( ' - 1 + \tau ' ) '$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	3 vs 7	4 vs 7

Omega Rank for R : cycles: {{1, 3}, {2, 5, 8}}, net cycles: 1 . order: 6

\$ [ [1, 1, 1, 0, 2, 1, 1, 1] , [1, 2, 1, 0, 2, 0, 1, 1] , [1, 2, 1, 0, 2, 0, 0, 2] , [1, 2, 1, 0, 2, 0, 0, 2] , [1, 2, 1, 0, 2, 0, 0, 2] , [1, 2, 1, 0, 2, 0, 0, 2] ] \$

$$[y_3, 2y_3 - y_1, y_3, 0, 2y_3, y_1, 2y_3 - y_2, y_2]$$

$$p = -s^3 + s^4 \quad p = -s^3 + s^5 \quad p = -s^3 + s^6 \quad p = -s^3 + s^7$$

Omega Rank for B : cycles: {{1, 4, 6}, {2, 3, 8}}, net cycles: 1 . order: 3

$$\$ [ [1, 1, 1, 2, 0, 1, 1, 1], [2, 1, 1, 2, 0, 1, 0, 1], [2, 1, 1, 1, 0, 2, 0, 1], [1, 1, 1, 2, 0, 2, 0, 1], [2, 1, 1, 2, 0, 1, 0, 1], [2, 1, 1, 1, 0, 2, 0, 1], [1, 1, 1, 2, 0, 2, 0, 1] ] \$$$

$$[5y_3 - y_1 - y_2 - y_4, y_3, y_3, y_1, 0, y_2, y_4, y_3]$$

$$p' = -s^3 + s^6 \quad p' = -s^2 + s^5 \quad p = -s^2 + s^5$$

Â» SYNC'D 3999/262144 , 0.01525497437

36 . Coloring, {2, 4, 6}

**R:** [3, 8, 1, 6, 7, 4, 5, 5]    **B:** [6, 3, 8, 1, 2, 7, 4, 2]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ '-4' (' - 5 + \tau^2 ')', -4' (' 5 + \tau ')'' (' - 1 + \tau ')', 4' (' 5 - 2\tau + \tau^2 ')', 4' (' 5 + 2\tau + \tau^2 ')', 4' \\ & (' 1 + \tau ')'' (' 5 + \tau^2 ')', 4' (' 5 + \tau + \tau^2 + \tau^3 ')', 4' (' 5 + 3\tau + 3\tau^2 + \tau^3 ')', -4' (' - 1 + \tau ')'' (' 5 + 2\tau + \tau^2 ')'' ]' \end{aligned}$$

For τ=1/2, [38, 22, 34, 50, 63, 47, 59, 25] . FixedPtCheck, [38, 22, 34, 50, 63, 47, 59, 25]

$$\det(A + \tau \Delta) = 1' (' \tau ')''^2 (' 1 + \tau^2 ')'' (' 1 + \tau ')'' (' - 1 + \tau ')'$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	2 vs 7	3 vs 7

Omega Rank for R : cycles: {{1, 3}, {4, 6}, {5, 7}}, net cycles: 2 . order: 2

$$\$ [ [1, 0, 1, 1, 2, 1, 1, 1], [1, 0, 1, 1, 2, 1, 2, 0], [1, 0, 1, 1, 2, 1, 2, 0], [1, 0, 1, 1, 2, 1, 2, 0], [1, 0, 1, 1, 2, 1, 2, 0], [1, 0, 1, 1, 2, 1, 2, 0], [1, 0, 1, 1, 2, 1, 2, 0] ] \$$$

$$[y_2, 0, y_2, y_2, 2y_2, y_2, 2y_2 - y_1, y_1]$$

$$p = -s^2 + s^4 \quad p = -s^2 + s^3 \quad p = -s^2 + s^7 \quad p = -s^2 + s^5 \quad p = -s^2 + s^6$$

Omega Rank for B : cycles:  $\{\{1, 4, 6, 7\}, \{2, 3, 8\}\}$ , net cycles: 2 .

$\$ [ [1, 2, 1, 1, 0, 1, 1, 1], [1, 1, 2, 1, 0, 1, 1, 1], [1, 1, 1, 1, 0, 1, 1, 2], [1, 2, 1, 1, 0, 1, 1, 1], [1, 1, 2, 1, 0, 1, 1, 1], [1, 1, 1, 1, 0, 1, 1, 2], [1, 2, 1, 1, 0, 1, 1, 1] ] \$$

$$[y_2, 4y_2 - y_1 - y_3, y_1, y_2, 0, y_2, y_2, y_3]$$

$$p' = s - s^4 \quad p' = s^2 - s^5 \quad p' = s^3 - s^6 \quad p = s - s^7$$

Â» SYNC'D 145/131072 , 0.001106262207

37 . Coloring,  $\{2, 4, 7\}$

**R**:  $[3, 8, 1, 6, 7, 7, 4, 5]$    **B**:  $[6, 3, 8, 1, 2, 4, 5, 2]$

' See graph

' ' See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

$$[ '4' ( '5 + 2\tau^2 + \tau^4 ' ) ' , 4' ( ' - 1 + \tau ' ) ' ^2 ( '5 + 2\tau + \tau^2 ' ) ' , 4' ( '5 - 4\tau + 6\tau^2 + \tau^4 ' ) ' , 4' ( '5 + 4\tau + 6\tau^2 + \tau^4 ' ) ' , -4' ( ' - 1 + \tau ' ) ' ( '1 + \tau ' ) ' ( '5 + \tau^2 ' ) ' , 4' ( '1 + \tau^2 ' ) ' ( '5 + 2\tau + \tau^2 ' ) ' , 4' ( '5 + \tau + \tau^2 + \tau^3 ' ) ' ( '1 + \tau ' ) ' , 4' ( ' - 1 + \tau ' ) ' ( ' - 5 + \tau - \tau^2 + \tau^3 ' ) ' ] '$$

For  $\tau=1/2$ ,  $[89, 25, 73, 137, 63, 125, 141, 37]$  . FixedPtCheck,  $[89, 25, 73, 137, 63, 125, 141, 37]$

$$\det(A + \tau \Delta) = 1' ( ' - 1 + \tau ' ) ' ( ' \tau ' ) ' ^2 ( '1 + \tau^2 ' ) ' ( '1 + \tau ' ) '$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	3 vs 7	4 vs 7

Omega Rank for R : cycles:  $\{\{1, 3\}, \{4, 6, 7\}\}$ , net cycles: 1 . order: 6

$\$ [ [1, 0, 1, 1, 1, 1, 2, 1], [1, 0, 1, 2, 1, 1, 2, 0], [1, 0, 1, 2, 0, 2, 2, 0], [1, 0, 1, 2, 0, 2, 2, 0], [1, 0, 1, 2, 0, 2, 2, 0], [1, 0, 1, 2, 0, 2, 2, 0], [1, 0, 1, 2, 0, 2, 2, 0] ] \$$

$$[y_1, 0, y_1, 2y_1 - y_3, 2y_1 - y_2, y_2, 2y_1, y_3]$$

$$p = -s^3 + s^4 \quad p = -s^3 + s^5 \quad p = -s^3 + s^6 \quad p = -s^3 + s^7$$

Omega Rank for B : cycles:  $\{\{1, 4, 6\}, \{2, 3, 8\}\}$ , net cycles: 1 . order: 3

$\$ [ [1, 2, 1, 1, 1, 1, 0, 1], [1, 2, 2, 1, 0, 1, 0, 1], [1, 1, 2, 1, 0, 1, 0, 2], [1, 2, 1, 1, 0, 1, 0, 2], [1, 2, 2, 1, 0, 1, 0, 1], [1, 1, 2, 1, 0, 1, 0, 2], [1, 2, 1, 1, 0, 1, 0, 2] ] \$$

$$[y_4, y_1, y_2, y_4, y_3, y_4, 0, 5y_4 - y_1 - y_2 - y_3]$$

$$p = -s^2 + s^5 \quad p' = -s^2 + s^5 \quad p'' = -s^3 + s^6$$

Â» SYNC'D 3999/262144 , 0.01525497437

38 . Coloring, {2, 4, 8}

**R:** [3, 8, 1, 6, 7, 7, 5, 2]    **B:** [6, 3, 8, 1, 2, 4, 4, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ '4' ( '5 - 2\tau + \tau^2 ' ) , 4' ( '5 + 2\tau + \tau^2 ' ) , -4' ( ' - 5 + \tau^2 ' ) , -4' ( '5 + \tau ' ) ( ' - 1 + \tau ' ) , 4' \\ & ( '5 + 3\tau + 3\tau^2 + \tau^3 ' ) , -4' ( '5 + 2\tau + \tau^2 ' ) ( ' - 1 + \tau ' ) , 4' ( '1 + \tau ' ) ( '5 + \tau^2 ' ) , 4' ( '5 + \tau \\ & + \tau^2 + \tau^3 ' ) ] \end{aligned}$$

For τ=1/2, [34, 50, 38, 22, 59, 25, 63, 47] . FixedPtCheck, [34, 50, 38, 22, 59, 25, 63, 47]

$$\det(A + \tau \Delta) = 1' ( '1 + \tau ' ) ( ' \tau ' )^2 ( '1 + \tau^2 ' ) ( ' - 1 + \tau ' )$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	2 vs 7	3 vs 7

Omega Rank for R : cycles: {{1, 3}, {2, 8}, {5, 7}}, net cycles: 2 . order: 2

$$\$ [ [1, 1, 1, 0, 1, 1, 2, 1], [1, 1, 1, 0, 2, 0, 2, 1], [1, 1, 1, 0, 2, 0, 2, 1], [1, 1, 1, 0, 2, 0, 2, 1], [1, 1, 1, 0, 2, 0, 2, 1], [1, 1, 1, 0, 2, 0, 2, 1], [1, 1, 1, 0, 2, 0, 2, 1] ] \$$$

$$[y_1, y_1, y_1, 0, -y_2 + 2y_1, y_2, 2y_1, y_1]$$

$$p = -s^2 + s^3 \quad p = -s^2 + s^4 \quad p = -s^2 + s^5 \quad p = -s^2 + s^6 \quad p = -s^2 + s^7$$

Omega Rank for B : cycles: {{1, 4, 6}, {2, 3, 5, 8}}, net cycles: 2 .

$$\$ [ [1, 1, 1, 2, 1, 1, 0, 1], [2, 1, 1, 1, 1, 0, 1], [1, 1, 1, 1, 1, 2, 0, 1], [1, 1, 1, 2, 1, 1, 0, 1], [2, 1, 1, 1, 1, 1, 0, 1], [1, 1, 1, 1, 1, 2, 0, 1], [1, 1, 1, 2, 1, 1, 0, 1] ] \$$$

$$[y_3, y_2, y_2, -y_3 + 4y_2 - y_1, y_2, y_1, 0, y_2]$$

$$p = -s + s^4 \quad p' = -s + s^4 \quad p'' = -s^2 + s^5 \quad p''' = -s + s^7$$

Â» SYNC'D 145/131072 , 0.001106262207

39 . Coloring, {2, 5, 6}

**R:** [3, 8, 1, 1, 2, 4, 5, 5] **B:** [6, 3, 8, 6, 7, 7, 4, 2]

' See graph

' ' See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

' [ '4' (' 1 +  $\tau$  ')', 4' (' 1 +  $\tau$  ')', 4' (' 1 +  $\tau$  ')', -4' (' - 1 +  $\tau$  ')', 4' (' 1 +  $\tau$  ')', -4' (' - 1 +  $\tau$  ')',  
-4' (' - 1 +  $\tau$  ')', 4' (' 1 +  $\tau$  ') ' ]'

For  $\tau=1/2$ , [3, 3, 3, 1, 3, 1, 1, 3] . FixedPtCheck, [3, 3, 3, 1, 3, 1, 1, 3]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	4 vs 6	3 vs 6

Omega Rank for R : cycles: {{1, 3}, {2, 5, 8}}, net cycles: 1 . order: 6

\$ [ [2, 1, 1, 1, 2, 0, 0, 1] , [2, 2, 2, 0, 1, 0, 0, 1] , [2, 1, 2, 0, 1, 0, 0, 2] , [2, 1, 2, 0, 2, 0, 0, 1] , [2, 2, 2, 0, 1,  
0, 0, 1] , [2, 1, 2, 0, 1, 0, 0, 2] ] \$

[ $y_2, 2y_2 - y_3 - y_4, y_2 - y_1, y_1, y_3, 0, 0, y_4$ ]

$p' = s^2 - s^5$   $p = -s^2 + s^5$

Omega Rank for B : cycles: {{4, 6, 7}, {2, 3, 8}}, net cycles: 2 . order: 3

\$ [ [0, 1, 1, 1, 0, 2, 2, 1] , [0, 1, 1, 2, 0, 1, 2, 1] , [0, 1, 1, 2, 0, 2, 1, 1] , [0, 1, 1, 1, 0, 2, 2, 1] , [0, 1, 1, 2, 0,  
1, 2, 1] , [0, 1, 1, 2, 0, 2, 1, 1] ] \$

[ $0, y_3, y_3, 5y_3 - y_1 - y_2, 0, y_1, y_2, y_3$ ]

$p = -s + s^4$   $p' = -s + s^4$   $p' = -s^2 + s^5$

Â» SYNC'D 525/32768 , 0.01602172852

40 . Coloring, {2, 5, 7}

**R:** [3, 8, 1, 1, 2, 7, 4, 5]    **B:** [6, 3, 8, 6, 7, 4, 5, 2]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

‘ [ ‘4‘ (‘5 - 2 $\tau$  +  $\tau^2$  ‘)‘ (‘1 +  $\tau$  ‘)‘ , 4‘ (‘5 -  $\tau$  -  $\tau^2$  +  $\tau^3$  ‘)‘ , 4‘ (‘5 +  $\tau$  +  $\tau^2$  +  $\tau^3$  ‘)‘ , -4‘ (‘- 1 +  $\tau$  ‘)‘ (‘5 +  $\tau^2$  ‘)‘ , 4‘ (‘5 - 2 $\tau$  +  $\tau^2$  ‘)‘ , 4‘ (‘- 1 +  $\tau$  ‘)‘ (‘- 5 +  $\tau$  ‘)‘ , -4‘ (‘5 +  $\tau$  ‘)‘ (‘- 1 +  $\tau$  ‘)‘ , -4‘ (‘- 5 +  $\tau^2$  ‘)‘ ] ‘

For  $\tau=1/2$ , [51, 35, 47, 21, 34, 18, 22, 38] . FixedPtCheck, [51, 35, 47, 21, 34, 18, 22, 38]

$\det(A + \tau \Delta) = 1 \cdot (‘- 1 + \tau ‘) \cdot (‘\tau ‘)^2 \cdot (‘1 + \tau^2 ‘) \cdot (‘1 + \tau ‘)$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	4 vs 7	2 vs 7

Omega Rank for R : cycles: {{1, 3}, {2, 5, 8}}, net cycles: 1 . order: 6

\$ [ [2, 1, 1, 1, 1, 0, 1, 1] , [2, 1, 2, 1, 1, 0, 0, 1] , [3, 1, 2, 0, 1, 0, 0, 1] , [2, 1, 3, 0, 1, 0, 0, 1] , [3, 1, 2, 0, 1, 0, 0, 1] , [2, 1, 3, 0, 1, 0, 0, 1] , [3, 1, 2, 0, 1, 0, 0, 1] ] \$

[5  $y_4 - y_1 - y_2 - y_3, y_4, y_1, y_2, y_4, 0, y_3, y_4$ ]

$p = -s^3 + s^5$      $p' = -s^3 + s^5$      $p = -s^3 + s^7$

Omega Rank for B : cycles: {{4, 6}, {2, 3, 8}, {5, 7}}, net cycles: 3 . order: 6

\$ [ [0, 1, 1, 1, 1, 2, 1, 1] , [0, 1, 1, 2, 1, 1, 1, 1] , [0, 1, 1, 1, 1, 2, 1, 1] , [0, 1, 1, 2, 1, 1, 1, 1] , [0, 1, 1, 1, 1, 2, 1, 1] , [0, 1, 1, 2, 1, 1, 1, 1] , [0, 1, 1, 2, 1, 1, 1, 1] ] \$

[0,  $y_2, y_2, 3y_2 - y_1, y_2, y_1, y_2, y_2$ ]

$p = -s + s^7$      $p = -s + s^5$      $p' = -s + s^5$      $p = -s + s^3$      $p' = -s + s^3$

Â» SYNC'D 2469/262144 , 0.009418487549

41 . Coloring, {2, 5, 8}

**R:** [3, 8, 1, 1, 2, 7, 5, 2]    **B:** [6, 3, 8, 6, 7, 4, 4, 5]

‘ See graph

‘ ‘ See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

$$\left[ 4' (1 + \tau)' (-5 - \tau - 3\tau^2 + \tau^3)', 4' (-5 + \tau)' (1 + \tau)'^3, 4' (-5 + \tau^2)' (1 + \tau)'^2, -4' (-1 + \tau)'^2 (5 + 2\tau + \tau^2)', 4' (5 + \tau)' (-1 + \tau)' (1 + \tau)'^2, 4' (-1 + \tau)' (5 - \tau + 3\tau^2 + \tau^3)', -4' (-1 + \tau)' (-5 + \tau^2)' (1 + \tau)', -4' (1 + \tau)'^2 (5 + 2\tau + \tau^2)' \right]$$

For  $\tau=1/2$ , [-147, -243, -171, -25, -99, -43, -57, -225] . FixedPtCheck, [147, 243, 171, 25, 99, 43, 57, 225]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	4 vs 6	5 vs 6

Omega Rank for R : cycles: {{1, 3}, {2, 8}}, net cycles: 1 . order: 4

$$\$ [ [2, 2, 1, 0, 1, 0, 1, 1], [1, 2, 2, 0, 1, 0, 0, 2], [2, 3, 1, 0, 0, 0, 0, 2], [1, 2, 2, 0, 0, 0, 0, 3], [2, 3, 1, 0, 0, 0, 0, 2], [1, 2, 2, 0, 0, 0, 0, 3] ] \$$$

$$[-4y_1 + 3y_2 + 3y_4, -5y_1 + 4y_2 - y_3 + 4y_4, y_1, 0, y_2, 0, y_3, y_4]$$

$$p = -s^3 + s^5 \quad p' = -s^3 + s^5$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: 0 . order: 6

$$\$ [ [0, 0, 1, 2, 1, 2, 1, 1], [0, 0, 0, 3, 1, 2, 1, 1], [0, 0, 0, 3, 1, 3, 1, 0], [0, 0, 0, 4, 0, 3, 1, 0], [0, 0, 0, 4, 0, 4, 0, 0], [0, 0, 0, 4, 0, 4, 0, 0] ] \$$$

$$[0, 0, -y_1 - y_2 + y_3 + y_4 + y_5, y_1, y_2, y_3, y_4, y_5]$$

$$p = -s^5 + s^6$$

Â» SYNC'D 463/65536 , 0.007064819336

42 . Coloring, {2, 6, 7}

**R**: [3, 8, 1, 1, 7, 4, 4, 5] **B**: [6, 3, 8, 6, 2, 7, 5, 2]

‘ See graph

‘ ‘ See pair graph



Ω for A+τΔ :

$$\begin{aligned} & [ 12(1 + \tau)^2(5 + 2\tau + 8\tau^2 - 2\tau^3 + 3\tau^4), 12(5 + \tau + 7\tau^2 + 3\tau^3)(-1 + \tau)^2, \\ & 12(5 - \tau + 3\tau^2 + \tau^3)(1 + 3\tau^2), -12(1 + \tau)^2(5 + 3\tau^2)(-1 + \tau), 4(1 + \tau)^2 \\ & (-5 + 3\tau - 7\tau^2 + \tau^3)(-1 + \tau), 4(1 + \tau)^2(-5 - \tau - 3\tau^2 + \tau^3)(-1 + \tau), \\ & -4(1 + \tau)^2(5 - \tau + 3\tau^2 + \tau^3)(-1 + \tau), -4(-1 + \tau)^2(5 + 10\tau^2 + \tau^4) ] \end{aligned}$$

For τ=1/2, [381, 61, 301, 207, 123, 147, 129, 121] . FixedPtCheck, [381, 61, 301, 207, 123, 147, 129, 121]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	5 vs 6	6 vs 6

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 6

$$\begin{aligned} \$ [ [2, 0, 1, 2, 1, 0, 1, 1], [3, 0, 2, 1, 1, 0, 1, 0], [3, 0, 3, 1, 0, 0, 1, 0], [4, 0, 3, 1, 0, 0, 0, 0], [4, 0, 4, 0, 0, \\ 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0] ] \$ \end{aligned}$$

$$[y_5, 0, y_4, y_3, y_2, 0, y_1, -y_5 + y_4 + y_3 + y_2 - y_1]$$

$$p = s^5 - s^6$$

Omega Rank for B : cycles: {{2, 3, 8}}, net cycles: 0 . order: 6

$$[0, y_1, y_2, 0, y_3, y_6, y_4, y_5]$$

$$\begin{aligned} B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, \\ 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [0, 0, 0, 0, 0, 0, 0, 0] \\ , [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, \\ 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [1/2, -1/4, -1/8, -1/8, 1/8, 0], [0, 0, 0, -1/8, 3/8, \\ -1/8], [0, 0, 0, -1/8, -1/8, 3/8], [1/2, -1/4, -1/8, -1/8, 1/8, 0], [0, 0, 0, 3/8, -1/8, -1/8], [0, 1/2, -1/4, -1/8, \\ -1/8, 1/8], [0, 0, 1/2, -1/8, -1/8, -1/8], [0, 0, 0, 3/8, -1/8, -1/8] ] \$ \times \$ [ [0, 2, 1, 0, 1, 2, 1, 1], [0, 2, 2, 0, 1, \\ 0, 2, 1], [0, 2, 2, 0, 2, 0, 0, 2], [0, 4, 2, 0, 0, 0, 0, 2], [0, 2, 4, 0, 0, 0, 0, 2], [0, 2, 2, 0, 0, 0, 0, 4] ] \$ \end{aligned}$$

Â» SYNC'D 555/8192 , 0.06774902344

43 . Coloring, {2, 6, 8}

**R:** [3, 8, 1, 1, 7, 4, 5, 2] **B:** [6, 3, 8, 6, 2, 7, 4, 5]

‘ See graph

‘ ‘ See pair graph

Ω for A+τΔ :

$$[4(1+\tau)^2(5+\tau^2), 4(5+\tau+\tau^2+\tau^3), 4(5+3\tau+3\tau^2+\tau^3), -4(-1+\tau)^2(5+2\tau+\tau^2), -4(-5+\tau^2), -4(5+\tau)^2(-1+\tau), 4(5-2\tau+\tau^2), 4(5+2\tau+\tau^2)]$$

For τ=1/2, [63, 47, 59, 25, 38, 22, 34, 50] . FixedPtCheck, [63, 47, 59, 25, 38, 22, 34, 50]

$$\det(A + \tau \Delta) = 1(\tau)^2(1+\tau^2)(-1+\tau)(1+\tau)$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	2 vs 7	3 vs 7

Omega Rank for R : cycles: {{1, 3}, {2, 8}, {5, 7}}, net cycles: 2 . order: 2

$$\$ [ [2, 1, 1, 1, 1, 0, 1, 1], [2, 1, 2, 0, 1, 0, 1, 1], [2, 1, 2, 0, 1, 0, 1, 1], [2, 1, 2, 0, 1, 0, 1, 1], [2, 1, 2, 0, 1, 0, 1, 1], [2, 1, 2, 0, 1, 0, 1, 1], [2, 1, 2, 0, 1, 0, 1, 1] ] \$$$

$$[2y_1, y_1, 2y_1 - y_2, y_2, y_1, 0, y_1, y_1]$$

$$p = -s^2 + s^5 \quad p = -s^2 + s^6 \quad p = -s^2 + s^7 \quad p = -s^2 + s^3 \quad p = -s^2 + s^4$$

Omega Rank for B : cycles: {{2, 3, 5, 8}, {4, 6, 7}}, net cycles: 2 .

$$\$ [ [0, 1, 1, 1, 1, 2, 1, 1], [0, 1, 1, 1, 1, 2, 1], [0, 1, 1, 2, 1, 1, 1, 1], [0, 1, 1, 1, 1, 2, 1, 1], [0, 1, 1, 1, 1, 1, 2, 1], [0, 1, 1, 2, 1, 1, 1, 1], [0, 1, 1, 1, 1, 2, 1, 1] ] \$$$

$$[0, y_3, y_3, -y_1 - y_2 + 4y_3, y_3, y_1, y_2, y_3]$$

$$p' = -s + s^4 \quad p = -s + s^4 \quad p' = -s^2 + s^5 \quad p = -s + s^7$$

Â» SYNC'D 145/131072 , 0.001106262207

44 . Coloring, {2, 7, 8}

**R:** [3, 8, 1, 1, 7, 7, 4, 2]    **B:** [6, 3, 8, 6, 2, 4, 5, 5]

‘ See graph

‘ ‘ See pair graph

Ω for A+τΔ :

$$[12(1+\tau)^2(5+3\tau^2), 12(5-3\tau+3\tau^2+3\tau^3), 12(5+\tau+7\tau^2+3\tau^3), -12(5+4\tau+3\tau^2)(-1+\tau), 4(-5+\tau^2)(-1+\tau), -4(5+2\tau+\tau^2)(-1+\tau)]$$

$$(-1)^4 \cdot (-4)^4 \cdot (5 + \tau)^4 \cdot (1 + \tau)^4 \cdot (-1 + \tau)^4 \cdot (5 - \tau + 3\tau^2 + \tau^3)^4$$

For  $\tau=1/2$ , [69, 37, 61, 31, 19, 25, 33, 43] . FixedPtCheck, [69, 37, 61, 31, 19, 25, 33, 43]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	4 vs 6	4 vs 6

Omega Rank for R : cycles: {{1, 3}, {2, 8}}, net cycles: 1 . order: 4

$$\$ [ [2, 1, 1, 1, 0, 0, 2, 1], [2, 1, 2, 2, 0, 0, 0, 1], [4, 1, 2, 0, 0, 0, 0, 1], [2, 1, 4, 0, 0, 0, 0, 1], [4, 1, 2, 0, 0, 0, 0, 1], [2, 1, 4, 0, 0, 0, 0, 1] ] \$$$

$$[-y_3 - y_4 - y_1 + 6y_2, y_2, y_3, y_4, 0, 0, y_1, y_2]$$

$$p' = -s^3 + s^5 \quad p = -s^3 + s^5$$

Omega Rank for B : cycles: {{2, 3, 5, 8}, {4, 6}}, net cycles: 2 . order: 4

$$\$ [ [0, 1, 1, 1, 2, 2, 0, 1], [0, 2, 1, 2, 1, 1, 0, 1], [0, 1, 2, 1, 1, 2, 0, 1], [0, 1, 1, 2, 1, 1, 0, 2], [0, 1, 1, 1, 2, 2, 0, 1], [0, 2, 1, 2, 1, 1, 0, 1] ] \$$$

$$[0, 4y_2 + 4y_1 - 5y_4 - y_3, y_2, 3y_2 + 3y_1 - 4y_4, y_1, y_4, 0, y_3]$$

$$p' = -s + s^5 \quad p = -s + s^5$$

Â» SYNC'D 179/16384 , 0.01092529297

45 . Coloring, {3, 4, 5}

**R:** [3, 3, 8, 6, 2, 7, 5, 5] **B:** [6, 8, 1, 1, 7, 4, 4, 2]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ (-4)^4 \cdot (-1 + \tau)^4 \cdot (5 - 4\tau + 6\tau^2 + \tau^4)^4 \cdot (1 + \tau)^4 \cdot (5 + 2\tau^2 + \tau^4)^4 \cdot (1 + \tau)^4 \cdot \\ & (1 + \tau^2)^4 \cdot (5 - 2\tau + \tau^2)^4 \cdot (-4)^4 \cdot (-5 + \tau - \tau^2 + \tau^3)^4 \cdot (-1 + \tau)^4 \cdot (5 - 4\tau + 3\tau^2)^4 \cdot \\ & (1 + \tau)^4 \cdot (-1 + \tau)^4 \cdot (5 + 3\tau^2)^4 \cdot (-4)^4 \cdot (1 + \tau)^4 \cdot (-1 + \tau)^4 \cdot (5 - 2\tau + \tau^2)^4 \cdot (1 + \tau)^4 \cdot \\ & (5 - \tau + 3\tau^2 + \tau^3)^4 ] \end{aligned}$$

For  $\tau=1/2$ , [73, 267, 255, 37, 270, 46, 102, 258] . FixedPtCheck, [73, 267, 255, 37, 270, 46, 102, 258]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	6 vs 6	2 vs 6

Omega Rank for R : cycles: {{2, 3, 5, 8}}, net cycles: 0 . order: 4

$$[0, y_6, y_1, 0, y_2, y_3, y_4, y_5]$$

$$\begin{aligned} R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, \\ 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [0, 0, 0, 0, 0, 0, 0, 0] \\ , [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, \\ 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, -3/32, 5/32, 13/32, -11/32], [0, 0, -3/32, \\ 5/32, 13/32, -11/32], [0, 0, -11/32, -3/32, 5/32, 13/32], [1, -1, -3/32, 5/32, -19/32, 21/32], [0, 0, 5/32, \\ 13/32, -11/32, -3/32], [0, 1, -11/32, -3/32, 5/32, -19/32], [0, 0, 13/32, -11/32, -3/32, 5/32], [0, 0, 13/32, \\ -11/32, -3/32, 5/32] ] \$ \times \$ [ [0, 1, 2, 0, 2, 1, 1, 1], [0, 2, 1, 0, 2, 0, 1, 2], [0, 2, 2, 0, 3, 0, 0, 1], [0, 3, 2, 0, \\ 1, 0, 0, 2], [0, 1, 3, 0, 2, 0, 0, 2], [0, 2, 1, 0, 2, 0, 0, 3] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{2, 8}, {1, 4, 6}}, net cycles: 1 . order: 6

$$\begin{aligned} \$ [ [2, 1, 0, 2, 0, 1, 1, 1], [2, 1, 0, 2, 0, 2, 0, 1], [2, 1, 0, 2, 0, 2, 0, 1], [2, 1, 0, 2, 0, 2, 0, 1], [2, 1, 0, 2, 0, \\ 2, 0, 1], [2, 1, 0, 2, 0, 2, 0, 1] ] \$ \end{aligned}$$

$$[2 y_2, y_2, 0, 2 y_2, 0, 2 y_2 - y_1, y_1, y_2]$$

$$p' = s^2 - s^4 \quad p' = s^3 - s^4 \quad p' = -s^4 + s^5 \quad p = s^2 - s^5$$

Â» SYNC'D 4447/262144 , 0.01696395874

46 . Coloring, {3, 4, 6}

**R**: [3, 3, 8, 6, 7, 4, 5, 5]    **B**: [6, 8, 1, 1, 2, 7, 4, 2]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} [ '4' (' - 5 + \tau^2 ')'' (' - 1 + \tau ')', -4' (' - 1 + \tau ')'' (' 5 + 2\tau + \tau^2 ')', -4' (' 5 + \tau ')'' (' 1 + \tau ')'' \\ (' - 1 + \tau ')', 4' (' 5 - \tau + 3\tau^2 + \tau^3 ')', 12' (' 1 + \tau ')'' (' 5 + 3\tau^2 ')', 12' (' 5 - 3\tau + 3\tau^2 + 3\tau^3 ')', \\ 12' (' 5 + \tau + 7\tau^2 + 3\tau^3 ')', -12' (' 5 + 4\tau + 3\tau^2 ')'' (' - 1 + \tau ')'' ]' \end{aligned}$$

For  $\tau=1/2$ , [19, 25, 33, 43, 69, 37, 61, 31] . FixedPtCheck, [19, 25, 33, 43, 69, 37, 61, 31]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	4 vs 6	4 vs 6

Omega Rank for R : cycles: {{4, 6}, {5, 7}}, net cycles: 1 . order: 4

\$ [ [0, 0, 2, 1, 2, 1, 1, 1] , [0, 0, 0, 1, 2, 1, 2, 2] , [0, 0, 0, 1, 4, 1, 2, 0] , [0, 0, 0, 1, 2, 1, 4, 0] , [0, 0, 0, 1, 4, 1, 2, 0] , [0, 0, 0, 1, 2, 1, 4, 0] ] \$

$$[0, 0, 6y_3 - y_2 - y_4 - y_1, y_3, y_2, y_3, y_4, y_1]$$

$$p' = s^3 - s^5 \quad p = s^3 - s^5$$

Omega Rank for B : cycles: {{1, 4, 6, 7}, {2, 8}}, net cycles: 2 . order: 4

\$ [ [2, 2, 0, 1, 0, 1, 1, 1] , [1, 1, 0, 1, 0, 2, 1, 2] , [1, 2, 0, 1, 0, 1, 2, 1] , [1, 1, 0, 2, 0, 1, 1, 2] , [2, 2, 0, 1, 0, 1, 1, 1] , [1, 1, 0, 1, 0, 2, 1, 2] ] \$

$$[y_2, y_3, 0, y_4, 0, 4y_2 - 5y_3 - y_4 + 4y_1, y_1, 3y_2 - 4y_3 + 3y_1]$$

$$p = -s + s^5 \quad p' = -s + s^5$$

Â» SYNC'D 179/16384 , 0.01092529297

47 . Coloring, {3, 4, 7}

**R**: [3, 3, 8, 6, 7, 7, 4, 5] **B**: [6, 8, 1, 1, 2, 4, 5, 2]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ '-4' ('-1+\tau')' ('5+2\tau^2+\tau^4')' , 4' ('-1+\tau')'^2 ('5+3\tau+3\tau^2+\tau^3')' , 4' ('-5+\tau \\ & -\tau^2+\tau^3')' ('-1+\tau')' ('1+\tau')' , -4' ('-5-\tau-3\tau^2+\tau^3')' ('1+\tau^2')' , -12' ('-1+\tau' \\ & )' ('1+\tau')' ('5+3\tau^2')' , 12' ('5-2\tau+8\tau^2+2\tau^3+3\tau^4')' , 4' ('1+\tau')' ('5-\tau+3\tau^2+\tau \\ & ^3')' , 4' ('-5-\tau-3\tau^2+\tau^3')' ('-1+\tau')' ]' \end{aligned}$$

For  $\tau=1/2$ , [89, 59, 111, 245, 138, 206, 258, 98] . FixedPtCheck, [89, 59, 111, 245, 138, 206, 258, 98]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	6 vs 6	4 vs 6

Omega Rank for R : cycles: {{4, 6, 7}}, net cycles: 0 . order: 6

$$[0, 0, y_5, y_4, y_1, y_2, y_6, y_3]$$

$$\begin{aligned} R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], \\ [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], \\ [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [1/2, -1/4, -1/8, -1/8, 1/8, 0], [1/2, -1/4, -1/8, -1/8, 1/8, 0], [0, 1/2, -1/4, -1/8, -1/8, 1/8], [0, 0, 0, -1/8, -1/8, 3/8], [0, 0, 0, 3/8, -1/8, -1/8], [0, 0, 0, 3/8, -1/8, -1/8], \\ [0, 0, 0, -1/8, 3/8, -1/8], [0, 0, 1/2, -1/8, -1/8, -1/8] ] \$ \times \$ [ [0, 0, 2, 1, 1, 1, 2, 1], [0, 0, 0, 2, 1, 1, 2, 2], [0, 0, 0, 2, 2, 2, 2, 0], [0, 0, 0, 2, 0, 2, 4, 0], [0, 0, 0, 4, 0, 2, 2, 0], [0, 0, 0, 2, 0, 4, 2, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{2, 8}, {1, 4, 6}}, net cycles: 1 . order: 6

$$\$ [ [2, 2, 0, 1, 1, 1, 0, 1], [1, 2, 0, 1, 0, 2, 0, 2], [1, 2, 0, 2, 0, 1, 0, 2], [2, 2, 0, 1, 0, 1, 0, 2], [1, 2, 0, 1, 0, 2, 0, 2], [1, 2, 0, 2, 0, 1, 0, 2] ] \$$$

$$[2y_2 + 2y_4 - y_1 - y_3, y_2 + y_4, 0, y_1, y_2, y_3, 0, y_4]$$

$$p^2 = s^2 - s^5 \quad p = -s^2 + s^5$$

$\hat{A}$ » SYNC'D 1665/32768 , 0.05081176758

48 . Coloring, {3, 4, 8}

**R**: [3, 3, 8, 6, 7, 7, 5, 2] **B**: [6, 8, 1, 1, 2, 4, 4, 5]

' See graph

' ' See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

$$[ '-4' (' - 1 + \tau ')', 4' (' 1 + \tau ')', 4' (' 1 + \tau ')', -4' (' - 1 + \tau ')', 4' (' 1 + \tau ')', -4' (' - 1 + \tau ')', 4' (' 1 + \tau ')', 4' (' 1 + \tau ')']$$

For  $\tau=1/2$ , [1, 3, 3, 1, 3, 1, 3, 3] . FixedPtCheck, [1, 3, 3, 1, 3, 1, 3, 3]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	4 vs 6	3 vs 6

Omega Rank for R : cycles: {{2, 3, 8}, {5, 7}}, net cycles: 1 . order: 6

\$ [ [0, 1, 2, 0, 1, 1, 2, 1] , [0, 1, 1, 0, 2, 0, 2, 2] , [0, 2, 1, 0, 2, 0, 2, 1] , [0, 1, 2, 0, 2, 0, 2, 1] , [0, 1, 1, 0, 2, 0, 2, 2] , [0, 2, 1, 0, 2, 0, 2, 1] ] \$

$$[0, -y_1 + 2y_3 - y_4, y_1, 0, -y_2 + y_3, y_2, y_3, y_4]$$

$$p' = -s^2 + s^5 \quad p = -s^2 + s^5$$

Omega Rank for B : cycles: {{2, 5, 8}, {1, 4, 6}}, net cycles: 2 . order: 3

\$ [ [2, 1, 0, 2, 1, 1, 0, 1] , [2, 1, 0, 1, 1, 2, 0, 1] , [1, 1, 0, 2, 1, 2, 0, 1] , [2, 1, 0, 2, 1, 1, 0, 1] , [2, 1, 0, 1, 1, 2, 0, 1] , [1, 1, 0, 2, 1, 2, 0, 1] ] \$

$$[5y_3 - y_1 - y_2, y_3, 0, y_1, y_3, y_2, 0, y_3]$$

$$p' = -s + s^4 \quad p = -s + s^4 \quad p' = -s^2 + s^5$$

Â» SYNC'D 525/32768 , 0.01602172852

49 . Coloring, {3, 5, 6}

**R:** [3, 3, 8, 1, 2, 4, 5, 5] **B:** [6, 8, 1, 6, 7, 7, 4, 2]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$\begin{aligned} & [ '-4' (' - 1 + \tau ')'' (' 1 + \tau ')'' (' 5 - 2\tau + \tau^2 ')', 4' (' 5 - \tau + 3\tau^2 + \tau^3 ')'' (' 1 + \tau ')', 12' (' 5 \\ & - 4\tau + 3\tau^2 ')'' (' 1 + \tau ')'^2, 12' (' 5 + 3\tau^2 ')'' (' - 1 + \tau ')'^2, 4' (' 1 + \tau^2 ')'' (' 1 + \tau ')'' (' 5 - 2\tau \\ & + \tau^2 ')', -4' (' - 5 + \tau - \tau^2 + \tau^3 ')'' (' - 1 + \tau ')'^2, -4' (' 5 - 4\tau + 6\tau^2 + \tau^4 ')'' (' - 1 + \tau ')', 4' (' \\ & 1 + \tau ')'' (' 5 + 2\tau^2 + \tau^4 ')'' ]' \end{aligned}$$

For τ=1/2, [102, 258, 270, 46, 255, 37, 73, 267] . FixedPtCheck, [102, 258, 270, 46, 255, 37, 73, 267]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	6 vs 6	2 vs 6

Omega Rank for R : cycles: {{2, 3, 5, 8}}, net cycles: 0 . order: 4

$$[y_1, y_2, y_3, y_4, y_6, 0, 0, y_5]$$

$R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 13/32, -11/32, -3/32, 5/32], [0, 0, 13/32, -11/32, -3/32, 5/32], [0, 0, 5/32, 13/32, -11/32, -3/32], [0, 1, -11/32, -3/32, 5/32, -19/32], [0, 0, -11/32, -3/32, 5/32, 13/32], [1, -1, -3/32, 5/32, -19/32, 21/32], [0, 0, -3/32, 5/32, 13/32, -11/32], [0, 0, -3/32, 5/32, 13/32, -11/32] ] \times \$ [ [1, 1, 2, 1, 2, 0, 0, 1], [1, 2, 2, 0, 1, 0, 0, 2], [0, 1, 3, 0, 2, 0, 0, 2], [0, 2, 1, 0, 2, 0, 0, 3], [0, 2, 2, 0, 3, 0, 0, 1], [0, 3, 2, 0, 1, 0, 0, 2] ] \$$

Omega Rank for B : cycles: {{2, 8}, {4, 6, 7}}, net cycles: 1 . order: 6

$\$ [ [1, 1, 0, 1, 0, 2, 2, 1], [0, 1, 0, 2, 0, 2, 2, 1], [0, 1, 0, 2, 0, 2, 2, 1], [0, 1, 0, 2, 0, 2, 2, 1], [0, 1, 0, 2, 0, 2, 2, 1], [0, 1, 0, 2, 0, 2, 2, 1] ] \$$

$$[2y_2 - y_1, y_2, 0, y_1, 0, 2y_2, 2y_2, y_2]$$

$$p = s^2 - s^6 \quad p' = s^3 - s^5 \quad p'' = s^4 - s^5 \quad p''' = s^2 - s^5$$

Â» SYNC'D 4447/262144 , 0.01696395874

50 . Coloring, {3, 5, 7}

**R:** [3, 3, 8, 1, 2, 7, 4, 5]    **B:** [6, 8, 1, 6, 7, 4, 5, 2]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$[ [ -4' ( ' 1 + \tau ' ) ' ( ' - 1 + \tau ' ) ' ( ' 5 - 2\tau + \tau^2 ' ) ' , 4' ( ' 5 + 2\tau^2 + \tau^4 ' ) ' , 4' ( ' 1 + \tau ' ) ' ( ' 5 - \tau - \tau^2 + \tau^3 ' ) ' , 4' ( ' - 1 + \tau ' ) ' ( ' - 5 + 3\tau - 3\tau^2 + \tau^3 ' ) ' , 4' ( ' 1 + \tau^2 ' ) ' ( ' 5 - 2\tau + \tau^2 ' ) ' , 4' ( ' - 1 + \tau ' ) ' ^2 ( ' 5 + \tau^2 ' ) ' , 4' ( ' - 5 + \tau - \tau^2 + \tau^3 ' ) ' ( ' - 1 + \tau ' ) ' , 4' ( ' 5 + 2\tau + 2\tau^2 - 2\tau^3 + \tau^4 ' ) ' ] ' ] '$

For τ=1/2, [51, 89, 105, 33, 85, 21, 37, 101] . FixedPtCheck, [51, 89, 105, 33, 85, 21, 37, 101]

$$\det(A + \tau \Delta) = 1' ( ' \tau ' ) ' ^2 ( ' 1 + \tau^2 ' ) ' ( ' - 1 + \tau ' ) ' ( ' 1 + \tau ' ) '$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	4 vs 7	2 vs 7

Omega Rank for R : cycles: {{2, 3, 5, 8}}, net cycles: 0 . order: 4



\$ [ [1, 1, 2, 1, 1, 0, 1, 1] , [1, 1, 2, 1, 1, 0, 0, 2] , [1, 1, 2, 0, 2, 0, 0, 2] , [0, 2, 2, 0, 2, 0, 0, 2] , [0, 2, 2, 0, 2, 0, 0, 2] , [0, 2, 2, 0, 2, 0, 0, 2] , [0, 2, 2, 0, 2, 0, 0, 2] ] \$

$$[-y_1 + y_3 + y_4, y_1, y_3 + y_4, y_3 + y_4 - y_2, y_2, 0, y_3, y_4]$$

$$p = -s^4 + s^6 \quad p = -s^4 + s^7 \quad p = -s^4 + s^5$$

Omega Rank for B : cycles: {{2, 8}, {4, 6}, {5, 7}}, net cycles: 2 . order: 2

\$ [ [1, 1, 0, 1, 1, 2, 1, 1] , [0, 1, 0, 2, 1, 2, 1, 1] , [0, 1, 0, 2, 1, 2, 1, 1] , [0, 1, 0, 2, 1, 2, 1, 1] , [0, 1, 0, 2, 1, 2, 1, 1] , [0, 1, 0, 2, 1, 2, 1, 1] , [0, 1, 0, 2, 1, 2, 1, 1] ] \$

$$[-y_1 + 2y_2, y_2, 0, y_1, y_2, 2y_2, y_2, y_2]$$

$$p = s^2 - s^7 \quad p' = s^5 - s^6 \quad p' = s^4 - s^6 \quad p' = s^3 - s^6 \quad p' = s^2 - s^6$$

Â» SYNC'D 285/262144 , 0.001087188721

51 . Coloring, {3, 5, 8}

**R:** [3, 3, 8, 1, 2, 7, 5, 2] **B:** [6, 8, 1, 6, 7, 4, 4, 5]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$\begin{aligned} & [ '-4' (' - 1 + \tau ')'' (' 1 + \tau ')'' (' - 5 - \tau - 3\tau^2 + \tau^3 ')', 4' (' 1 + \tau ')''^3 (' - 5 + \tau^2 ')', 12' (' 1 + \tau ')''^2 (' - 5 - 3\tau - 3\tau^2 + 3\tau^3 ')', 12' (' - 1 + \tau ')''^3 (' 5 + 4\tau + 3\tau^2 ')', -4' (' - 1 + \tau ')'' (' - 5 - \tau + \tau^2 + \tau^3 ')'' (' 1 + \tau ')''^2, -4' (' - 1 + \tau ')''^2 (' 5 + 2\tau^2 + \tau^4 ')', 4' (' - 1 + \tau ')''^2 (' 1 + \tau ')'' (' - 5 - 3\tau - \tau^2 + \tau^3 ')', 4' (' 1 + \tau ')''^3 (' - 5 + \tau - \tau^2 + \tau^3 ')'' ]' \end{aligned}$$

For τ=1/2, [-294, -1026, -990, -62, -369, -89, -159, -999] . FixedPtCheck, [294, 1026, 990, 62, 369, 89, 159, 999]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	5 vs 6	4 vs 6

Omega Rank for R : cycles: {{2, 3, 8}}, net cycles: -1 . order: 3

\$ [ [1, 2, 2, 0, 1, 0, 1, 1] , [0, 2, 3, 0, 1, 0, 0, 2] , [0, 3, 2, 0, 0, 0, 0, 3] , [0, 3, 3, 0, 0, 0, 0, 2] , [0, 2, 3, 0, 0, 0, 0, 3] , [0, 3, 2, 0, 0, 0, 0, 3] ] \$

$$[y_4, y_1, y_2, 0, y_3, 0, y_4, y_5]$$

$$p = -s^3 + s^6$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: -1 . order: 4

$$\$ [ [1, 0, 0, 2, 1, 2, 1, 1], [0, 0, 0, 3, 1, 3, 1, 0], [0, 0, 0, 4, 0, 3, 1, 0], [0, 0, 0, 4, 0, 4, 0, 0], [0, 0, 0, 4, 0, 4, 0, 0], [0, 0, 0, 4, 0, 4, 0, 0] ] \$$$

$$[y_3, 0, 0, -y_1 + y_2 + y_4, y_1, y_2, y_4, y_3]$$

$$p = -s^4 + s^5 \quad p = -s^4 + s^6$$

Â» SYNC'D 855/65536 , 0.01304626465

52 . Coloring, {3, 6, 7}

**R:** [3, 3, 8, 1, 7, 4, 4, 5]    **B:** [6, 8, 1, 6, 2, 7, 5, 2]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ '-12' ('1 + \tau')'' ('5 + 2\tau + 8\tau^2 - 2\tau^3 + 3\tau^4)' , 12' ('-1 + \tau')'' ('1 + 3\tau^2')'' ('5 + 2\tau + \tau^2')' , \\ & 4' ('1 + \tau')'' ('-5 - 2\tau - 12\tau^2 + 2\tau^3 + \tau^4)' , 4' ('1 + \tau')''^2 ('-5 + 3\tau - 7\tau^2 + \tau^3')' , \\ & 4' ('1 + \tau')'' ('-5 + 3\tau - 16\tau^2 + 4\tau^3 - 3\tau^4 + \tau^5)' , 4' ('1 + \tau')'' ('-1 + \tau')'' ('1 + \tau^2')'' ('5 + 2\tau + \tau^2')' , \\ & -4' ('1 + \tau')'' ('5 - 3\tau + 10\tau^2 + 2\tau^3 + \tau^4 + \tau^5)' , -4' ('5 + 2\tau + 19\tau^2 + 7\tau^4 - 2\tau^5 + \tau^6')'' ]' \end{aligned}$$

For τ=1/2, [-762, -350, -834, -738, -687, -375, -609, -713] . FixedPtCheck, [762, 350, 834, 738, 687, 375, 609, 713]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	6 vs 6	5 vs 6

Omega Rank for R : cycles: {{1, 3, 4, 5, 7, 8}}, net cycles: 1 . order: 6

$$[y_1, 0, y_4, y_2, y_3, 0, y_6, y_5]$$

$$\begin{aligned} R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, \\ 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], \\ [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] \end{aligned}$$

0, 0], [0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1]] \$ = \$ [ [7/16, -1/16, 7/16, -1/16, -9/16, -1/16], [7/16, -1/16, 7/16, -1/16, -9/16, -1/16], [-1/16, 7/16, -1/16, 7/16, -1/16, -9/16], [-1/16, 7/16, -1/16, -9/16, -1/16, 7/16], [-1/16, -9/16, -1/16, 7/16, -1/16, 7/16], [7/16, -1/16, -9/16, -1/16, 7/16, -1/16], [7/16, -1/16, -9/16, -1/16, 7/16, -1/16], [-9/16, -1/16, 7/16, -1/16, 7/16, -1/16]] \$ x \$ [ [1, 0, 2, 2, 1, 0, 1, 1], [2, 0, 1, 1, 1, 0, 1, 2], [1, 0, 2, 1, 2, 0, 1, 1], [1, 0, 1, 1, 1, 0, 2, 2], [1, 0, 1, 2, 2, 0, 1, 1], [2, 0, 1, 1, 1, 0, 2, 1]] \$

Omega Rank for B : cycles: {{2, 8}}, net cycles: 0 . order: 6

\$ [ [1, 2, 0, 0, 1, 2, 1, 1], [0, 2, 0, 0, 1, 1, 2, 2], [0, 3, 0, 0, 2, 0, 1, 2], [0, 4, 0, 0, 1, 0, 0, 3], [0, 4, 0, 0, 0, 0, 0, 4], [0, 4, 0, 0, 0, 0, 0, 4]] \$

$[y_1, -y_1 + y_5 + y_4 - y_2 + y_3, 0, 0, y_5, y_4, y_2, y_3]$

$$p = -s^5 + s^6$$

Â» SYNC'D 2665/65536 , 0.04066467285

53 . Coloring, {3, 6, 8}

**R:** [3, 3, 8, 1, 7, 4, 5, 2] **B:** [6, 8, 1, 6, 2, 7, 4, 5]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

' [ '-4' (' 1 + τ ')'' (' - 1 + τ ')'' (' 5 + τ <sup>2</sup> ')', 4' (' 1 + τ <sup>2</sup> ')'' (' 5 + 2τ + τ <sup>2</sup> ')', 4' (' 5 + τ + τ <sup>2</sup> + τ <sup>3</sup> ')'' (' 1 + τ ')', 4' (' - 5 + τ - τ <sup>2</sup> + τ <sup>3</sup> ')'' (' - 1 + τ ')', 4' (' 5 + 2τ <sup>2</sup> + τ <sup>4</sup> ')', 4' (' - 1 + τ ')', 4' (' 5 + 2τ + τ <sup>2</sup> ')', 4' (' 5 - 4τ + 6τ <sup>2</sup> + τ <sup>4</sup> ')', 4' (' 5 + 4τ + 6τ <sup>2</sup> + τ <sup>4</sup> ')'' ]'

For τ=1/2, [63, 125, 141, 37, 89, 25, 73, 137] . FixedPtCheck, [63, 125, 141, 37, 89, 25, 73, 137]

$$\det(A + \tau \Delta) = 1' (' \tau ')'^2 (' 1 + \tau^2 ')'' (' 1 + \tau ')'' (' - 1 + \tau ')'$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	3 vs 7	4 vs 7

Omega Rank for R : cycles: {{5, 7}, {2, 3, 8}}, net cycles: 1 . order: 6

\$ [ [1, 1, 2, 1, 1, 0, 1, 1], [1, 1, 2, 0, 1, 0, 1, 2], [0, 2, 2, 0, 1, 0, 1, 2], [0, 2, 2, 0, 1, 0, 1, 2], [0, 2, 2, 0, 1, 0, 1, 2], [0, 2, 2, 0, 1, 0, 1, 2]] \$

$[y_1, -y_1 + 2y_2, 2y_2, 2y_2 - y_3, y_2, 0, y_2, y_3]$

$$p = -s^3 + s^4 \quad p = -s^3 + s^5 \quad p = -s^3 + s^6 \quad p = -s^3 + s^7$$

Omega Rank for B : cycles: {{4, 6, 7}, {2, 5, 8}}, net cycles: 1 . order: 3

$$\$ [ [1, 1, 0, 1, 1, 2, 1, 1], [0, 1, 0, 1, 1, 2, 2, 1], [0, 1, 0, 2, 1, 1, 2, 1], [0, 1, 0, 2, 1, 2, 1, 1], [0, 1, 0, 1, 1, 2, 2, 1], [0, 1, 0, 2, 1, 1, 2, 1], [0, 1, 0, 2, 1, 2, 1, 1] ] \$$$

$$[y_4, y_3, 0, y_2, y_3, y_1, -y_4 + 5y_3 - y_2 - y_1, y_3]$$

$$p' = -s^3 + s^6 \quad p' = -s^2 + s^5 \quad p' = -s^2 + s^5$$

Â» SYNC'D 3999/262144 , 0.01525497437

54 . Coloring, {3, 7, 8}

**R:** [3, 3, 8, 1, 7, 7, 4, 2]    **B:** [6, 8, 1, 6, 2, 4, 5, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ ' -12' ( ' 1 + \tau ' )'' ( ' 5 + 3\tau^2 ' )'' ( ' - 1 + \tau ' )', 12' ( ' 5 - 2\tau + 8\tau^2 + 2\tau^3 + 3\tau^4 ' )', 4' ( ' 5 - \tau + \\ & 3\tau^2 + \tau^3 ' )'' ( ' 1 + \tau ' )', 4' ( ' - 5 - \tau - 3\tau^2 + \tau^3 ' )'' ( ' - 1 + \tau ' )', -4' ( ' 5 + 2\tau^2 + \tau^4 ' )'' ( ' - 1 + \\ & \tau ' )', 4' ( ' 5 + 3\tau + 3\tau^2 + \tau^3 ' )'' ( ' - 1 + \tau ' )'^2, 4' ( ' 1 + \tau ' )'' ( ' - 5 + \tau - \tau^2 + \tau^3 ' )'' ( ' - 1 + \tau ' \\ & )', -4' ( ' 1 + \tau^2 ' )'' ( ' - 5 - \tau - 3\tau^2 + \tau^3 ' )'' ]' \end{aligned}$$

For τ=1/2, [138, 206, 258, 98, 89, 59, 111, 245] . FixedPtCheck, [138, 206, 258, 98, 89, 59, 111, 245]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	6 vs 6	4 vs 6

Omega Rank for R : cycles: {{2, 3, 8}}, net cycles: 0 . order: 6

$$[y_1, y_2, y_3, y_4, 0, 0, y_5, y_6]$$

$$\begin{aligned} R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, \\ 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0] \\ , [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], \\ [0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, 3/8, -1/8, -1/8], [0, 0, 0, 3/8, -1/8, -1/8] \\ , [0, 0, 0, -1/8, 3/8, -1/8], [0, 0, 1/2, -1/8, -1/8, -1/8], [1/2, -1/4, -1/8, -1/8, 1/8, 0], [1/2, -1/4, -1/8, -1/8, \\ 1/8, 0], [0, 1/2, -1/4, -1/8, -1/8, 1/8], [0, 0, 0, -1/8, -1/8, 3/8] ] \$ \times \$ [ [1, 1, 2, 1, 0, 0, 2, 1], [1, 1, 2, 2, 0, \\ 0, 0, 2], [2, 2, 2, 0, 0, 0, 0, 2], [0, 2, 4, 0, 0, 0, 0, 2], [0, 2, 2, 0, 0, 0, 0, 4], [0, 4, 2, 0, 0, 0, 0, 2] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{2, 5, 8}, {4, 6}}, net cycles: 1 . order: 6

\$ [ [1, 1, 0, 1, 2, 2, 0, 1] , [0, 2, 0, 2, 1, 2, 0, 1] , [0, 1, 0, 2, 1, 2, 0, 2] , [0, 1, 0, 2, 2, 2, 0, 1] , [0, 2, 0, 2, 1, 2, 0, 1] , [0, 1, 0, 2, 1, 2, 0, 2] ] \$

$$[-y_1 + y_3, 2y_3 - y_2 - y_4, 0, y_1, y_2, y_3, 0, y_4]$$

$$p = -s^2 + s^5 \quad p' = -s^2 + s^5$$

Â» SYNC'D 1665/32768 , 0.05081176758

55 . Coloring, {4, 5, 6}

**R**: [3, 3, 1, 6, 2, 4, 5, 5]    **B**: [6, 8, 8, 1, 7, 7, 4, 2]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

[ '12' ('5 + τ + 7τ<sup>2</sup> + 3τ<sup>3</sup> ')', -12' ('-1 + τ ')'' ('5 + 4τ + 3τ<sup>2</sup> ')', 12' ('5 + 3τ<sup>2</sup> ')'' ('1 + τ ')', 12' ('5 - 3τ + 3τ<sup>2</sup> + 3τ<sup>3</sup> ')', -4' ('5 + τ ')'' ('-1 + τ ')'' ('1 + τ ')', 4' ('5 - τ + 3τ<sup>2</sup> + τ<sup>3</sup> ')', 4' ('-1 + τ ')'' ('-5 + τ<sup>2</sup> ')', -4' ('-1 + τ ')'' ('5 + 2τ + τ<sup>2</sup> ')'' ]'

For τ=1/2, [61, 31, 69, 37, 33, 43, 19, 25] . FixedPtCheck, [61, 31, 69, 37, 33, 43, 19, 25]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	4 vs 6	4 vs 6

Omega Rank for R : cycles: {{1, 3}, {4, 6}}, net cycles: 1 . order: 4

\$ [ [1, 1, 2, 1, 2, 1, 0, 0] , [2, 2, 2, 1, 0, 1, 0, 0] , [2, 0, 4, 1, 0, 1, 0, 0] , [4, 0, 2, 1, 0, 1, 0, 0] , [2, 0, 4, 1, 0, 1, 0, 0] , [4, 0, 2, 1, 0, 1, 0, 0] ] \$

$$[-y_1 - y_2 - y_3 + 6y_4, y_1, y_2, y_4, y_3, y_4, 0, 0]$$

$$p' = s^3 - s^5 \quad p = -s^3 + s^5$$

Omega Rank for B : cycles: {{1, 4, 6, 7}, {2, 8}}, net cycles: 2 . order: 4

\$ [ [1, 1, 0, 1, 0, 1, 2, 2] , [1, 2, 0, 2, 0, 1, 1, 1] , [2, 1, 0, 1, 0, 1, 1, 2] , [1, 2, 0, 1, 0, 2, 1, 1] , [1, 1, 0, 1, 0, 1, 2, 2] , [1, 2, 0, 2, 0, 1, 1, 1] ] \$

$$[y_2, 3y_2 + 3y_4 - 4y_3, 0, y_1, 0, 4y_2 - y_1 + 4y_4 - 5y_3, y_4, y_3]$$

$$p' = s - s^5 \quad p = s - s^5$$

Â» SYNC'D 179/16384 , 0.01092529297

56 . Coloring, {4, 5, 7}

**R:** [3, 3, 1, 6, 2, 7, 4, 5]   **B:** [6, 8, 8, 1, 7, 4, 5, 2]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ ' 4 ' ( ' 5 + \tau + \tau^2 + \tau^3 ' ) , -4 ' ( ' 5 + \tau^2 ' ) ' ( ' -1 + \tau ' ) , 4 ' ( ' 1 + \tau ' ) ' ( ' 5 - 2\tau + \tau^2 ' ) , 4 ' \\ & ( ' 5 - \tau - \tau^2 + \tau^3 ' ) , -4 ' ( ' 5 + \tau ' ) ' ( ' -1 + \tau ' ) , -4 ' ( ' -5 + \tau^2 ' ) , 4 ' ( ' 5 - 2\tau + \tau^2 ' ) , 4 ' ( ' -5 \\ & + \tau ' ) ' ( ' -1 + \tau ' ) ' ] \end{aligned}$$

For τ=1/2, [47, 21, 51, 35, 22, 38, 34, 18] . FixedPtCheck, [47, 21, 51, 35, 22, 38, 34, 18]

$$\det(A + \tau \Delta) = 1 ' ( ' 1 + \tau ' ) ' ( ' \tau ' ) ^2 ' ( ' 1 + \tau^2 ' ) ' ( ' -1 + \tau ' ) '$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	4 vs 7	2 vs 7

Omega Rank for R : cycles: {{1, 3}, {4, 6, 7}}, net cycles: 1 . order: 6

$$\$ [ [1, 1, 2, 1, 1, 1, 1, 0], [2, 1, 2, 1, 0, 1, 1, 0], [2, 0, 3, 1, 0, 1, 1, 0], [3, 0, 2, 1, 0, 1, 1, 0], [2, 0, 3, 1, 0, 1, 1, 0], [3, 0, 2, 1, 0, 1, 1, 0], [2, 0, 3, 1, 0, 1, 1, 0] ] \$$$

$$[-y_1 - y_2 + 5y_4 - y_3, y_1, y_2, y_4, y_3, y_4, y_4, 0]$$

$$p = -s^3 + s^7 \quad p = -s^3 + s^5 \quad p' = -s^3 + s^5$$

Omega Rank for B : cycles: {{2, 8}, {1, 4, 6}, {5, 7}}, net cycles: 3 . order: 6

$$\$ [ [1, 1, 0, 1, 1, 1, 1, 2], [1, 2, 0, 1, 1, 1, 1, 1], [1, 1, 0, 1, 1, 1, 1, 2], [1, 2, 0, 1, 1, 1, 1, 1], [1, 1, 0, 1, 1, 1, 1, 2], [1, 2, 0, 1, 1, 1, 1, 1], [1, 1, 0, 1, 1, 1, 1, 2] ] \$$$

$$[y_1, 3y_1 - y_2, 0, y_1, y_1, y_1, y_1, y_2]$$

$$p' = -s + s^5 \quad p = -s + s^7 \quad p' = -s + s^3 \quad p = -s + s^5 \quad p = -s + s^3$$

Â» SYNC'D 2469/262144 , 0.009418487549

57 . Coloring, {4, 5, 8}

**R:** [3, 3, 1, 6, 2, 7, 5, 2]   **B:** [6, 8, 8, 1, 7, 4, 4, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ '12' ( '1 + 3\tau^2' ) ' ( '5 - \tau + 3\tau^2 + \tau^3' ) ' , -12' ( '5 + 3\tau^2' ) ' ( ' - 1 + \tau ' ) ' ( '1 + \tau ' ) ' ^2 , \\ & 12' ( '1 + \tau ' ) ' ( '5 + 2\tau + 8\tau^2 - 2\tau^3 + 3\tau^4' ) ' , 12' ( '5 + \tau + 7\tau^2 + 3\tau^3' ) ' ( ' - 1 + \tau ' ) ' ^2 , -4' ( ' - \\ & 1 + \tau ' ) ' ( '1 + \tau ' ) ' ( '5 - \tau + 3\tau^2 + \tau^3' ) ' , -4' ( ' - 1 + \tau ' ) ' ( '5 + 10\tau^2 + \tau^4' ) ' , 4' ( ' - 5 + 3\tau - \\ & 7\tau^2 + \tau^3' ) ' ( ' - 1 + \tau ' ) ' ( '1 + \tau ' ) ' , 4' ( ' - 5 - \tau - 3\tau^2 + \tau^3' ) ' ( ' - 1 + \tau ' ) ' ( '1 + \tau ' ) ' ] \end{aligned}$$

For τ=1/2, [301, 207, 381, 61, 129, 121, 123, 147] . FixedPtCheck, [301, 207, 381, 61, 129, 121, 123, 147]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	5 vs 6	6 vs 6

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 6

$$\$ [ [1, 2, 2, 0, 1, 1, 1, 0], [2, 1, 3, 0, 1, 0, 1, 0], [3, 1, 3, 0, 1, 0, 0, 0], [3, 1, 4, 0, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0] ] \$$$

$$[-y_1 + y_2 + y_3 + y_4 - y_5, y_1, y_2, 0, y_3, y_4, y_5, 0]$$

$$p = -s^5 + s^6$$

Omega Rank for B : cycles: {{1, 4, 6}}, net cycles: 0 . order: 6

$$[y_1, 0, 0, y_2, y_3, y_4, y_5, y_6]$$

$$\begin{aligned} B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], \\ [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], \\ [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, -1/8, -1/8, 3/8], [1/2, -1/4, -1/8, -1/8, 1/8, 0], [1/2, -1/4, -1/8, -1/8, 1/8, 0], \\ [0, 0, 0, -1/8, 3/8, -1/8], [0, 0, 1/2, -1/8, -1/8, -1/8], [0, 0, 0, 3/8, -1/8, -1/8], [0, 0, 0, 3/8, -1/8, -1/8], [0, 1/2, -1/4, -1/8, -1/8, 1/8] ] \$ \times \$ [ [1, 0, 0, 2, 1, 1, 1, 2], [2, 0, 0, 2, 2, 1, 1, 0], \\ [2, 0, 0, 2, 0, 2, 2, 0], [2, 0, 0, 4, 0, 2, 0, 0], [4, 0, 0, 2, 0, 2, 0, 0], [2, 0, 0, 2, 0, 4, 0, 0] ] \$ \end{aligned}$$

Â» SYNC'D 555/8192 , 0.06774902344

58 . Coloring, {4, 6, 7}

**R:** [3, 3, 1, 6, 7, 4, 4, 5]    **B:** [6, 8, 8, 1, 2, 7, 5, 2]

' See graph

' ' See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

' [ '4' (' - 5 +  $\tau^2$  ')'' (' 1 +  $\tau$  ')'^ 2 , -4' (' - 1 +  $\tau$  ')'^ 2 ' (' 5 + 2 $\tau$  +  $\tau^2$  ')'^ , 4' (' - 5 -  $\tau$  - 3 $\tau^2$  +  $\tau^3$  ')'' (' 1 +  $\tau$  ')'^ , 4' (' 1 +  $\tau$  ')'^ 3 ' (' - 5 +  $\tau$  ')'^ , -4' (' - 5 +  $\tau^2$  ')'' (' 1 +  $\tau$  ')'' (' - 1 +  $\tau$  ')'^ , -4' (' 1 +  $\tau$  ')'^ 2 ' (' 5 + 2 $\tau$  +  $\tau^2$  ')'^ , 4' (' 5 +  $\tau$  ')'' (' 1 +  $\tau$  ')'^ 2 ' (' - 1 +  $\tau$  ')'^ , 4' (' 5 -  $\tau$  + 3 $\tau^2$  +  $\tau^3$  ')'' (' - 1 +  $\tau$  ')'^ ]'

For  $\tau=1/2$ , [-171, -25, -147, -243, -57, -225, -99, -43] . FixedPtCheck, [171, 25, 147, 243, 57, 225, 99, 43]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	4 vs 6	5 vs 6

Omega Rank for R : cycles: {{1, 3}, {4, 6}}, net cycles: 1 . order: 4

\$ [ [1, 0, 2, 2, 1, 1, 1, 0] , [2, 0, 1, 2, 0, 2, 1, 0] , [1, 0, 2, 3, 0, 2, 0, 0] , [2, 0, 1, 2, 0, 3, 0, 0] , [1, 0, 2, 3, 0, 2, 0, 0] , [2, 0, 1, 2, 0, 3, 0, 0] ] \$

$[y_1, 0, -4y_1 + 3y_3 + 3y_2, y_4, -5y_1 - y_4 + 4y_3 + 4y_2, y_3, y_2, 0]$

$p' = s^3 - s^5$      $p = s^3 - s^5$

Omega Rank for B : cycles: {{2, 8}}, net cycles: 0 . order: 6

\$ [ [1, 2, 0, 0, 1, 1, 1, 2] , [0, 3, 0, 0, 1, 1, 1, 2] , [0, 3, 0, 0, 1, 0, 1, 3] , [0, 4, 0, 0, 1, 0, 0, 3] , [0, 4, 0, 0, 0, 0, 0, 4] , [0, 4, 0, 0, 0, 0, 0, 4] ] \$

$[-y_1 + y_2 + y_3 - y_4 + y_5, y_1, 0, 0, y_2, y_3, y_4, y_5]$

$p = -s^5 + s^6$

Â» SYNC'D 463/65536 , 0.007064819336



59 . Coloring, {4, 6, 8}

**R:** [3, 3, 1, 6, 7, 4, 5, 2]    **B:** [6, 8, 8, 1, 2, 7, 4, 5]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

‘ [ ‘4‘ (‘5 + 3 $\tau$  + 3 $\tau^2$  +  $\tau^3$  ‘)‘ , -4‘ (‘- 1 +  $\tau$  ‘)‘ (‘5 + 2 $\tau$  +  $\tau^2$  ‘)‘ , 4‘ (‘1 +  $\tau$  ‘)‘ (‘5 +  $\tau^2$  ‘)‘ ,  
4‘ (‘5 +  $\tau$  +  $\tau^2$  +  $\tau^3$  ‘)‘ , 4‘ (‘5 - 2 $\tau$  +  $\tau^2$  ‘)‘ , 4‘ (‘5 + 2 $\tau$  +  $\tau^2$  ‘)‘ , -4‘ (‘- 5 +  $\tau^2$  ‘)‘ , -4‘ (‘5 +  $\tau$  ‘  
)‘ (‘- 1 +  $\tau$  ‘)‘ ]‘

For  $\tau=1/2$ , [59, 25, 63, 47, 34, 50, 38, 22] . FixedPtCheck, [59, 25, 63, 47, 34, 50, 38, 22]

$\det(A + \tau \Delta) = 1 \cdot (-1 + \tau) \cdot (\tau)^2 \cdot (1 + \tau^2) \cdot (1 + \tau)$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	2 vs 7	3 vs 7

Omega Rank for R : cycles: {{1, 3}, {4, 6}, {5, 7}}, net cycles: 2 . order: 2

\$ [ [1, 1, 2, 1, 1, 1, 1, 0] , [2, 0, 2, 1, 1, 1, 1, 0] , [2, 0, 2, 1, 1, 1, 1, 0] , [2, 0, 2, 1, 1, 1, 1, 0] , [2, 0, 2, 1, 1, 1, 1, 0] , [2, 0, 2, 1, 1, 1, 1, 0] , [2, 0, 2, 1, 1, 1, 1, 0] ] \$

$[-y_2 + 2y_1, y_2, 2y_1, y_1, y_1, y_1, y_1, 0]$

$p = -s^2 + s^5$     $p = -s^2 + s^3$     $p = -s^2 + s^4$     $p = -s^2 + s^6$     $p = -s^2 + s^7$

Omega Rank for B : cycles: {{1, 4, 6, 7}, {2, 5, 8}}, net cycles: 2 .

\$ [ [1, 1, 0, 1, 1, 1, 1, 2] , [1, 1, 0, 1, 2, 1, 1, 1] , [1, 2, 0, 1, 1, 1, 1, 1] , [1, 1, 0, 1, 1, 1, 1, 2] , [1, 1, 0, 1, 2, 1, 1, 1] , [1, 2, 0, 1, 1, 1, 1, 1] , [1, 1, 0, 1, 1, 1, 1, 2] ] \$

$[y_2, 4y_2 - y_1 - y_3, 0, y_2, y_1, y_2, y_2, y_3]$

$p = -s + s^4$     $p' = -s + s^4$     $p' = -s^2 + s^5$     $p = -s + s^7$

Â» SYNC'D 145/131072 , 0.001106262207

60 . Coloring, {4, 7, 8}

**R:** [3, 3, 1, 6, 7, 7, 4, 2]    **B:** [6, 8, 8, 1, 2, 4, 5, 5]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

‘ [ ‘  $4^{\prime} (1 + \tau^{\prime})^{\prime}$  ,  $-4^{\prime} (-1 + \tau^{\prime})^{\prime}$  ,  $4^{\prime} (1 + \tau^{\prime})^{\prime}$  ,  $4^{\prime} (1 + \tau^{\prime})^{\prime}$  ,  $-4^{\prime} (-1 + \tau^{\prime})^{\prime}$  ,  $4^{\prime} (1 + \tau^{\prime})^{\prime}$  ,  $4^{\prime} (1 + \tau^{\prime})^{\prime}$  ,  $-4^{\prime} (-1 + \tau^{\prime})^{\prime}$  ] ‘

For  $\tau=1/2$ , [3, 1, 3, 3, 1, 3, 3, 1] . FixedPtCheck, [3, 1, 3, 3, 1, 3, 3, 1]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	4 vs 6	3 vs 6

Omega Rank for **R** : cycles: {{1, 3}, {4, 6, 7}}, net cycles: 1 . order: 6

\$ [ [1, 1, 2, 1, 0, 1, 2, 0] , [2, 0, 2, 2, 0, 1, 1, 0] , [2, 0, 2, 1, 0, 2, 1, 0] , [2, 0, 2, 1, 0, 1, 2, 0] , [2, 0, 2, 2, 0, 1, 1, 0] , [2, 0, 2, 1, 0, 2, 1, 0] ] \$

$[y_3, y_4, y_3 + y_4, 2y_3 + 2y_4 - y_1 - y_2, 0, y_1, y_2, 0]$

$p = -s^2 + s^5$      $p' = -s^2 + s^5$

Omega Rank for **B** : cycles: {{2, 5, 8}, {1, 4, 6}}, net cycles: 2 . order: 3

\$ [ [1, 1, 0, 1, 2, 1, 0, 2] , [1, 2, 0, 1, 2, 1, 0, 1] , [1, 2, 0, 1, 1, 1, 0, 2] , [1, 1, 0, 1, 2, 1, 0, 2] , [1, 2, 0, 1, 2, 1, 0, 1] , [1, 2, 0, 1, 1, 1, 0, 2] ] \$

$[y_3, y_1, 0, y_3, 5y_3 - y_1 - y_2, y_3, 0, y_2]$

$p' = -s + s^4$      $p' = s^2 - s^5$      $p = -s + s^4$

$\hat{A} \gg \text{SYNC'D } 525/32768 , 0.01602172852$

61 . Coloring, {5, 6, 7}

**R:** [3, 3, 1, 1, 2, 4, 4, 5]    **B:** [6, 8, 8, 6, 7, 7, 5, 2]

‘ See graph

‘ ‘ See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ 4(1+\tau)^2(5-2\tau+\tau^2), -4(5-\tau+3\tau^2+\tau^3)(-1+\tau), -4(1+\tau)(-5-\tau-3\tau^2+\tau^3), \\ & 4(-5+\tau^2)(1+\tau)(-1+\tau), 4(-5+2\tau-4\tau^2-2\tau^3+\tau^4)(-1+\tau), \\ & -4(1+\tau)(5-\tau-\tau^2+\tau^3)(-1+\tau), 4(-1+\tau)^2(5+\tau+\tau^2+\tau^3), \\ & 4(1+\tau^2)(-5+\tau^2)(-1+\tau) ] \end{aligned}$$

For  $\tau=1/2$ , [306, 86, 294, 114, 83, 105, 47, 95] . FixedPtCheck, [306, 86, 294, 114, 83, 105, 47, 95]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	4 vs 5	3 vs 5

Omega Rank for R : cycles: {{1, 3}}, net cycles: -1 . order: 4

$$\$ [ [2, 1, 2, 2, 1, 0, 0, 0], [4, 1, 3, 0, 0, 0, 0, 0], [3, 0, 5, 0, 0, 0, 0, 0], [5, 0, 3, 0, 0, 0, 0, 0], [3, 0, 5, 0, 0, 0, 0, 0] ] \$$$

$$[y_1, y_2, y_3, 2y_4, y_4, 0, 0, 0]$$

$$p = -s^3 + s^5$$

Omega Rank for B : cycles: {{2, 8}, {5, 7}}, net cycles: 1 . order: 2

$$\$ [ [0, 1, 0, 0, 1, 2, 2, 2], [0, 2, 0, 0, 2, 0, 3, 1], [0, 1, 0, 0, 3, 0, 2, 2], [0, 2, 0, 0, 2, 0, 3, 1], [0, 1, 0, 0, 3, 0, 2, 2] ] \$$$

$$[0, y_1, 0, 0, -5y_1 - y_2 + 4y_3, y_2, y_3, -4y_1 + 3y_3]$$

$$p = s^2 - s^4 \quad p' = s^2 - s^4$$

Â» SYNC'D 9/256 , 0.03515625000

62 . Coloring, {5, 6, 8}

**R:** [3, 3, 1, 1, 2, 4, 5, 2] **B:** [6, 8, 8, 6, 7, 7, 4, 5]

' See graph

' ' See pair graph

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ -4' ( ' 1 + \tau ' )'' ( ' 5 + \tau + 10\tau^2 - 2\tau^3 + \tau^4 + \tau^5 ' )', 4' ( ' 1 + \tau ' )'^2 ( ' - 1 + \tau ' )'' ( ' 5 - \tau - \tau \\ & 2 + \tau^3 ' )', -4' ( ' 1 + \tau ' )'^2 ( ' 5 + 2\tau^2 + \tau^4 ' )', 4' ( ' - 1 + \tau ' )'' ( ' 5 - 3\tau + 10\tau^2 + 2\tau^3 + \tau^4 + \tau \\ & 5 ' )', -4' ( ' 1 + \tau ' )'' ( ' 5 + \tau + \tau^2 + \tau^3 ' )'' ( ' - 1 + \tau ' )'^2, 4' ( ' 5 - \tau + 12\tau^2 - \tau^4 + \tau^5 ' )'' ( ' - 1 + \\ & \tau ' )', 4' ( ' 1 + \tau^2 ' )'' ( ' - 5 + \tau^2 ' )'' ( ' - 1 + \tau ' )'^2, 4' ( ' 5 - 3\tau + \tau^2 + \tau^3 ' )'' ( ' 1 + \tau ' )'^2 ( ' - 1 \\ & + \tau ' )' ]' \end{aligned}$$

For  $\tau=1/2$ , [-753, -315, -801, -203, -141, -239, -95, -279] . FixedPtCheck, [753, 315, 801, 203, 141, 239, 95, 279]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	3 vs 5	5 vs 5

Omega Rank for R : cycles: {{1, 3}}, net cycles: -1 . order: 4

$$\$ [ [2, 2, 2, 1, 1, 0, 0, 0], [3, 1, 4, 0, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0], [0, 0, 0] ] \$$$

$$[-y_1 + y_2 + 2 y_3, y_1, y_2, y_3, 0, 0, 0]$$

$$p = -s^3 + s^4 \quad p = -s^3 + s^5$$

Omega Rank for B : cycles: {{4, 6, 7}}, net cycles: 0 . order: 3

$$[0, 0, 0, y_5, y_4, y_3, y_2, y_1]$$

$$\begin{aligned} B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], \\ [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], \\ [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], \\ [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, -5/8, 3/8, 3/8], [1/2, -1/4, 3/8, -1/8, -3/8], [1/2, -1/4, 3/8, -1/8, -3/8], \\ [0, 0, -5/8, 3/8, 3/8], [0, 0, 3/8, -5/8, 3/8], [0, 0, 3/8, -5/8, 3/8], [0, 0, 3/8, 3/8, -5/8], [0, 1/2, -5/8, 3/8, -1/8] ] \$ \times \$ [ [0, 0, 0, 1, 1, 2, 2, 2], [0, 0, 0, 2, 2, 1, 3, 0], [0, 0, 0, 3, 0, 2, 3, 0], \\ [0, 0, 0, 3, 0, 3, 2, 0], [0, 0, 0, 2, 0, 3, 3, 0] ] \$ \end{aligned}$$

$\hat{A}$ » SYNC'D 3/64 , 0.04687500000

63 . Coloring, {5, 7, 8}

**R**: [3, 3, 1, 1, 2, 7, 4, 2]    **B**: [6, 8, 8, 6, 7, 4, 5, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{bmatrix} -4(-1+\tau)^2(-5-\tau-3\tau^2+\tau^3), & 4(-1+\tau)^2(-5+\tau^2), & 4(-1+\tau)^2(5-2\tau+\tau^2), & -4(-1+\tau)^2(5-\tau+3\tau^2+\tau^3), & 4(-1+\tau)^2(5+\tau+\tau^2+\tau^3), & 4(-1+\tau)^2(-5+\tau^2), & 4(-1+\tau)^2(-5+2\tau-4\tau^2-2\tau^3+\tau^4), & -4(-1+\tau)^2(5-\tau-\tau^2+\tau^3), & 4(-1+\tau)^2 \end{bmatrix}$$

For τ=1/2, [294, 114, 306, 86, 47, 95, 83, 105] . FixedPtCheck, [294, 114, 306, 86, 47, 95, 83, 105]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	4 vs 5	3 vs 5

Omega Rank for R : cycles: {{1, 3}}, net cycles: -1 . order: 4

$$\$ [ [2, 2, 2, 1, 0, 0, 1, 0], [3, 0, 4, 1, 0, 0, 0, 0], [5, 0, 3, 0, 0, 0, 0, 0], [3, 0, 5, 0, 0, 0, 0, 0], [5, 0, 3, 0, 0, 0, 0, 0] ] \$$$

$$[y_1, 2y_4, y_2, y_3, 0, 0, y_4, 0]$$

$$p = -s^3 + s^5$$

Omega Rank for B : cycles: {{4, 6}, {5, 7}}, net cycles: 1 . order: 2

$$\$ [ [0, 0, 0, 1, 2, 2, 1, 2], [0, 0, 0, 2, 3, 1, 2, 0], [0, 0, 0, 1, 2, 2, 3, 0], [0, 0, 0, 2, 3, 1, 2, 0], [0, 0, 0, 1, 2, 2, 3, 0] ] \$$$

$$[0, 0, 0, y_1, y_2, -4y_1 + 3y_2, -5y_1 + 4y_2 - y_3, y_3]$$

$$p' = -s^2 + s^4 \quad p = -s^2 + s^4$$

Â» SYNC'D 9/256 , 0.03515625000

64 . Coloring, {6, 7, 8}

**R:** [3, 3, 1, 1, 7, 4, 4, 2] **B:** [6, 8, 8, 6, 2, 7, 5, 5]

' See graph

' ' See pair graph

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ -4' ( ' 1 + \tau ' ) ^ { 2 } ( ' 5 + 2\tau ^ 2 + \tau ^ 4 ' ) , 4' ( ' 5 - 3\tau + 10\tau ^ 2 + 2\tau ^ 3 + \tau ^ 4 + \tau ^ 5 ' ) ^ { 2 } ( ' - 1 + \tau ' ) ^ { 2 } , \\ & -4' ( ' 1 + \tau ' ) ^ { 2 } ( ' 5 + \tau + 10\tau ^ 2 - 2\tau ^ 3 + \tau ^ 4 + \tau ^ 5 ' ) , 4' ( ' 1 + \tau ' ) ^ { 2 } ( ' 5 - \tau - \tau ^ 2 + \tau ^ 3 ' ) ^ { 2 } ( ' - 1 + \tau ' ) ^ { 2 } , \\ & 4' ( ' 1 + \tau ^ 2 ' ) ( ' - 5 + \tau ^ 2 ' ) ( ' - 1 + \tau ' ) ^ { 2 } , 4' ( ' 1 + \tau ' ) ^ { 2 } ( ' 5 - 3\tau + \tau ^ 2 + \tau ^ 3 ' ) ^ { 2 } ( ' - 1 + \tau ' ) ^ { 2 } , \\ & -4' ( ' 5 + \tau + \tau ^ 2 + \tau ^ 3 ' ) ( ' 1 + \tau ' ) ( ' - 1 + \tau ' ) ^ { 2 } , 4' ( ' - 1 + \tau ' ) ( ' 5 - \tau + 12\tau ^ 2 - \tau ^ 4 + \tau ^ 5 ' ) ] \end{aligned}$$

For  $\tau=1/2$ , [-801, -203, -753, -315, -95, -279, -141, -239] . FixedPtCheck, [801, 203, 753, 315, 95, 279, 141, 239]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	3 vs 5	5 vs 5

Omega Rank for R : cycles: {{1, 3}}, net cycles: -1 . order: 4

$$\$ [ [ 2, 1, 2, 2, 0, 0, 1, 0 ] , [ 4, 0, 3, 1, 0, 0, 0, 0 ] , [ 4, 0, 4, 0, 0, 0, 0, 0 ] , [ 4, 0, 4, 0, 0, 0, 0, 0 ] , [ 4, 0, 4, 0, 0, 0, 0, 0 ] , [ 4, 0, 4, 0, 0, 0, 0, 0 ] ] \$$$

$$[-2 y_3 + y_1 + y_2, y_3, y_1, y_2, 0, 0, y_3, 0]$$

$$p = -s^3 + s^4 \quad p = -s^3 + s^5$$

Omega Rank for B : cycles: {{2, 5, 8}}, net cycles: 0 . order: 3

$$[0, y_1, 0, 0, y_2, y_3, y_4, y_5]$$

$$\begin{aligned} B = \$ [ [ 0, 0, 0, 0, 0, 1, 0, 0 ] , [ 0, 0, 0, 0, 0, 0, 0, 1 ] , [ 0, 0, 0, 0, 0, 0, 0, 1 ] , [ 0, 0, 0, 0, 0, 1, 0, 0 ] , [ 0, 1, 0, 0, 0, 0, 0, 0 ] , [ 0, 0, 0, 0, 0, 0, 1, 0 ] , [ 0, 0, 0, 0, 1, 0, 0, 0 ] , [ 0, 0, 0, 0, 1, 0, 0, 0 ] ] \$ \times \$ [ [ 0, 0, 0, 0, 0, 0, 0, 0 ] , [ 0, 1, 0, 0, 0, 0, 0, 0 ] , [ 0, 0, 0, 0, 0, 0, 0, 0 ] , [ 0, 0, 0, 0, 0, 0, 0, 0 ] , [ 0, 0, 0, 0, 1, 0, 0, 0 ] , [ 0, 0, 0, 0, 0, 1, 0, 0 ] , [ 0, 0, 0, 0, 0, 0, 1, 0 ] , [ 0, 0, 0, 0, 0, 0, 0, 1 ] ] \$ = \$ [ [ 1/2, -1/4, 3/8, -1/8, -3/8 ] , [ 0, 0, -5/8, 3/8, 3/8 ] , [ 0, 0, -5/8, 3/8, 3/8 ] , [ 1/2, -1/4, 3/8, -1/8, -3/8 ] , [ 0, 0, 3/8, 3/8, -5/8 ] , [ 0, 1/2, -5/8, 3/8, -1/8 ] , [ 0, 0, 3/8, -5/8, 3/8 ] , [ 0, 0, 3/8, -5/8, 3/8 ] ] \$ \times \$ [ [ 0, 1, 0, 0, 2, 2, 1, 2 ] , [ 0, 2, 0, 0, 3, 0, 2, 1 ] , [ 0, 3, 0, 0, 3, 0, 0, 2 ] , [ 0, 3, 0, 0, 2, 0, 0, 3 ] , [ 0, 2, 0, 0, 3, 0, 0, 3 ] ] \$ \end{aligned}$$

$\hat{A}$ » SYNC'D 3/64 , 0.04687500000

65 . Coloring, {2, 3, 4, 5}

**R**: [3, 8, 8, 6, 2, 7, 5, 5]   **B**: [6, 3, 1, 1, 7, 4, 4, 2]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$[ 2^{(-1+\tau)^2}, 2^{(1+\tau)^2}, -2^{(1+\tau)^2}(-1+\tau)^2, 2^{(-1+\tau)^2}, 2^{(1+\tau)^2} ]^2, 2^{(-1+\tau)^2}, -2^{(1+\tau)^2}(-1+\tau)^2, 2^{(1+\tau)^2} ]^2$$

For τ=1/2, [1, 9, 3, 1, 9, 1, 3, 9] . FixedPtCheck, [1, 9, 3, 1, 9, 1, 3, 9]

$$\det(A + \tau \Delta) = 1^{(-1+\tau)^2} (\tau)^2 (-1+\tau)^2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	5 vs 6	5 vs 6

Omega Rank for R : cycles: {{2, 5, 8}}, net cycles: -1 . order: 3

$$\$ [ [0, 1, 1, 0, 2, 1, 1, 2], [0, 2, 0, 0, 3, 0, 1, 2], [0, 3, 0, 0, 3, 0, 0, 2], [0, 3, 0, 0, 2, 0, 0, 3], [0, 2, 0, 0, 3, 0, 0, 3], [0, 3, 0, 0, 3, 0, 0, 2] ] \$$$

$$[0, y_1, y_3, 0, y_2, y_3, y_4, y_5]$$

$$p = -s^3 + s^6$$

Omega Rank for B : cycles: {{1, 4, 6}}, net cycles: -1 . order: 3

$$\$ [ [2, 1, 1, 2, 0, 1, 1, 0], [3, 0, 1, 2, 0, 2, 0, 0], [3, 0, 0, 2, 0, 3, 0, 0], [2, 0, 0, 3, 0, 3, 0, 0], [3, 0, 0, 3, 0, 2, 0, 0], [3, 0, 0, 2, 0, 3, 0, 0] ] \$$$

$$[y_4, y_5, y_3, y_2, 0, y_1, y_5, 0]$$

$$p = -s^3 + s^6$$

Â» SYNC'D 1269/32768 , 0.03872680664

66 . Coloring, {2, 3, 4, 6}

**R:** [3, 8, 8, 6, 7, 4, 5, 5] **B:** [6, 3, 1, 1, 2, 7, 4, 2]

‘ See graph

‘ ‘ See pair graph

Ω for A+τΔ :

$$[ -4^{(-1+\tau)^2} (5-\tau+3\tau^2+\tau^3)^2, -4^{(1+\tau)^2} (-1+\tau)^2 (5+2\tau+\tau^2)^2, -4^{(1+\tau)^2} (-1+\tau)^2 (5-2\tau+\tau^2)^2, 4^{(5+10\tau^2+\tau^4)^2}, 4^{(1+\tau)^2} (5+4\tau+6\tau^2+\tau^4)^2, 4^{(5-3\tau+10\tau^2+2\tau^3+\tau^4+\tau^5)^2}, 4^{(5+3\tau+16\tau^2+4\tau^3+3\tau^4+\tau^5)^2}, -4^{(1+\tau)^2} ]^2$$

$$+ \tau')^2 (-1 + \tau')^2 (5 + \tau^2)'$$

For  $\tau=1/2$ , [86, 150, 102, 242, 411, 203, 359, 189]. FixedPtCheck, [86, 150, 102, 242, 411, 203, 359, 189]

$$\det(A + \tau \Delta) = 1' (1 + \tau')^2 (\tau')^2 (-1 + \tau')^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	3 vs 6	6 vs 6

Omega Rank for R : cycles: {{4, 6}, {5, 7}}, net cycles: 1 . order: 4

$$\$ [ [0, 0, 1, 1, 2, 1, 1, 2], [0, 0, 0, 1, 3, 1, 2, 1], [0, 0, 0, 1, 3, 1, 3, 0], [0, 0, 0, 1, 3, 1, 3, 0], [0, 0, 0, 1, 3, 1, 3, 0], [0, 0, 0, 1, 3, 1, 3, 0] ] \$$$

$$[0, 0, 3y_3 - y_2, y_3, y_2, y_3, 3y_3 - y_1, y_1]$$

$$p = -s^3 + s^5 \quad p = -s^3 + s^6 \quad p = -s^3 + s^4$$

Omega Rank for B : cycles: {{1, 4, 6, 7}}, net cycles: 0 . order: 4

$$[y_1, y_2, y_3, y_4, 0, y_5, y_6, 0]$$

$$\begin{aligned} \mathbf{B} = \$ [ [0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 0, 5/32, 13/32, -11/32, -3/32], [0, 1/2, -11/32, -3/32, 5/32, -3/32], [0, 0, 13/32, -11/32, -3/32, 5/32], [0, 0, 13/32, -11/32, -3/32, 5/32], [1/2, -1/4, -3/32, 5/32, -3/32, -3/32], [0, 0, -3/32, 5/32, 13/32, -11/32], [0, 0, -11/32, -3/32, 5/32, 13/32], [1/2, -1/4, -3/32, 5/32, -3/32, -3/32] ] \$ \times \$ [ [2, 2, 1, 1, 0, 1, 1, 0], [2, 0, 2, 1, 0, 2, 1, 0], [3, 0, 0, 1, 0, 2, 2, 0], [1, 0, 0, 2, 0, 3, 2, 0], [2, 0, 0, 2, 0, 1, 3, 0], [2, 0, 0, 3, 0, 2, 1, 0] ] \$ \end{aligned}$$

Â» SYNC'D 59/4096 , 0.01440429688

67 . Coloring, {2, 3, 4, 7}

**R**: [3, 8, 8, 6, 7, 7, 4, 5]    **B**: [6, 3, 1, 1, 2, 4, 5, 2]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$\begin{aligned} & [ '2' (-5 + \tau - 10\tau^2 - 2\tau^3 - \tau^4 + \tau^5)' (-1 + \tau)', 2' (-1 + \tau')^2 (1 + \tau')^2 (5 + 3\tau + 3\tau^2 + \tau^3)', -2' (-1 + \tau')^2 (5 - 4\tau + 6\tau^2 + \tau^4)' (1 + \tau)', 2' (5 + 2\tau + 19\tau^2 + 7\tau^4 - 2\tau^5 + \tau^6)', -2' (-1 + \tau')^2 (5 + 4\tau + 6\tau^2 + \tau^4)' (1 + \tau)', 2' (5 - 2\tau + 19\tau^2 + 7\tau^4 + 2\tau \end{aligned}$$



$$5 + \tau^6, 2(1 + \tau)^2(5 + \tau + 10\tau^2 - 2\tau^3 + \tau^4 + \tau^5), 2(-1 + \tau)^2(-5 + 3\tau - 3\tau^2 + \tau^3)(1 + \tau)^2$$

For  $\tau=1/2$ , [233, 177, 219, 713, 411, 593, 753, 297] . FixedPtCheck, [233, 177, 219, 713, 411, 593, 753, 297]

$$\det(A + \tau \Delta) = 1(1 + \tau)^2(\tau)^2(-1 + \tau)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	6 vs 6	6 vs 6

Omega Rank for R : cycles: {{4, 6, 7}}, net cycles: 0 . order: 6

$$[0, 0, y_1, y_2, y_3, y_4, y_5, y_6]$$

$$\begin{aligned} R = & \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], \\ & [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \times \$ [ [0, 0, 0, 0, 0, 0, 0, 0], \\ & [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, \\ & 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [1, -2, 3, -5/8, 11/8, -21/8], [0, 1, -2, 3/8, -5/8, \\ & 11/8], [0, 1, -2, 3/8, -5/8, 11/8], [0, 0, 0, -5/8, 3/8, 3/8], [0, 0, 0, 3/8, -5/8, 3/8], [0, 0, 0, 3/8, -5/8, 3/8], \\ & [0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 1, -5/8, 3/8, -5/8] ] \times \$ [ [0, 0, 1, 1, 1, 1, 2, 2], [0, 0, 0, 2, 2, 1, 2, 1], [0, 0, \\ & 0, 2, 1, 2, 3, 0], [0, 0, 0, 3, 0, 2, 3, 0], [0, 0, 0, 3, 0, 3, 2, 0], [0, 0, 0, 2, 0, 3, 3, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 4, 6}}, net cycles: 0 . order: 6

$$[y_6, y_3, y_4, y_5, y_2, y_1, 0, 0]$$

$$\begin{aligned} B = & \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, \\ & 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], \\ & [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, \\ & 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 1, -5/8, 3/8, -5/8], \\ & [0, 0, 0, 3/8, -5/8, 3/8], [0, 0, 0, 3/8, -5/8, 3/8], [0, 1, -2, 3/8, -5/8, 11/8], [0, 0, 0, -5/8, 3/8, 3/8], [1, -2, 3, \\ & -5/8, 11/8, -21/8], [0, 1, -2, 3/8, -5/8, 11/8] ] \times \$ [ [2, 2, 1, 1, 1, 1, 0, 0], [2, 1, 2, 1, 0, 2, 0, 0], [3, 0, 1, \\ & 2, 0, 2, 0, 0], [3, 0, 0, 2, 0, 3, 0, 0], [2, 0, 0, 3, 0, 3, 0, 0], [3, 0, 0, 3, 0, 2, 0, 0] ] \$ \end{aligned}$$

Â» SYNC'D 665/16384 , 0.04058837891

68 . Coloring, {2, 3, 4, 8}

**R:** [3, 8, 8, 6, 7, 7, 5, 2] **B:** [6, 3, 1, 1, 2, 4, 4, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{bmatrix} 4(-1+\tau)^2(5+2\tau+\tau^2), & 4(1+\tau)^2(5-2\tau+\tau^2), & 4(1+\tau)(-5+\tau^2)(-1+\tau), \\ -4(5-\tau+3\tau^2+\tau^3)(-1+\tau), & 4(5-3\tau+\tau^2+\tau^3)(1+\tau)^2, & -4(5-2\tau+2\tau^2+2\tau^3+\tau^4)(-1+\tau), \\ 4(1+\tau)(5+2\tau+\tau^2), & 4(5-\tau-\tau^2+\tau^3)(1+\tau)^2 \end{bmatrix}$$

For  $\tau=1/2$ , [50, 306, 114, 86, 279, 77, 267, 315] . FixedPtCheck, [50, 306, 114, 86, 279, 77, 267, 315]

$$\det(A + \tau \Delta) = (-\tau)^2 (1 + \tau)^2 (-1 + \tau)^2$$

$\Delta$ -Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	2 vs 6	6 vs 6

Omega Rank for R : cycles: {{2, 8}, {5, 7}}, net cycles: 0 . order: 2

$$\$ [ [0, 1, 1, 0, 1, 1, 2, 2], [0, 2, 0, 0, 2, 0, 2, 2], [0, 2, 0, 0, 2, 0, 2, 2], [0, 2, 0, 0, 2, 0, 2, 2], [0, 2, 0, 0, 2, 0, 2, 2], [0, 2, 0, 0, 2, 0, 2, 2] ] \$$$

$$[0, -y_1 + y_2, y_1, 0, -y_1 + y_2, y_1, y_2, y_2]$$

$$p = -s^2 + s^3 \quad p = -s^2 + s^5 \quad p = -s^2 + s^4 \quad p = -s^2 + s^6$$

Omega Rank for B : cycles: {{1, 4, 6}}, net cycles: 0 . order: 6

$$[y_1, y_2, y_3, y_4, y_5, y_6, 0, 0]$$

$$\begin{aligned} B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], \\ [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1, 0, 0] ] \$ = \$ [ [0, 0, 0, -5/8, 3/8, 3/8], [0, 0, 1, 3/8, -5/8, -5/8], \\ [0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, 3/8, 3/8, -5/8], [0, 1, -1, -5/8, -5/8, 11/8], [0, 0, 0, 3/8, -5/8, 3/8], [0, 0, 0, 3/8, -5/8, 3/8], [1, -1, 0, -5/8, 11/8, -5/8] ] \$ \times \$ [ [2, 1, 1, 2, 1, 1, 0, 0], [3, 1, 1, 1, 0, 2, 0, 0], [2, 0, 1, 2, 0, 3, 0, 0], [3, 0, 0, 3, 0, 2, 0, 0], [3, 0, 0, 2, 0, 3, 0, 0], [2, 0, 0, 3, 0, 3, 0, 0] ] \$ \end{aligned}$$

Â» SYNC'D 1409/65536 , 0.02149963379

69 . Coloring, {2, 3, 5, 6}

**R:** [3, 8, 8, 1, 2, 4, 5, 5]    **B:** [6, 3, 1, 6, 7, 7, 4, 2]

‘ See graph

‘ ‘ See pair graph

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ 1^2 (1+\tau)^2 (-1+\tau)^2 (5+2\tau+\tau^2), 1^2 (1+\tau)^2 (5-\tau+3\tau^2+\tau^3), -1^2 (1+\tau)^2 (-1+\tau)^2 (5-2\tau+\tau^2), \\ & -1^2 (-1+\tau)^2 (-5-\tau-3\tau^2+\tau^3), 3^2 (1+\tau)^2 (5+2\tau+8\tau^2-2\tau^3+3\tau^4), \\ & -3^2 (-1+\tau)^3 (5+4\tau+3\tau^2), -3^2 (-1+\tau)^2 (5-2\tau+8\tau^2+2\tau^3+3\tau^4), \\ & 3^2 (5-4\tau+3\tau^2) (1+\tau)^3 ] \end{aligned}$$

For  $\tau=1/2$ , [75, 387, 153, 49, 381, 31, 103, 405] . FixedPtCheck, [75, 387, 153, 49, 381, 31, 103, 405]

$$\det(A + \tau \Delta) = 0$$

Delta Range :  $[y_2, -y_1 - y_3 - y_5, -y_2 - y_4 - y_6, y_1, y_3, y_4, y_5, y_6]$

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

$$\begin{aligned} & \$ [ [1, 1, 1, 1, 2, 0, 0, 2], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, \\ & 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$ [ [1, 1, 1, 1, 0, 2, 2, 0], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], \\ & [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$ [ [0, 0, 0, 0, 1, -1, -1, 1], [0, 0, 0, \\ & 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$ \end{aligned}$$

$$[0, 0, 0, 0, y_1, -y_1, -y_1, y_1]$$

$$p^2 = s^2 \quad p = s^2$$

$$S+ \quad \backslash ; \quad S- \quad \backslash ; \quad NM$$

$$\begin{aligned} & \$ [ [0, 0, 0, 0, 0, 0, 1, 1], [1, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 1, 0, 0], [0, 0, 1, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, \\ & 0, 0, 1], [1, 0, 0, 1, 0, 0, 0, 0], [0, 1, 0, 0, 0, 1, 0, 0], [0, 1, 1, 0, 0, 0, 0, 0] ] \$ [ [1, 0, 0, 1, 0, 0, 0, 0], \\ & [0, 1, 0, 0, 0, 0, 0, 1], [0, 1, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 1, 0, 0], [0, 0, 1, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 1, 0, \\ & 0], [1, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1, 1] ] \$ [ [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, \\ & 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], \\ & [0, 0, 0, 0, 0, 0, 0, 0] ] \$ \end{aligned}$$

CmmCk true, true, true

$$p^3 = s^3 \quad p^4 = s^4 \quad p^5 = s^5$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
1 vs 6	6 vs 6	6 vs 6	6 vs 6	6 vs 6

Omega Rank for R : cycles: {{2, 5, 8}}, net cycles: 0 . order: 6

$$[y_1, y_2, y_3, y_4, y_5, 0, 0, y_6]$$

$$\begin{aligned} R = & \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, \\ & 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0, 0, 0] \\ & , [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, \\ & 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 1, -5/8, 3/8, -5/8], [0, 0, 0, 3/8, -5/8, 3/8], \end{aligned}$$

$[0, 0, 0, 3/8, -5/8, 3/8], [0, 1, -1, 3/8, -5/8, 3/8], [0, 0, 0, -5/8, 3/8, 3/8], [1, -1, 0, -5/8, 3/8, 3/8], [0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, 3/8, 3/8, -5/8]$  ] \$ x \$ [  $[1, 1, 1, 1, 2, 0, 0, 2], [1, 2, 1, 0, 2, 0, 0, 2], [0, 2, 1, 0, 2, 0, 0, 3], [0, 2, 0, 0, 3, 0, 0, 3], [0, 3, 0, 0, 3, 0, 0, 2], [0, 3, 0, 0, 2, 0, 0, 3]$  ] \$

Omega Rank for B : cycles: {{4, 6, 7}}, net cycles: 0 . order: 6

$$[y_1, y_2, y_3, y_4, 0, y_6, y_5, 0]$$

$B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] x $ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] = $ [ [0, 0, 0, 3/8, -5/8, 3/8], [0, 1, -1, 3/8, -5/8, 3/8], [0, 0, 1, -5/8, 3/8, -5/8], [0, 0, 0, 3/8, -5/8, 3/8], [0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, -5/8, 3/8, 3/8], [1, -1, 0, -5/8, 3/8, 3/8] ] x $ [ [1, 1, 1, 1, 0, 2, 2, 0], [1, 0, 1, 2, 0, 2, 2, 0], [1, 0, 0, 2, 0, 3, 2, 0], [0, 0, 0, 2, 0, 3, 3, 0], [0, 0, 0, 3, 0, 2, 3, 0], [0, 0, 0, 3, 0, 3, 2, 0] ] $$

Â» SYNC'D 15/512 , 0.02929687500

70 . Coloring, {2, 3, 5, 7}

**R:** [3, 8, 8, 1, 2, 7, 4, 5] **B:** [6, 3, 1, 6, 7, 4, 5, 2]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$[ ' 1 ' ( ' 1 + \tau ' ) ' ( ' - 1 + \tau ' ) ' ( ' - 5 + 3\tau - 3\tau^2 + \tau^3 ' ) ' , 1 ' ( ' 1 + \tau ' ) ' ( ' 5 + 2\tau^2 + \tau^4 ' ) ' , -1 ' ( ' 5 + \tau + \tau^2 + \tau^3 ' ) ' ( ' 1 + \tau ' ) ' ( ' - 1 + \tau ' ) ' , -1 ' ( ' 1 + \tau^2 ' ) ' ( ' - 1 + \tau ' ) ' ( ' 5 - 2\tau + \tau^2 ' ) ' , 3 ' ( ' 5 + 2\tau + 8\tau^2 - 2\tau^3 + 3\tau^4 ' ) ' , 3 ' ( ' 5 + 3\tau^2 ' ) ' ( ' - 1 + \tau ' ) '^2 , 1 ' ( ' - 1 + \tau ' ) ' ( ' - 5 - \tau - 3\tau^2 + \tau^3 ' ) ' , 1 ' ( ' 1 + \tau ' ) '^2 ( ' 5 - 2\tau + \tau^2 ' ) ' ]'$

For  $\tau=1/2$ , [99, 267, 141, 85, 254, 46, 98, 306] . FixedPtCheck, [99, 267, 141, 85, 254, 46, 98, 306]

$\det(A + \tau \Delta) = 0$

Delta Range :  $[y_2, -y_1 - y_3 - y_5, -y_2 - y_4 - y_6, y_1, y_3, y_4, y_5, y_6]$

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

$\$ [ [1, 1, 1, 1, 1, 0, 1, 2], [2, 1, 2, 3, 3, 2, 1, 2], [5, 5, 5, 3, 5, 3, 3, 3], [3, 5, 4, 4, 4, 4, 3, 5], [8, 7, 6, 7, 10, 9, 8, 9], [17, 17, 17, 15, 17, 17, 15, 13] ] \$ [ [1, 1, 1, 1, 1, 2, 1, 0], [2, 3, 2, 1, 1, 2, 3, 2], [3, 3, 3, 5, 3, 5, 5, 5], [5, 3, 4, 4, 4, 4, 5, 3], [8, 9, 10, 9, 6, 7, 8, 7], [15, 15, 15, 17, 15, 15, 17, 19] ] \$ [ [0, 0, 0, 0, 0, -1, 0, 1], [0, -1, 0, 1, 1, 0, -1, 0], [1, 1, 1, -1, 1, -1, -1, -1], [-1, 1, 0, 0, 0, 0, -1, 1], [0, -1, -2, -1, 2, 1, 0, 1]$

, [1, 1, 1, -1, 1, 1, -1, -3] ] \$

$$[y_1 + y_3 + y_4, -y_3 - y_4 - y_2, y_1, y_2, y_3, -2y_1 - y_3 - y_4 - y_5, y_4, y_5]$$

$$p = s - 2s^3 + 8s^6$$

S+ \; S- \; NM  
 \$ [ [5, 5, 7, 5, 5, 9, 5, 3], [2, 4, 4, 1, 2, 2, 3, 4], [5, 3, 7, 7, 5, 3, 5, 9], [5, 2, 1, 3, 3, 4, 2, 2], [3, 5, 2, 1, 3, 3, 3, 2], [3, 3, 7, 9, 3, 7, 9, 3], [3, 2, 2, 4, 3, 2, 3, 3], [5, 7, 5, 5, 9, 3, 3, 7] ] \$ \$ [ [7, 7, 5, 3, 5, 9, 5, 3], [1, 3, 5, 2, 2, 2, 3, 4], [7, 5, 5, 5, 5, 3, 5, 9], [4, 1, 2, 4, 3, 4, 2, 2], [2, 4, 3, 2, 3, 3, 3, 2], [5, 5, 5, 7, 3, 7, 9, 3], [2, 1, 3, 5, 3, 2, 3, 3], [7, 9, 3, 3, 9, 3, 3, 7] ] \$ \$ [ [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$

CmmCk true, true, true

$\Delta$ -Rank	A+(1/2) $\Delta$	A-(1/2) $\Delta$	R	B
5 vs 6	7 vs 7	7 vs 7	7 vs 7	4 vs 7

Omega Rank for R : cycles: {{2, 5, 8}}, net cycles: 0 . order: 6

$$[y_1, y_2, y_3, y_4, y_5, 0, y_6, y_7]$$

R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, 1, -5/8, 3/8, -5/8], [0, 0, 0, 0, 3/8, -5/8, 3/8], [0, 0, 0, 0, 3/8, -5/8, 3/8], [0, 0, 1, -1, 3/8, -5/8, 3/8], [0, 0, 0, 0, -5/8, 3/8, 3/8], [1, -1, 0, 0, 3/8, 3/8, -5/8], [0, 1, -1, 0, -5/8, 3/8, 3/8], [0, 0, 0, 0, 3/8, 3/8, -5/8] ] \$ x \$ [ [1, 1, 1, 1, 1, 0, 1, 2], [1, 1, 1, 1, 2, 0, 0, 2], [1, 2, 1, 0, 2, 0, 0, 2], [0, 2, 1, 0, 2, 0, 0, 3], [0, 2, 0, 0, 3, 0, 0, 3], [0, 3, 0, 0, 3, 0, 0, 2], [0, 3, 0, 0, 2, 0, 0, 3] ] \$

Omega Rank for B : cycles: {{4, 6}, {5, 7}}, net cycles: 1 . order: 4

\$ [ [1, 1, 1, 1, 1, 2, 1, 0], [1, 0, 1, 2, 1, 2, 1, 0], [1, 0, 0, 2, 1, 3, 1, 0], [0, 0, 0, 3, 1, 3, 1, 0], [0, 0, 0, 3, 1, 3, 1, 0], [0, 0, 0, 3, 1, 3, 1, 0], [0, 0, 0, 3, 1, 3, 1, 0] ] \$

$$[-y_1 - y_2 + 3y_4, y_1, 3y_4 - y_3, y_2, y_4, y_3, y_4, 0]$$

$$p = -s^4 + s^5 \quad p = -s^4 + s^7 \quad p = -s^4 + s^6$$

Â» SYNC'D 135/8192 , 0.01647949219

71 . Coloring, {2, 3, 5, 8}

**R:** [3, 8, 8, 1, 2, 7, 5, 2]    **B:** [6, 3, 1, 6, 7, 4, 4, 5]

‘ See graph

‘ ‘ See pair graph

Ω for A+τΔ :

$$\left[ \begin{matrix} 1^{\epsilon} (1 + \tau^{\epsilon})^{\epsilon} (-1 + \tau^{\epsilon})^{\epsilon 2}, 1^{\epsilon} (1 + \tau^{\epsilon})^{\epsilon 3}, -1^{\epsilon} (1 + \tau^{\epsilon})^{\epsilon 2} (-1 + \tau^{\epsilon})^{\epsilon}, -1^{\epsilon} (-1 + \tau^{\epsilon})^{\epsilon 3}, \\ -1^{\epsilon} (1 + \tau^{\epsilon})^{\epsilon 2} (-1 + \tau^{\epsilon})^{\epsilon}, -1^{\epsilon} (-1 + \tau^{\epsilon})^{\epsilon 3}, 1^{\epsilon} (1 + \tau^{\epsilon})^{\epsilon} (-1 + \tau^{\epsilon})^{\epsilon 2}, 1^{\epsilon} (1 + \tau^{\epsilon})^{\epsilon 3} \end{matrix} \right]^{\epsilon}$$

For τ=1/2, [3, 27, 9, 1, 9, 1, 3, 27] . FixedPtCheck, [3, 27, 9, 1, 9, 1, 3, 27]

$$\det(A + \tau \Delta) = 0$$

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

$$\begin{aligned} & \$ [ [1, 2, 1, 0, 1, 0, 1, 2], [1, 3, 1, 3, 1, 3, 1, 3], [3, 2, 1, 2, 1, 2, 3, 2], [5, 3, 5, 3, 5, 3, 5, 3], [3, 4, 5, 4, 5, \\ & 4, 3, 4], [7, 9, 7, 9, 7, 9, 7, 9] ] \$ \$ [ [1, 0, 1, 2, 1, 2, 1, 0], [3, 1, 3, 1, 3, 1, 3, 1], [1, 2, 3, 2, 3, 2, 1, 2], \\ & [3, 5, 3, 5, 3, 5, 3, 5], [5, 4, 3, 4, 3, 4, 5, 4], [9, 7, 9, 7, 9, 7, 9, 7] ] \$ \$ [ [0, 1, 0, -1, 0, -1, 0, 1], [-1, 1, -1, \\ & 1, -1, 1, -1, 1], [1, 0, -1, 0, -1, 0, 1, 0], [1, -1, 1, -1, 1, -1, 1, -1], [-1, 0, 1, 0, 1, 0, -1, 0], [-1, 1, -1, 1, -1, 1, \\ & -1, 1] ] \$ \end{aligned}$$

$$[y_2, y_3, -y_2 - y_3 - y_1, y_1, -y_2 - y_3 - y_1, y_1, y_2, y_3]$$

$$p = s^2 - 4s^6$$

$$S+ \quad \backslash ; \quad S- \quad \backslash ; \quad NM$$

$$\begin{aligned} & \$ [ [7, 5, 3, 5, 3, 5, 7, 5], [5, 7, 5, 3, 5, 3, 5, 7], [3, 5, 7, 5, 7, 5, 3, 5], [5, 3, 5, 7, 5, 7, 5, 3], [3, 5, 7, 5, 7, \\ & 5, 3, 5], [5, 3, 5, 7, 5, 7, 5, 3], [7, 5, 3, 5, 3, 5, 7, 5], [5, 7, 5, 3, 5, 3, 5, 7] ] \$ \$ [ [7, 5, 3, 5, 3, 5, 7, 5], \\ & [5, 7, 5, 3, 5, 3, 5, 7], [3, 5, 7, 5, 7, 5, 3, 5], [5, 3, 5, 7, 5, 7, 5, 3], [3, 5, 7, 5, 7, 5, 3, 5], [5, 3, 5, 7, 5, 7, 5, \\ & 3], [7, 5, 3, 5, 3, 5, 7, 5], [5, 7, 5, 3, 5, 3, 5, 7] ] \$ \$ [ [9, 4, 3, 6, 6, 3, 0, 5], [4, 9, 6, 5, 3, 4, 5, 0], [3, 6, \\ & 9, 4, 0, 5, 6, 3], [6, 5, 4, 9, 5, 0, 3, 4], [6, 3, 0, 5, 9, 4, 3, 6], [3, 4, 5, 0, 4, 9, 6, 5], [0, 5, 6, 3, 3, 6, 9, 4], \\ & [5, 0, 3, 4, 6, 5, 4, 9] ] \$ \end{aligned}$$

CmmCk true, true, true

$$p' = s^2 + 2s^4 \quad p' = s^3 + 2s^5$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
3 vs 6	3 vs 6	3 vs 6	3 vs 6	3 vs 6

Omega Rank for R : cycles: {{2, 8}}, net cycles: -1 . order: 4

\$ [ [1, 2, 1, 0, 1, 0, 1, 2], [0, 3, 1, 0, 1, 0, 0, 3], [0, 4, 0, 0, 0, 0, 0, 4], [0, 4, 0, 0, 0, 0, 0, 4], [0, 4, 0, 0, 0, 0, 0, 4], [0, 4, 0, 0, 0, 0, 0, 4], [0, 4, 0, 0, 0, 0, 0, 4] ] \$

[y<sub>2</sub>, y<sub>3</sub>, y<sub>1</sub>, 0, y<sub>1</sub>, 0, y<sub>2</sub>, y<sub>3</sub>]

$$p = -s^3 + s^5 \quad p = -s^3 + s^4 \quad p = -s^3 + s^6$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: -1 . order: 4

\$ [ [1, 0, 1, 2, 1, 2, 1, 0], [1, 0, 0, 3, 0, 3, 1, 0], [0, 0, 0, 4, 0, 4, 0, 0], [0, 0, 0, 4, 0, 4, 0, 0], [0, 0, 0, 4, 0, 4, 0, 0], [0, 0, 0, 4, 0, 4, 0, 0], [0, 0, 0, 4, 0, 4, 0, 0] ] \$

[y<sub>3</sub>, 0, y<sub>1</sub>, y<sub>2</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, 0]

$$p = -s^3 + s^4 \quad p = -s^3 + s^5 \quad p = -s^3 + s^6$$

Â« NOT SYNC'D Â»

Nullspace of {ΩΔ<sup>i</sup>} :

[0, x<sub>1</sub>, x<sub>2</sub>, x<sub>3</sub>, 2x<sub>2</sub>, -4x<sub>1</sub> + 2x<sub>3</sub>]

For A+2Δ : [-3y<sub>1</sub> - 3y<sub>5</sub> - y<sub>4</sub>, y<sub>1</sub>, -3y<sub>1</sub> - y<sub>2</sub> - 3y<sub>5</sub>, 9y<sub>1</sub> + 9y<sub>5</sub> - y<sub>3</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>]

For A-2Δ : [-3y<sub>1</sub> - 3y<sub>3</sub> - y<sub>4</sub>, 9y<sub>1</sub> + 9y<sub>3</sub> - y<sub>5</sub>, -3y<sub>1</sub> - 3y<sub>3</sub> - y<sub>2</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>]

Range of {ΩΔ<sup>i</sup>} : [μ<sub>3</sub>, μ<sub>2</sub>, μ<sub>1</sub>, -μ<sub>3</sub> - μ<sub>2</sub> - μ<sub>1</sub>, μ<sub>1</sub>, -μ<sub>3</sub> - μ<sub>2</sub> - μ<sub>1</sub>, μ<sub>3</sub>, μ<sub>2</sub>]

rank of M is 8 , rank of N is 5

M \ ; N

\$ [ [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \$ [ [0, 5, 6, 3, 3, 6, 9, 4], [5, 0, 3, 4, 6, 5, 4, 9], [6, 3, 0, 5, 9, 4, 3, 6], [3, 4, 5, 0, 4, 9, 6, 5], [3, 6, 9, 4, 0, 5, 6, 3], [6, 5, 4, 9, 5, 0, 3, 4], [9, 4, 3, 6, 6, 3, 0, 5], [4, 9, 6, 5, 3, 4, 5, 0] ] \$

Check is ΩΔN zero? true, πΔ= [0, 1, 0, -1, 0, -1, 0, 1]

ker M, [0, 0, 0, 0, 0, 0, 0, 0]

Range M, [x<sub>5</sub>, x<sub>4</sub>, x<sub>1</sub>, x<sub>2</sub>, x<sub>3</sub>, x<sub>6</sub>, x<sub>7</sub>, x<sub>8</sub>]

τ= 32 , r'= 1/2

Ranges





0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0] ] \$  
[D, C, C, '3', '2', '3', '1', '7', '2', E, E, D, '1', '7', '8', '8'], ['6', '3', '7', '3', B, '7', B, C, '10', '6', '5', '5', '10', '8', '8', C] with invariant measure [2, 1, 1, 0, 2, 1, 2, 1, 0, 0, 2, 2, 2, 1, 0, 1]

N by blocks, check: true . ' See partition graph.

' ' See level-2 partition graph.

'

<b>Sandwich</b>	
<b>Coloring</b>	{2, 3, 5, 8}
<b>Rank</b>	2
<b>R,B</b>	[3, 8, 8, 1, 2, 7, 5, 2], [6, 3, 1, 6, 7, 4, 4, 5]
$\pi_2$	[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0]
$u_2$	[5, 6, 3, 3, 6, 9, 4, 3, 4, 6, 5, 4, 9, 5, 9, 4, 3, 6, 4, 9, 6, 5, 5, 6, 3, 3, 4, 5] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

72 . Coloring, {2, 3, 6, 7}

**R:** [3, 8, 8, 1, 7, 4, 4, 5]    **B:** [6, 3, 1, 6, 2, 7, 5, 2]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

' [ '-1' (' - 5 -  $\tau$  - 3 $\tau^2$  +  $\tau^3$  '), -1' (' - 1 +  $\tau$  ')'' (' 5 + 2 $\tau$  +  $\tau^2$  '), 1' (' 5 -  $\tau$  + 3 $\tau^2$  +  $\tau^3$  '), 1' (' 1 +  $\tau$  ')'' (' 5 - 2 $\tau$  +  $\tau^2$  '), -1' (' - 5 -  $\tau$  - 3 $\tau^2$  +  $\tau^3$  '), -1' (' - 1 +  $\tau$  ')'' (' 5 + 2 $\tau$  +  $\tau^2$  '), 1' (' 5 -  $\tau$  + 3 $\tau^2$  +  $\tau^3$  '), 1' (' 1 +  $\tau$  ')'' (' 5 - 2 $\tau$  +  $\tau^2$  ')'' ]'

For  $\tau=1/2$ , [49, 25, 43, 51, 49, 25, 43, 51] . FixedPtCheck, [49, 25, 43, 51, 49, 25, 43, 51]

$\det(A + \tau \Delta) = 0$

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

[1, 1, 1, 1, 1, 1, 1]

+ \; - \; Δ

\$ [ [1, 0, 1, 2, 1, 0, 1, 2], [3, 1, 3, 1, 3, 1, 3, 1], [1, 2, 3, 2, 1, 2, 3, 2], [3, 5, 3, 5, 3, 5, 3, 5], [5, 4, 3, 4, 5, 4, 3, 4], [9, 7, 9, 7, 9, 7, 9, 7] ] \$ \$ [ [1, 2, 1, 0, 1, 2, 1, 0], [1, 3, 1, 3, 1, 3, 1, 3], [3, 2, 1, 2, 3, 2, 1, 2], [5, 3, 5, 3, 5, 3, 5, 3], [3, 4, 5, 4, 3, 4, 5, 4], [7, 9, 7, 9, 7, 9, 7, 9] ] \$ \$ [ [0, -1, 0, 1, 0, -1, 0, 1], [1, -1, 1, -1, 1, -1, 1, -1], [-1, 0, 1, 0, -1, 0, 1, 0], [-1, 1, -1, 1, -1, 1, -1, 1], [1, 0, -1, 0, 1, 0, -1, 0], [1, -1, 1, -1, 1, -1, 1, -1] ] \$

[y<sub>3</sub>, -y<sub>3</sub> - y<sub>1</sub> - y<sub>2</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, -y<sub>3</sub> - y<sub>1</sub> - y<sub>2</sub>, y<sub>1</sub>, y<sub>2</sub>]

$$p' = s^3 + 2s^5 \quad p = s^2 - 4s^6$$

S+ \; S- \; NM

\$ [ [7, 5, 3, 5, 3, 5, 7, 5], [5, 7, 5, 3, 5, 3, 5, 7], [3, 5, 7, 5, 7, 5, 3, 5], [5, 3, 5, 7, 5, 7, 5, 3], [3, 5, 7, 5, 7, 5, 3, 5], [5, 3, 5, 7, 5, 7, 5, 3], [7, 5, 3, 5, 3, 5, 7, 5], [5, 7, 5, 3, 5, 3, 5, 7] ] \$ \$ [ [7, 5, 3, 5, 3, 5, 7, 5], [5, 7, 5, 3, 5, 3, 5, 7], [3, 5, 7, 5, 7, 5, 3, 5], [5, 3, 5, 7, 5, 7, 5, 3], [3, 5, 7, 5, 7, 5, 3, 5], [5, 3, 5, 7, 5, 7, 5, 3], [7, 5, 3, 5, 3, 5, 7, 5], [5, 7, 5, 3, 5, 3, 5, 7] ] \$ \$ [ [5, 2, 3, 4, 0, 3, 2, 1], [2, 5, 4, 1, 3, 0, 1, 4], [3, 4, 5, 2, 2, 1, 0, 3], [4, 1, 2, 5, 1, 4, 3, 0], [0, 3, 2, 1, 5, 2, 3, 4], [3, 0, 1, 4, 2, 5, 4, 1], [2, 1, 0, 3, 3, 4, 5, 2], [1, 4, 3, 0, 4, 1, 2, 5] ] \$

CmmCk true, true, true

$$p' = s^2 + 2s^4$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
3 vs 6	3 vs 6	3 vs 6	3 vs 6	3 vs 6

Omega Rank for R : cycles: {{1, 3, 4, 5, 7, 8}}, net cycles: 1 . order: 6

\$ [ [1, 0, 1, 2, 1, 0, 1, 2], [2, 0, 1, 1, 2, 0, 1, 1], [1, 0, 2, 1, 1, 0, 2, 1], [1, 0, 1, 2, 1, 0, 1, 2], [2, 0, 1, 1, 2, 0, 1, 1], [1, 0, 2, 1, 1, 0, 2, 1] ] \$

[y<sub>1</sub>, 0, y<sub>2</sub>, y<sub>3</sub>, y<sub>1</sub>, 0, y<sub>2</sub>, y<sub>3</sub>]

$$p' = -s + s^4 \quad p' = -s^2 + s^5 \quad p = -s + s^4$$

Omega Rank for B : cycles: {{1, 2, 3, 5, 6, 7}}, net cycles: 1 . order: 6

\$ [ [1, 2, 1, 0, 1, 2, 1, 0], [1, 1, 2, 0, 1, 1, 2, 0], [2, 1, 1, 0, 2, 1, 1, 0], [1, 2, 1, 0, 1, 2, 1, 0], [1, 1, 2, 0, 1, 1, 2, 0], [2, 1, 1, 0, 2, 1, 1, 0] ] \$

[y<sub>2</sub>, y<sub>1</sub>, y<sub>3</sub>, 0, y<sub>2</sub>, y<sub>1</sub>, y<sub>3</sub>, 0]

$$p = -s + s^4 \quad p' = -s + s^4 \quad p' = -s^2 + s^5$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[0, x_1, x_2, x_3, 2x_2, -4x_1 + 2x_3]$$

$$\text{For } A+2\Delta : [y_2, y_3, y_4, y_5, -y_2 - 3y_5 - 3y_1, -y_3 + 9y_5 + 9y_1, -y_4 - 3y_5 - 3y_1, y_1]$$

$$\text{For } A-2\Delta : [-3y_1 - y_2 - 3y_3, y_1, -3y_1 - 3y_3 - y_4, 9y_1 + 9y_3 - y_5, y_2, y_3, y_4, y_5]$$

$$\text{Range of } \{\Omega\Delta^i\} : [-\mu_1 - \mu_2 - \mu_3, \mu_1, \mu_2, \mu_3, -\mu_1 - \mu_2 - \mu_3, \mu_1, \mu_2, \mu_3]$$

rank of M is 8 , rank of N is 5

$$M \quad \setminus ; \quad N$$

$$\begin{aligned} & \$ [ [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, \\ & 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0] ] \$ \quad \$ [ [0, 3, 2, 1, 5, 2, 3, 4], \\ & [3, 0, 1, 4, 2, 5, 4, 1], [2, 1, 0, 3, 3, 4, 5, 2], [1, 4, 3, 0, 4, 1, 2, 5], [5, 2, 3, 4, 0, 3, 2, 1], [2, 5, 4, 1, 3, 0, 1, \\ & 4], [3, 4, 5, 2, 2, 1, 0, 3], [4, 1, 2, 5, 1, 4, 3, 0] ] \$ \end{aligned}$$

$$\text{Check is } \Omega\Delta N \text{ zero? } \text{true}, \pi\Delta = [0, -1, 0, 1, 0, -1, 0, 1]$$

$$\ker M, [0, 0, 0, 0, 0, 0, 0, 0]$$

$$\text{Range } M, [x_8, x_6, x_7, x_5, x_3, x_4, x_2, x_1]$$

$$\tau = 32, r' = 1/2$$

Ranges

$$\text{Action of } R \text{ on ranges, } [[3], [4], [4], [1]]$$

$$\text{Action of } B \text{ on ranges, } [[2], [3], [1], [2]]$$

$$\beta(\{1, 5\}) = 1/4$$

$$\beta(\{2, 6\}) = 1/4$$

$$\beta(\{3, 7\}) = 1/4$$

$$\beta(\{4, 8\}) = 1/4$$

$$\ker N, [\mu_3, \mu_2, \mu_1, -\mu_3 - \mu_2 - \mu_1, \mu_3, \mu_2, \mu_1, -\mu_3 - \mu_2 - \mu_1]$$

Range of N

$$[y_1 - y_2 + y_5, y_1 + y_5 - y_3, y_1 + y_5 - y_4, y_1, y_2, y_3, y_4, y_5]$$

Partitions

$$\text{Action of } R \text{ on partitions, } [[2], [3], [1], [3]]$$

$$\text{Action of } B \text{ on partitions, } [[3], [3], [4], [2]]$$

$$\alpha(\{\{4, 5, 6, 7\}, \{1, 2, 3, 8\}\}) = 1/5$$

$$\alpha(\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}) = 1/5$$

$$\alpha(\{\{2, 3, 5, 8\}, \{1, 4, 6, 7\}\}) = 2/5$$

$$\alpha(\{\{1, 3, 4, 6\}, \{2, 5, 7, 8\}\}) = 1/5$$

$$b_1 = \{2, 3, 5, 8\} \text{ ' , ' } b_2 = \{4, 5, 6, 7\} \text{ ' , ' } b_3 = \{1, 2, 3, 4\} \text{ ' , ' } b_4 = \{1, 2, 3, 8\} \text{ ' , ' } b_5 = \{1, 4, 6, 7\} \text{ ' , ' }$$

$$b_6 = \{5, 6, 7, 8\} \text{ ' , ' } b_7 = \{1, 3, 4, 6\} \text{ ' , ' } b_8 = \{2, 5, 7, 8\}$$

Action of R and B on the blocks of the partitions: \$ [ [0, 0, 0, 1, 0, 0, 0, 1] , [0, 0, 0, 0, 1, 1, 0, 0] , [1, 0, 0, 0, 1, 0, 0, 0] , [1, 0, 1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 1, 0] , [1, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 1, 0, 1, 0, 0, 0] , [1, 0, 0, 0, 0, 1, 0, 0] ] \$ = \$ [ [0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 1, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0] ] \$ + \$ [ [0, 0, 0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 1, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0] ] \$  
 ['4', '6', '5', '3', '2', '1', '5', '1'], ['8', '5', '1', '1', '7', '5', '3', '6'] with invariant measure [2, 1, 1, 1, 2, 1, 1, 1]

N by blocks, check: true . ' See partition graph.

' ' See level-2 partition graph.

'

<b>Sandwich</b>	
<b>Coloring</b>	{2, 3, 6, 7}
<b>Rank</b>	2
<b>R,B</b>	[3, 8, 8, 1, 7, 4, 4, 5], [6, 3, 1, 6, 2, 7, 5, 2]
$\pi_2$	[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]
$u_2$	[3, 2, 1, 5, 2, 3, 4, 1, 4, 2, 5, 4, 1, 3, 3, 4, 5, 2, 4, 1, 2, 5, 3, 2, 1, 1, 4, 3] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

73 . Coloring, {2, 3, 6, 8}

**R:** [3, 8, 8, 1, 7, 4, 5, 2]    **B:** [6, 3, 1, 6, 2, 7, 4, 5]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

' [ '1' ('1 +  $\tau$ ') (' - 1 +  $\tau$ ') (' - 5 +  $\tau$  -  $\tau^2$  +  $\tau^3$ '), 1' ('1 +  $\tau$ ') ('1 +  $\tau^2$ ') ('5 + 2 $\tau$  +  $\tau^2$ '), -1' ('1 +  $\tau$ ') (' - 1 +  $\tau$ ') ('5 + 3 $\tau$  + 3 $\tau^2$  +  $\tau^3$ '), -1' ('5 + 2 $\tau^2$  +  $\tau^4$ ') (' - 1 +  $\tau$ '), 1' ('1 +  $\tau$ ') ('5 -  $\tau$  + 3 $\tau^2$  +  $\tau^3$ '), 1' ('5 + 2 $\tau$  +  $\tau^2$ ') (' - 1 +  $\tau$ ')<sup>2</sup>, 3' ('5 - 2 $\tau$  + 8 $\tau^2$  + 2 $\tau^3$  + 3 $\tau^4$ '), 3' ('1 +  $\tau$ ')<sup>2</sup> ('5 + 3 $\tau^2$ ') ]'

For  $\tau=1/2$ , [111, 375, 177, 89, 258, 50, 206, 414] . FixedPtCheck, [111, 375, 177, 89, 258, 50, 206, 414]

$$\det(A + \tau \Delta) = 0$$

Delta Range :  $[y_2, -y_1 - y_3 - y_5, -y_2 - y_4 - y_6, y_1, y_3, y_4, y_5, y_6]$

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

$\$ [ [1, 1, 1, 1, 1, 0, 1, 2], [2, 3, 2, 1, 1, 2, 3, 2], [3, 5, 3, 3, 5, 5, 3, 5], [4, 4, 3, 5, 3, 5, 4, 4], [10, 9, 8, 9, 8, 7, 6, 7], [17, 15, 17, 17, 15, 13, 17, 17] ] \$$   
 $\$ [ [1, 1, 1, 1, 1, 2, 1, 0], [2, 1, 2, 3, 3, 2, 1, 2], [5, 3, 5, 5, 3, 3, 5, 3], [4, 4, 5, 3, 5, 3, 4, 4], [6, 7, 8, 7, 8, 9, 10, 9], [15, 17, 15, 15, 17, 19, 15, 15] ] \$$   
 $\$ [ [0, 0, 0, 0, 0, -1, 0, 1], [0, 1, 0, -1, -1, 0, 1, 0], [-1, 1, -1, -1, 1, 1, -1, 1], [0, 0, -1, 1, -1, 1, 0, 0], [2, 1, 0, 1, 0, -1, -2, -1], [1, -1, 1, 1, -1, -3, 1, 1] ] \$$

$$[-y_1 - y_3 - y_5, -y_2 - 2y_1 - y_3 - y_5, y_1, y_2, -y_4 + 2y_1 + y_3 + y_5, y_3, y_4, y_5]$$

$$p = s - 2s^3 - 8s^6$$

$$S+ \quad \backslash ; \quad S- \quad \backslash ; \quad NM$$

$\$ [ [5, 5, 7, 5, 5, 9, 5, 3], [2, 4, 4, 1, 2, 2, 3, 4], [5, 3, 7, 7, 5, 3, 5, 9], [5, 2, 1, 3, 3, 4, 2, 2], [3, 5, 2, 1, 3, 3, 3, 2], [3, 3, 7, 9, 3, 7, 9, 3], [3, 2, 2, 4, 3, 2, 3, 3], [5, 7, 5, 5, 9, 3, 3, 7] ] \$$   
 $\$ [ [7, 7, 5, 3, 5, 9, 5, 3], [1, 3, 5, 2, 2, 2, 3, 4], [7, 5, 5, 5, 5, 3, 5, 9], [4, 1, 2, 4, 3, 4, 2, 2], [2, 4, 3, 2, 3, 3, 3, 2], [5, 5, 5, 7, 3, 7, 9, 3], [2, 1, 3, 5, 3, 2, 3, 3], [7, 9, 3, 3, 9, 3, 3, 7] ] \$$   
 $\$ [ [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$$

CmmCk true, true, true

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 6	7 vs 7	7 vs 7	4 vs 7	7 vs 7

Omega Rank for R : cycles:  $\{\{2, 8\}, \{5, 7\}\}$ , net cycles: 1 . order: 4

$\$ [ [1, 1, 1, 1, 1, 0, 1, 2], [1, 2, 1, 0, 1, 0, 1, 2], [0, 2, 1, 0, 1, 0, 1, 3], [0, 3, 0, 0, 1, 0, 1, 3], [0, 3, 0, 0, 1, 0, 1, 3], [0, 3, 0, 0, 1, 0, 1, 3], [0, 3, 0, 0, 1, 0, 1, 3] ] \$$

$$[3y_3 - y_4, -y_1 - y_2 + 3y_3, y_1, y_2, y_3, 0, y_3, y_4]$$

$$p = -s^4 + s^7 \quad p = -s^4 + s^5 \quad p = -s^4 + s^6$$

Omega Rank for B : cycles:  $\{\{4, 6, 7\}\}$ , net cycles: 0 . order: 6

$$[y_3, y_4, y_2, y_1, y_5, y_6, y_7, 0]$$

$B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$$   
 $\times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$$

0, 0], [0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 0, 0, 0, 3/8, -5/8, 3/8], [0, 0, 1, -1, 3/8, -5/8, 3/8], [0, 0, 0, 1, -5/8, 3/8, -5/8], [0, 0, 0, 0, 3/8, -5/8, 3/8], [0, 1, -1, 0, -5/8, 3/8, 3/8], [0, 0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, 0, -5/8, 3/8, 3/8], [1, -1, 0, 0, 3/8, 3/8, -5/8] ] \$ x \$ [ [1, 1, 1, 1, 1, 2, 1, 0], [1, 1, 1, 1, 0, 2, 0], [1, 0, 1, 2, 0, 2, 2, 0], [1, 0, 0, 2, 0, 3, 2, 0], [0, 0, 0, 2, 0, 3, 3, 0], [0, 0, 0, 3, 0, 2, 3, 0], [0, 0, 0, 3, 0, 3, 2, 0] ] \$

Â» SYNC'D 135/8192 , 0.01647949219

74 . Coloring, {2, 3, 7, 8}

**R:** [3, 8, 8, 1, 7, 7, 4, 2]    **B:** [6, 3, 1, 6, 2, 4, 5, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

' [ '3' ('1 + τ')'' ('-5 + τ - 7τ<sup>2</sup> + 3τ<sup>3</sup>')'' ('-1 + τ')', 3' ('1 + τ')'' ('5 - 2τ + 8τ<sup>2</sup> + 2τ<sup>3</sup> + 3τ<sup>4</sup>')', -3' ('1 + τ')'' ('5 + τ + 7τ<sup>2</sup> + 3τ<sup>3</sup>')'' ('-1 + τ')', -3' ('-1 + τ')'' ('5 + 2τ + 8τ<sup>2</sup> - 2τ<sup>3</sup> + 3τ<sup>4</sup>')', -1' ('5 - τ + 3τ<sup>2</sup> + τ<sup>3</sup>')'' ('1 + τ')'' ('-1 + τ')', 1' ('5 + 3τ + 7τ<sup>2</sup> + τ<sup>3</sup>')'' ('-1 + τ')'<sup>2</sup>, 1' ('1 + τ')'' ('-1 + τ')'' ('-5 - τ - 3τ<sup>2</sup> + τ<sup>3</sup>')', -1' ('1 + τ')'<sup>2</sup> ('-5 + 3τ - 7τ<sup>2</sup> + τ<sup>3</sup>')'' ]'

For τ=1/2, [141, 309, 183, 127, 129, 67, 147, 369] . FixedPtCheck, [141, 309, 183, 127, 129, 67, 147, 369]

det(A + τ Δ) = 0

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

[1, 1, 1, 1, 1, 1, 1, 1]

+ \ ; - \ ; Δ

\$ [ [1, 1, 1, 1, 0, 0, 2, 2], [1, 2, 1, 2, 0, 1, 0, 1], [3, 3, 1, 1, 3, 1, 1, 3], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$ \$ [ [1, 1, 1, 1, 2, 2, 0, 0], [1, 0, 1, 0, 2, 1, 2, 1], [1, 1, 3, 3, 1, 3, 3, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$ \$ [ [0, 0, 0, 0, -1, -1, 1, 1], [0, 1, 0, 1, -1, 0, -1, 0], [1, 1, -1, -1, 1, -1, -1, 1], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$

[y<sub>3</sub>, -y<sub>3</sub> + y<sub>2</sub> - y<sub>1</sub>, -y<sub>3</sub>, -3 y<sub>3</sub> + y<sub>2</sub> - y<sub>1</sub>, 4 y<sub>3</sub> - 2 y<sub>2</sub> + y<sub>1</sub>, -y<sub>2</sub>, y<sub>1</sub>, y<sub>2</sub>]

p' = s<sup>4</sup>    p' = s<sup>5</sup>    p = s<sup>4</sup>

S+ \ ; S- \ ; NM

\$ [ [1, 1, 1, 1, 0, 1, 2, 1], [1, 1, 1, 1, 1, 0, 1, 2], [1, 1, 1, 1, 2, 1, 0, 1], [1, 1, 1, 1, 1, 2, 1, 0], [0, 1, 2, 1, 1, 1, 1, 1], [1, 0, 1, 2, 1, 1, 1, 1], [2, 1, 0, 1, 1, 1, 1, 1], [1, 2, 1, 0, 1, 1, 1, 1] ] \$ \$ [ [1, 1, 1, 1, 0, 1, 2, 1], [1, 1, 1, 1, 1, 0, 1, 2], [1, 1, 1, 1, 2, 1, 0, 1], [1, 1, 1, 1, 1, 2, 1, 0], [0, 1, 2, 1, 1, 1, 1, 1], [1, 0, 1, 2, 1, 1, 1, 1], [2, 1, 0, 1, 1, 1, 1, 1], [1, 2, 1, 0, 1, 1, 1, 1] ] \$

1], [2, 1, 0, 1, 1, 1, 1, 1], [1, 2, 1, 0, 1, 1, 1, 1]] \$ \$ [ [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0]] \$

CmmCk true, true, true

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
3 vs 6	6 vs 6	6 vs 6	6 vs 6	6 vs 6

Omega Rank for R : cycles: {{2, 8}}, net cycles: 0 . order: 6

$$[y_1, y_4, y_5, y_6, 0, 0, y_2, y_3]$$

$R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, 1/2, -3/16, -3/16], [0, 0, 0, 0, 5/16, -3/16], [0, 0, 0, 0, 5/16, -3/16], [0, 0, 1/2, -1/4, -3/16, 1/16], [1/2, -1/4, -1/8, -1/16, -1/16, 1/8], [1/2, -1/4, -1/8, -1/16, -1/16, 1/8], [0, 1/2, -1/4, -1/8, 1/16, -1/16], [0, 0, 0, 0, -3/16, 5/16] ] \$ \times \$ [ [1, 1, 1, 1, 0, 0, 2, 2], [1, 2, 1, 2, 0, 0, 0, 2], [2, 2, 1, 0, 0, 0, 0, 3], [0, 3, 2, 0, 0, 0, 0, 3], [0, 3, 0, 0, 0, 0, 0, 5], [0, 5, 0, 0, 0, 0, 0, 3] ] \$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: 0 . order: 6

$$[y_1, y_2, y_3, y_4, y_5, y_6, 0, 0]$$

$B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 0, 0, 0, 5/16, -3/16], [0, 0, 1/2, -1/4, -3/16, 1/16], [0, 0, 0, 1/2, -3/16, -3/16], [0, 0, 0, 0, 5/16, -3/16], [0, 1/2, -1/4, -1/8, 1/16, -1/16], [0, 0, 0, 0, -3/16, 5/16], [1/2, -1/4, -1/8, -1/16, -1/16, 1/8], [1/2, -1/4, -1/8, -1/16, -1/16, 1/8] ] \$ \times \$ [ [1, 1, 1, 1, 2, 2, 0, 0], [1, 2, 1, 2, 0, 2, 0, 0], [1, 0, 2, 2, 0, 3, 0, 0], [2, 0, 0, 3, 0, 3, 0, 0], [0, 0, 0, 3, 0, 5, 0, 0], [0, 0, 0, 5, 0, 3, 0, 0] ] \$$

Â» SYNC'D 77/1024 , 0.07519531250

75 . Coloring, {2, 4, 5, 6}

**R:** [3, 8, 1, 6, 2, 4, 5, 5] **B:** [6, 3, 8, 1, 7, 7, 4, 2]

‘ See graph

‘ ‘ See pair graph

Ω for A+τΔ :

‘ [ ‘ 6 ‘ ( ‘ 5 + 3τ <sup>2</sup> ‘ ) ‘ , 6 ‘ ( ‘ 5 + 4τ + 3τ <sup>2</sup> ‘ ) ‘ , 2 ‘ ( ‘ 5 + 2τ + τ <sup>2</sup> ‘ ) ‘ , 2 ‘ ( ‘ 5 - 2τ + τ <sup>2</sup> ‘ ) ‘ , 2 ‘ ( ‘ 1 + τ ‘ ) ‘ ( ‘ 5 + τ <sup>2</sup> ‘ ) ‘ , -2 ‘ ( ‘ - 5 + τ - τ <sup>2</sup> + τ <sup>3</sup> ‘ ) ‘ , -2 ‘ ( ‘ - 1 + τ ‘ ) ‘ ( ‘ 5 + 2τ + τ <sup>2</sup> ‘ ) ‘ , 2 ‘ ( ‘ 5 + 3τ + 3τ <sup>2</sup> + τ <sup>3</sup> ‘ ) ‘ ] ‘

For τ=1/2, [46, 62, 50, 34, 63, 37, 25, 59] . FixedPtCheck, [46, 62, 50, 34, 63, 37, 25, 59]

det(A + τ Δ) = 0

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

[1, 1, 1, 1, 1, 1, 1, 1]

+ \ ; - \ ; Δ

\$ [ [1, 1, 1, 1, 2, 1, 0, 1] , [2, 3, 2, 3, 1, 2, 1, 2] , [3, 3, 3, 5, 3, 5, 5, 5] , [3, 3, 4, 4, 5, 5, 4, 4] , [8, 9, 8, 9, 8, 9, 6, 7] , [15, 17, 15, 19, 13, 17, 15, 17] ] \$ \$ [ [1, 1, 1, 1, 0, 1, 2, 1] , [2, 1, 2, 1, 3, 2, 3, 2] , [5, 5, 5, 3, 5, 3, 3, 3] , [5, 5, 4, 4, 3, 3, 4, 4] , [8, 7, 8, 7, 8, 7, 10, 9] , [17, 15, 17, 13, 19, 15, 17, 15] ] \$ \$ [ [0, 0, 0, 0, 1, 0, -1, 0] , [0, 1, 0, 1, -1, 0, -1, 0] , [-1, -1, -1, 1, -1, 1, 1, 1] , [-1, -1, 0, 0, 1, 1, 0, 0] , [0, 1, 0, 1, 0, 1, -2, -1] , [-1, 1, -1, 3, -3, 1, -1, 1] ] \$

[-y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, y<sub>2</sub> - y<sub>3</sub> - y<sub>5</sub>, y<sub>1</sub>, y<sub>2</sub>, -2 y<sub>2</sub> - y<sub>4</sub> + y<sub>3</sub> + y<sub>5</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>]

p = s <sup>2</sup> + s <sup>3</sup> - 4s <sup>6</sup>

S+ \ ; S- \ ; NM

\$ [ [4, 3, 2, 3, 2, 4, 4, 2] , [5, 9, 7, 3, 7, 3, 5, 9] , [2, 3, 4, 3, 3, 3, 3, 3] , [7, 3, 5, 9, 7, 7, 5, 5] , [5, 7, 7, 5, 9, 5, 3, 7] , [3, 2, 3, 4, 2, 5, 4, 1] , [7, 5, 5, 7, 5, 5, 7, 7] , [3, 4, 3, 2, 3, 2, 3, 4] ] \$ \$ [ [4, 3, 2, 3, 3, 3, 3, 3] , [5, 9, 7, 3, 5, 5, 7, 7] , [2, 3, 4, 3, 4, 2, 2, 4] , [7, 3, 5, 9, 5, 9, 7, 3] , [5, 7, 7, 5, 7, 7, 5, 5] , [3, 2, 3, 4, 3, 4, 3, 2] , [7, 5, 5, 7, 3, 7, 9, 5] , [3, 4, 3, 2, 4, 1, 2, 5] ] \$ \$ [ [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] ] \$

CmmCk true, true, true

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 6	7 vs 7	7 vs 7	3 vs 7	4 vs 7

Omega Rank for R : cycles: {{1, 3}, {4, 6}, {2, 5, 8}}, net cycles: 3 . order: 6

\$ [ [1, 1, 1, 1, 2, 1, 0, 1] , [1, 2, 1, 1, 1, 1, 0, 1] , [1, 1, 1, 1, 1, 1, 0, 2] , [1, 1, 1, 1, 2, 1, 0, 1] , [1, 2, 1, 1, 1, 1, 0, 1] , [1, 1, 1, 1, 1, 1, 0, 2] , [1, 1, 1, 1, 2, 1, 0, 1] ] \$

[y<sub>2</sub>, 4 y<sub>2</sub> - y<sub>1</sub> - y<sub>3</sub>, y<sub>2</sub>, y<sub>2</sub>, y<sub>1</sub>, y<sub>2</sub>, 0, y<sub>3</sub>]



$$p = -s + s^7 \quad p = -s + s^4 \quad p' = -s + s^4 \quad p' = -s^2 + s^5$$

Omega Rank for B : cycles: {{1, 4, 6, 7}, {2, 3, 8}}, net cycles: 2 .

$$\$ [ [1, 1, 1, 1, 0, 1, 2, 1], [1, 1, 1, 2, 0, 1, 1, 1], [2, 1, 1, 1, 0, 1, 1, 1], [1, 1, 1, 1, 0, 2, 1, 1], [1, 1, 1, 1, 0, 1, 2, 1], [1, 1, 1, 2, 0, 1, 1, 1], [2, 1, 1, 1, 0, 1, 1, 1] ] \$$$

$$[5y_4 - y_1 - y_2 - y_3, y_4, y_4, y_1, 0, y_2, y_3, y_4]$$

$$p' = -s^2 + s^6 \quad p = -s + s^5 \quad p' = -s + s^5$$

Â» SYNC'D 19/2048 , 0.009277343750

76 . Coloring, {2, 4, 5, 7}

**R:** [3, 8, 1, 6, 2, 7, 4, 5]    **B:** [6, 3, 8, 1, 7, 4, 5, 2]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[ '1, 1, 1, 1, 1, 1, 1, 1 ' ]$$

For τ=1/2, [1, 1, 1, 1, 1, 1, 1, 1] . FixedPtCheck, [1, 1, 1, 1, 1, 1, 1, 1]

$$\det(A + \tau \Delta) = 1' ( ' \tau ' )' 2 ' ( ' 1 + \tau ' )' 2$$

Delta Range : [y2, -y1 - y3 - y5, -y2 - y4 - y6, y1, y3, y4, y5, y6]

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

$$\$ [ [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$ \quad \$ [ [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$ \quad \$ [ [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$$$

$$[0, 0, 0, 0, 0, 0, 0, 0]$$

$$p' = s^4 \quad p' = s^5 \quad p' = s^2 \quad p' = s \quad p' = s^3 \quad p = s$$

$$S+ \quad \backslash ; \quad S- \quad \backslash ; \quad NM$$

$$\$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \quad \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0] ]$$

0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0]] \$ \$ [ [7, 6, 6, 6, 6, 6, 6, 6], [6, 7, 6, 6, 6, 6, 6, 6], [6, 6, 7, 6, 6, 6, 6, 6], [6, 6, 6, 7, 6, 6, 6, 6], [6, 6, 6, 6, 7, 6, 6, 6], [6, 6, 6, 6, 6, 7, 6, 6], [6, 6, 6, 6, 6, 6, 7, 6], [6, 6, 6, 6, 6, 6, 6, 7]] \$

CmmCk true, true, true

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
0 vs 6	1 vs 8	1 vs 8	1 vs 8	1 vs 8

Omega Rank for R : cycles: {{1, 3}, {4, 6, 7}, {2, 5, 8}}, net cycles: 3 . order: 6

\$ [ [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1]] \$

[y<sub>1</sub>, y<sub>1</sub>, y<sub>1</sub>, y<sub>1</sub>, y<sub>1</sub>, y<sub>1</sub>, y<sub>1</sub>, y<sub>1</sub>]

p' = -1 + s p' = -1 + s<sup>2</sup> p' = -1 + s<sup>3</sup> p' = -1 + s<sup>4</sup> p' = -1 + s<sup>6</sup> p' = -1 + s<sup>7</sup> p' = -1 + s<sup>5</sup>

Omega Rank for B : cycles: {{5, 7}, {2, 3, 8}, {1, 4, 6}}, net cycles: 3 . order: 6

\$ [ [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1]] \$

[y<sub>1</sub>, y<sub>1</sub>, y<sub>1</sub>, y<sub>1</sub>, y<sub>1</sub>, y<sub>1</sub>, y<sub>1</sub>, y<sub>1</sub>]

p' = -1 + s<sup>4</sup> p' = -1 + s p' = -1 + s<sup>2</sup> p' = -1 + s<sup>7</sup> p' = -1 + s<sup>3</sup> p' = -1 + s<sup>5</sup> p' = -1 + s<sup>6</sup>

Â« NOT SYNC'D Â»

Nullspace of { $\Omega\Delta^i$ } :

[x<sub>1</sub>, x<sub>2</sub>, x<sub>6</sub>, x<sub>3</sub>, x<sub>4</sub>, x<sub>5</sub>]

For A+2 $\Delta$ : [-y<sub>1</sub> - y<sub>2</sub> - y<sub>3</sub> - y<sub>5</sub> - y<sub>4</sub> - y<sub>6</sub> - y<sub>7</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>5</sub>, y<sub>4</sub>, y<sub>6</sub>, y<sub>7</sub>]

For A-2 $\Delta$ : [-y<sub>1</sub> - y<sub>2</sub> - y<sub>3</sub> - y<sub>5</sub> - y<sub>4</sub> - y<sub>6</sub> - y<sub>7</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>, y<sub>7</sub>]

Range of { $\Omega\Delta^i$ }: [0, 0, 0, 0, 0, 0, 0, 0]

rank of M is 8 , rank of N is 8

M \ ; N

\$ [ [0, 1, 1, 1, 1, 1, 1, 1], [1, 0, 1, 1, 1, 1, 1, 1], [1, 1, 0, 1, 1, 1, 1, 1], [1, 1, 1, 0, 1, 1, 1, 1], [1, 1, 1, 1, 0, 1, 1, 1], [1, 1, 1, 1, 1, 0, 1, 1], [1, 1, 1, 1, 1, 1, 0, 1], [1, 1, 1, 1, 1, 1, 1, 0]] \$ \$ [ [0, 1, 1, 1, 1, 1, 1, 1], [1, 0, 1, 1, 1, 1, 1, 1], [1, 1, 0, 1, 1, 1, 1, 1], [1, 1, 1, 0, 1, 1, 1, 1], [1, 1, 1, 1, 0, 1, 1, 1], [1, 1, 1, 1, 1, 0, 1, 1], [1, 1, 1, 1, 1, 1, 0, 1], [1, 1, 1, 1, 1, 1, 1, 0]] \$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [0, 0, 0, 0, 0, 0, 0, 0]$

ker M,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range M,  $[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$

$\tau = 8$ ,  $r' = 7/8$

Ranges

Action of R on ranges,  $[[1]]$

Action of B on ranges,  $[[1]]$

$\beta(\{1, 2, 3, 4, 5, 6, 7, 8\}) = 1/1$

ker N,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range of N

$[y_5, y_4, y_2, y_3, y_1, y_7, y_8, y_6]$

Partitions

$\alpha(\{\{8\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}\}) = 1/1$

$b1 = \{8\}$ ,  $b2 = \{1\}$ ,  $b3 = \{2\}$ ,  $b4 = \{3\}$ ,  $b5 = \{4\}$ ,  $b6 = \{5\}$ ,  $b7 = \{6\}$ ,  $b8 = \{7\}$

Action of R and B on the blocks of the partitions:  $\$ [ [0, 0, 1, 1, 0, 0, 0, 0], [0, 0, 0, 1, 1, 0, 0, 0], [1, 0, 0, 0, 0, 1, 0, 0], [0, 1, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 1], [1, 0, 0, 0, 0, 0, 0, 1], [0, 1, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 1, 0] ] \$ = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0] ] \$ + \$ [ [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0] ] \$$

$['3', '4', '6', '2', '8', '1', '5', '7'], ['4', '5', '1', '3', '7', '8', '2', '6']$  with invariant measure  $[1, 1, 1, 1, 1, 1, 1, 1]$

N by blocks, check: *true*. ' See partition graph.

' ' See level-8 partition graph.

'



$[0, 0, 0, 0], [1, 0, -1, 0, 0, -1, 0, 1], [-1, 1, 1, 1, -1, -1, -1, 1], [0, 0, -1, 1, -1, 1, 0, 0], [-2, -1, 0, -1, 0, 1, 2, 1], [1, 1, -1, -3, 1, 1, 1, -1]$  ] \$

$$[y_3, -y_1 - 2y_5 + y_2 + y_4, -y_3 - y_2 - y_4, y_1, y_5 - y_2 - y_4, y_2, y_5, y_4]$$

$$p = s - 2s^3 - 8s^6$$

S+ \ ; S- \ ; NM  
 $\$ [ [3, 3, 3, 2, 2, 4, 3, 2], [3, 7, 9, 3, 5, 5, 5, 7], [3, 2, 3, 3, 2, 1, 3, 5], [9, 3, 3, 7, 7, 9, 3, 3], [5, 9, 5, 3, 7, 7, 5, 3], [2, 2, 3, 4, 1, 3, 5, 2], [5, 3, 5, 9, 7, 5, 5, 5], [3, 4, 2, 2, 4, 1, 2, 4] ] \$ \ [ [3, 3, 3, 2, 3, 5, 2, 1], [3, 7, 9, 3, 3, 3, 7, 9], [3, 2, 3, 3, 3, 2, 2, 4], [9, 3, 3, 7, 5, 7, 5, 5], [5, 9, 5, 3, 5, 5, 7, 5], [2, 2, 3, 4, 2, 4, 4, 1], [5, 3, 5, 9, 5, 3, 7, 7], [3, 4, 2, 2, 5, 2, 1, 3] ] \$ \ [ [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$$

CmmCk true, true, true

$\Delta$ -Rank	A+(1/2) $\Delta$	A-(1/2) $\Delta$	R	B
5 vs 6	7 vs 7	7 vs 7	4 vs 7	7 vs 7

Omega Rank for R : cycles: {{1, 3}, {2, 8}}, net cycles: 1 . order: 4

$\$ [ [1, 2, 1, 0, 1, 1, 1, 1], [1, 2, 1, 0, 1, 0, 1, 2], [1, 3, 1, 0, 1, 0, 0, 2], [1, 3, 1, 0, 0, 0, 0, 3], [1, 3, 1, 0, 0, 0, 0, 3], [1, 3, 1, 0, 0, 0, 0, 3], [1, 3, 1, 0, 0, 0, 0, 3] ] \$$

$$[y_1, 3y_1 - y_4, y_1, 0, 3y_1 - y_2 - y_3, y_2, y_4, y_3]$$

$$p' = s^5 - s^6 \quad p = s^4 - s^7 \quad p' = s^4 - s^6$$

Omega Rank for B : cycles: {{1, 4, 6}}, net cycles: 0 . order: 6

$$[y_1, 0, y_3, y_2, y_5, y_6, y_7, y_4]$$

$B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, 0, -5/8, 3/8, 3/8], [1, -1, 0, 0, 3/8, 3/8, -5/8], [0, 1, -1, 0, -5/8, 3/8, 3/8], [0, 0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, 1, -5/8, 3/8, -5/8], [0, 0, 0, 0, 3/8, -5/8, 3/8], [0, 0, 0, 0, 3/8, -5/8, 3/8], [0, 0, 1, -1, 3/8, -5/8, 3/8] ] \$ \times \$ [ [1, 0, 1, 2, 1, 1, 1, 1], [2, 0, 0, 2, 1, 1, 1, 1], [2, 0, 0, 2, 1, 2, 1, 0], [2, 0, 0, 3, 0, 2, 1, 0], [3, 0, 0, 3, 0, 2, 0, 0], [3, 0, 0, 2, 0, 3, 0, 0], [2, 0, 0, 3, 0, 3, 0, 0] ] \$$

Â» SYNC'D 135/8192 , 0.01647949219

78 . Coloring, {2, 4, 6, 7}

**R:** [3, 8, 1, 6, 7, 4, 4, 5]    **B:** [6, 3, 8, 1, 2, 7, 5, 2]

‘ See graph

‘ ‘ See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ ' 1 ' ( ' 1 + \tau ' ) ' ( ' 5 - \tau + 3\tau^2 + \tau^3 ' ) ' , 1 ' ( ' - 1 + \tau ' ) ' ^2 ' ( ' 5 + 2\tau + \tau^2 ' ) ' , 3 ' ( ' 5 - 2\tau + 8\tau^2 \\ & + 2\tau^3 + 3\tau^4 ' ) ' , 3 ' ( ' 5 + 3\tau^2 ' ) ' ( ' 1 + \tau ' ) ' ^2 , 1 ' ( ' 1 + \tau ' ) ' ( ' - 1 + \tau ' ) ' ( ' - 5 + \tau - \tau^2 + \tau^3 ' \\ & ) ' , 1 ' ( ' 1 + \tau^2 ' ) ' ( ' 1 + \tau ' ) ' ( ' 5 + 2\tau + \tau^2 ' ) ' , -1 ' ( ' 5 + 3\tau + 3\tau^2 + \tau^3 ' ) ' ( ' 1 + \tau ' ) ' ( ' - 1 + \tau \\ & ' ) ' , -1 ' ( ' - 1 + \tau ' ) ' ( ' 5 + 2\tau^2 + \tau^4 ' ) ' ] ' \end{aligned}$$

For  $\tau=1/2$ , [258, 50, 206, 414, 111, 375, 177, 89] . FixedPtCheck, [258, 50, 206, 414, 111, 375, 177, 89]

$$\det(A + \tau \Delta) = 0$$

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

\$ [ [1, 0, 1, 2, 1, 1, 1, 1] , [1, 2, 3, 2, 2, 3, 2, 1] , [5, 5, 3, 5, 3, 5, 3, 3] , [3, 5, 4, 4, 4, 4, 3, 5] , [8, 7, 6, 7, 10, 9, 8, 9] , [15, 13, 17, 17, 17, 15, 17, 17] ] \$ \$ [ [1, 2, 1, 0, 1, 1, 1, 1] , [3, 2, 1, 2, 2, 1, 2, 3] , [3, 3, 5, 3, 5, 3, 5, 5] , [5, 3, 4, 4, 4, 4, 5, 3] , [8, 9, 10, 9, 6, 7, 8, 7] , [17, 19, 15, 15, 15, 17, 15, 15] ] \$ \$ [ [0, -1, 0, 1, 0, 0, 0, 0] , [-1, 0, 1, 0, 0, 1, 0, -1] , [1, 1, -1, 1, -1, 1, -1, -1] , [-1, 1, 0, 0, 0, -1, 1] , [0, -1, -2, -1, 2, 1, 0, 1] , [-1, -3, 1, 1, 1, -1, 1, 1] ] \$

$$[-y_3 - y_4 - y_1, -y_2 - 2y_5 - y_3 - y_4, y_1, y_2, y_5 + y_3 + y_4, y_3, y_5, y_4]$$

$$p = s - 2s^3 - 8s^6$$

$$S+ \quad \backslash ; \quad S- \quad \backslash ; \quad NM$$

\$ [ [3, 3, 3, 2, 3, 5, 2, 1] , [3, 7, 9, 3, 3, 3, 7, 9] , [3, 2, 3, 3, 3, 2, 2, 4] , [9, 3, 3, 7, 5, 7, 5, 5] , [5, 9, 5, 3, 5, 5, 7, 5] , [2, 2, 3, 4, 2, 4, 4, 1] , [5, 3, 5, 9, 5, 3, 7, 7] , [3, 4, 2, 2, 5, 2, 1, 3] ] \$ \$ [ [3, 3, 3, 2, 2, 4, 3, 2] , [3, 7, 9, 3, 5, 5, 5, 7] , [3, 2, 3, 3, 2, 1, 3, 5] , [9, 3, 3, 7, 7, 9, 3, 3] , [5, 9, 5, 3, 7, 7, 5, 3] , [2, 2, 3, 4, 1, 3, 5, 2] , [5, 3, 5, 9, 7, 5, 5, 5] , [3, 4, 2, 2, 4, 1, 2, 4] ] \$ \$ [ [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] ] \$

CmmCk true, true, true

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 6	7 vs 7	7 vs 7	4 vs 7	7 vs 7

Omega Rank for R : cycles: {{1, 3}, {4, 6}}, net cycles: 1 . order: 4

$\$ [ [1, 0, 1, 2, 1, 1, 1, 1], [1, 0, 1, 2, 1, 2, 1, 0], [1, 0, 1, 3, 0, 2, 1, 0], [1, 0, 1, 3, 0, 3, 0, 0], [1, 0, 1, 3, 0, 3, 0, 0], [1, 0, 1, 3, 0, 3, 0, 0], [1, 0, 1, 3, 0, 3, 0, 0] ] \$$

$[y_1, 0, y_1, 3y_1 - y_2, y_2, 3y_1 - y_3 - y_4, y_3, y_4]$

$$p' = -s^4 + s^5 \quad p = s^4 - s^5 \quad p' = -s^4 + s^6$$

Omega Rank for B : cycles: {{2, 3, 8}}, net cycles: 0 . order: 6

$[y_1, y_2, y_3, 0, y_4, y_5, y_6, y_7]$

$B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 1, -1, 0, -5/8, 3/8, 3/8], [0, 0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, 0, -5/8, 3/8, 3/8], [1, -1, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, 0, 3/8, -5/8, 3/8], [0, 0, 1, -1, 3/8, -5/8, 3/8], [0, 0, 0, 1, -5/8, 3/8, -5/8], [0, 0, 0, 0, 3/8, -5/8, 3/8] ] \$ \times \$ [ [1, 2, 1, 0, 1, 1, 1, 1], [0, 2, 2, 0, 1, 1, 1, 1], [0, 2, 2, 0, 1, 0, 1, 2], [0, 3, 2, 0, 1, 0, 0, 2], [0, 3, 3, 0, 0, 0, 0, 2], [0, 2, 3, 0, 0, 0, 0, 3], [0, 3, 2, 0, 0, 0, 0, 3] ] \$$

$\hat{A} \gg \text{SYNC'D } 135/8192, 0.01647949219$

79 . Coloring, {2, 4, 6, 8}

**R:** [3, 8, 1, 6, 7, 4, 5, 2]    **B:** [6, 3, 8, 1, 2, 7, 4, 5]

' See graph

' ' See pair graph

,

$\Omega$  for  $A + \tau \Delta$  :

' [ '1, 1, 1, 1, 1, 1, 1, 1' ]'

For  $\tau=1/2$ , [1, 1, 1, 1, 1, 1, 1, 1] . FixedPtCheck, [1, 1, 1, 1, 1, 1, 1, 1]

$$\det(A + \tau \Delta) = 1' (\tau')^2 (\tau + 1)^2$$

Delta Range :  $[y_2, -y_1 - y_3 - y_5, -y_2 - y_4 - y_6, y_1, y_3, y_4, y_5, y_6]$

[1, 1, 1, 1, 1, 1, 1, 1]

+ \ ; - \ ;  $\Delta$

\$ [ [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$ \$ [ [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$ \$ [ [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$

[0, 0, 0, 0, 0, 0, 0, 0]

$p' = s^5$   $p' = s^4$   $p' = s^3$   $p' = s^2$   $p' = s$   $p = s$   
 $S^+$  \ ;  $S^-$  \ ; NM

\$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \$ [ [7, 6, 6, 6, 6, 6, 6, 6], [6, 7, 6, 6, 6, 6, 6, 6], [6, 6, 7, 6, 6, 6, 6, 6], [6, 6, 6, 7, 6, 6, 6, 6], [6, 6, 6, 6, 7, 6, 6, 6], [6, 6, 6, 6, 6, 7, 6, 6], [6, 6, 6, 6, 6, 6, 7, 6], [6, 6, 6, 6, 6, 6, 6, 7] ] \$

CmmCk true, true, true

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
0 vs 6	1 vs 8	1 vs 8	1 vs 8	1 vs 8

Omega Rank for R : cycles: {{1, 3}, {2, 8}, {4, 6}, {5, 7}}, net cycles: 4 . order: 2

\$ [ [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$

$[y_1, y_1, y_1, y_1, y_1, y_1, y_1, y_1]$

$p' = -1 + s$   $p' = -1 + s^2$   $p' = -1 + s^3$   $p' = -1 + s^4$   $p' = -1 + s^5$   $p' = -1 + s^6$   $p' = -1 + s^7$

Omega Rank for B : cycles: {{2, 3, 5, 8}, {1, 4, 6, 7}}, net cycles: 2 . order: 4

\$ [ [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$

$[y_1, y_1, y_1, y_1, y_1, y_1, y_1, y_1]$

$p' = -1 + s^3$   $p' = -1 + s^4$   $p' = -1 + s^5$   $p' = -1 + s^6$   $p' = -1 + s^2$   $p' = -1 + s^7$

$\hat{A} \ll \text{NOT SYNC'D } \hat{A} \gg$

Nullspace of  $\{\Omega\Delta^i\}$  :

$[x_1, x_2, x_3, x_4, x_5, x_6]$

For  $A+2\Delta$  :  $[-y_1 - y_2 - y_3 - y_4 - y_5 - y_6 - y_7, y_1, y_2, y_3, y_4, y_5, y_6, y_7]$

For  $A-2\Delta$  :  $[y_7, -y_1 - y_2 - y_3 - y_4 - y_5 - y_6 - y_7, y_1, y_2, y_3, y_4, y_5, y_6]$



Range of  $\{\Omega\Delta^i\}$ :  $[0, 0, 0, 0, 0, 0, 0, 0]$

rank of M is 8 , rank of N is 8

M \ ; N

$\$ [ [0, 1, 1, 1, 1, 1, 1, 1], [1, 0, 1, 1, 1, 1, 1, 1], [1, 1, 0, 1, 1, 1, 1, 1], [1, 1, 1, 0, 1, 1, 1, 1], [1, 1, 1, 1, 0, 1, 1, 1], [1, 1, 1, 1, 1, 0, 1, 1], [1, 1, 1, 1, 1, 1, 0, 1], [1, 1, 1, 1, 1, 1, 1, 0] ] \$$   $\$ [ [0, 1, 1, 1, 1, 1, 1, 1], [1, 0, 1, 1, 1, 1, 1, 1], [1, 1, 0, 1, 1, 1, 1, 1], [1, 1, 1, 0, 1, 1, 1, 1], [1, 1, 1, 1, 0, 1, 1, 1], [1, 1, 1, 1, 1, 0, 1, 1], [1, 1, 1, 1, 1, 1, 0, 1], [1, 1, 1, 1, 1, 1, 1, 0] ] \$$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [0, 0, 0, 0, 0, 0, 0, 0]$

ker M,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range M,  $[x_3, x_2, x_1, x_7, x_5, x_6, x_4, x_8]$

$\tau = 8$  ,  $r' = 7/8$

Ranges

Action of R on ranges,  $[[1]]$

Action of B on ranges,  $[[1]]$

$\beta(\{1, 2, 3, 4, 5, 6, 7, 8\}) = 1/1$

ker N,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range of N

$[y_1, y_2, y_3, y_4, y_5, y_8, y_6, y_7]$

Partitions

$\alpha(\{\{8\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}\}) = 1/1$

$b1 = \{8\}$  ‘ , ‘  $b2 = \{1\}$  ‘ , ‘  $b3 = \{2\}$  ‘ , ‘  $b4 = \{3\}$  ‘ , ‘  $b5 = \{4\}$  ‘ , ‘  $b6 = \{5\}$  ‘ , ‘  $b7 = \{6\}$  ‘ , ‘  $b8 = \{7\}$

Action of R and B on the blocks of the partitions:  $\$ [ [0, 0, 1, 1, 0, 0, 0, 0], [0, 0, 0, 1, 1, 0, 0, 0], [1, 0, 0, 0, 1, 0, 0, 0], [0, 1, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 1], [1, 0, 0, 0, 0, 0, 0, 1], [0, 1, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 1, 0] ] \$ = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0] ] \$ + \$ [ [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0] ] \$$

$[‘3’, ‘4’, ‘1’, ‘2’, ‘7’, ‘8’, ‘5’, ‘6’], [‘4’, ‘5’, ‘6’, ‘3’, ‘8’, ‘1’, ‘2’, ‘7’]$  with invariant measure  $[1, 1, 1, 1, 1, 1, 1, 1]$

N by blocks, check: true . ‘ See partition graph.

‘ ‘ See level-8 partition graph.

<b>Right Group</b>	
<b>Coloring</b>	{2, 4, 6, 8}
<b>Rank</b>	8
<b>R,B</b>	[3, 8, 1, 6, 7, 4, 5, 2], [6, 3, 8, 1, 2, 7, 4, 5]
$\pi_2$	[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
$u_2$	[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] (dim 4)
<b>wpp</b>	[1, 1, 1, 1, 1, 1, 1, 1]
$\pi_8$	[1]
$u_8$	[1]

80 . Coloring, {2, 4, 7, 8}

**R:** [3, 8, 1, 6, 7, 7, 4, 2]   **B:** [6, 3, 8, 1, 2, 4, 5, 5]

‘ See graph

‘ ‘ See pair graph

$\Omega$  for  $A+\tau\Delta$  :

[ ‘2‘ (‘5 + 2 $\tau$  +  $\tau^2$ ‘)‘ , 2‘ (‘5 - 2 $\tau$  +  $\tau^2$ ‘)‘ , 6‘ (‘5 + 3 $\tau^2$ ‘)‘ , 6‘ (‘5 + 4 $\tau$  + 3 $\tau^2$ ‘)‘ , -2‘ (‘ - 1 +  $\tau$ ‘)‘ (‘5 + 2 $\tau$  +  $\tau^2$ ‘)‘ , 2‘ (‘5 + 3 $\tau$  + 3 $\tau^2$  +  $\tau^3$ ‘)‘ , 2‘ (‘1 +  $\tau$ ‘)‘ (‘5 +  $\tau^2$ ‘)‘ , -2‘ (‘ - 5 +  $\tau$  -  $\tau^2$  +  $\tau^3$ ‘)‘ ]‘

For  $\tau=1/2$ , [50, 34, 46, 62, 25, 59, 63, 37] . FixedPtCheck, [50, 34, 46, 62, 25, 59, 63, 37]

$\det(A + \tau \Delta) = 0$

Delta Range : [ $y_2, -y_1 - y_3 - y_5, -y_2 - y_4 - y_6, y_1, y_3, y_4, y_5, y_6$ ]

[1, 1, 1, 1, 1, 1, 1, 1]

+ \; - \; \Delta

\$ [ [1, 1, 1, 1, 0, 1, 2, 1] , [2, 3, 2, 3, 1, 2, 1, 2] , [3, 5, 3, 3, 5, 5, 3, 5] , [4, 4, 3, 3, 4, 4, 5, 5] , [8, 9, 8, 9, 6, 7, 8, 9] , [15, 19, 15, 17, 15, 17, 13, 17] ] \$ \$ [ [1, 1, 1, 1, 2, 1, 0, 1] , [2, 1, 2, 1, 3, 2, 3, 2] , [5, 3, 5, 5, 3, 3, 5, 3] , [4, 4, 5, 5, 4, 4, 3, 3] , [8, 7, 8, 7, 10, 9, 8, 7] , [17, 13, 17, 15, 17, 15, 19, 15] ] \$ \$ [ [0, 0, 0, 0, -1, 0, 1, 0] , [0, 1, 0, 1, -1, 0, -1, 0] , [-1, 1, -1, -1, 1, 1, -1, 1] , [0, 0, -1, -1, 0, 0, 1, 1] , [0, 1, 0, 1, -2, -1, 0, 1] , [-1, 3, -1, 1, -1, 1, -3, 1] ] \$

$$[-y_1 - y_3 - y_5, y_3 + y_5 + y_2, y_1, y_2, -y_3 - y_5 - 2y_2 - y_4, y_3, y_4, y_5]$$

$$p = s^2 + s^3 - 4s^6$$

S+ \ ; S- \ ; NM  
 \$ [ [4, 3, 2, 3, 3, 3, 3, 3] , [5, 9, 7, 3, 5, 5, 7, 7] , [2, 3, 4, 3, 4, 2, 2, 4] , [7, 3, 5, 9, 5, 9, 7, 3] , [5, 7, 7, 5, 7, 7, 5, 5] , [3, 2, 3, 4, 3, 4, 3, 2] , [7, 5, 5, 7, 3, 7, 9, 5] , [3, 4, 3, 2, 4, 1, 2, 5] ] \$ \$ [ [4, 3, 2, 3, 2, 4, 4, 2] , [5, 9, 7, 3, 7, 3, 5, 9] , [2, 3, 4, 3, 3, 3, 3, 3] , [7, 3, 5, 9, 7, 7, 5, 5] , [5, 7, 7, 5, 9, 5, 3, 7] , [3, 2, 3, 4, 2, 5, 4, 1] , [7, 5, 5, 7, 5, 5, 7, 7] , [3, 4, 3, 2, 3, 2, 3, 4] ] \$ \$ [ [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] ] \$  
 CmmCk true, true, true

$\Delta$ -Rank	A+(1/2) $\Delta$	A-(1/2) $\Delta$	R	B
5 vs 6	7 vs 7	7 vs 7	3 vs 7	4 vs 7

Omega Rank for R : cycles: {{1, 3}, {2, 8}, {4, 6, 7}}, net cycles: 3 . order: 6

\$ [ [1, 1, 1, 1, 0, 1, 2, 1] , [1, 1, 1, 2, 0, 1, 1, 1] , [1, 1, 1, 1, 0, 2, 1, 1] , [1, 1, 1, 1, 0, 1, 2, 1] , [1, 1, 1, 2, 0, 1, 1, 1] , [1, 1, 1, 1, 0, 2, 1, 1] , [1, 1, 1, 1, 0, 1, 2, 1] ] \$

$$[y_3, y_3, y_3, 4y_3 - y_1 - y_2, 0, y_1, y_2, y_3]$$

$$p' = -s^2 + s^5 \quad p = -s + s^4 \quad p' = -s + s^4 \quad p = -s + s^7$$

Omega Rank for B : cycles: {{2, 3, 5, 8}, {1, 4, 6}}, net cycles: 2 .

\$ [ [1, 1, 1, 1, 2, 1, 0, 1] , [1, 2, 1, 1, 1, 1, 0, 1] , [1, 1, 2, 1, 1, 1, 0, 1] , [1, 1, 1, 1, 1, 1, 0, 2] , [1, 1, 1, 1, 2, 1, 0, 1] , [1, 2, 1, 1, 1, 1, 0, 1] , [1, 1, 2, 1, 1, 1, 0, 1] ] \$

$$[y_3, 5y_3 - y_1 - y_2 - y_4, y_1, y_3, y_2, y_3, 0, y_4]$$

$$p' = s^2 - s^6 \quad p = -s + s^5 \quad p' = s - s^5$$

Â» SYNC'D 19/2048 , 0.009277343750

81 . Coloring, {2, 5, 6, 7}

R: [3, 8, 1, 1, 2, 4, 4, 5] B: [6, 3, 8, 6, 7, 7, 5, 2]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{bmatrix} 1' (1 + \tau)' (5 + 2\tau + \tau^2)' , 1' (5 - \tau + 3\tau^2 + \tau^3)' , 1' (5 + 3\tau + 7\tau^2 + \tau^3)' , -1' \\ (5 + \tau)' (1 + \tau)' (-1 + \tau)' , -1' (-5 + 3\tau - 7\tau^2 + \tau^3)' , 1' (1 + \tau)' (-5 + \tau)' (-1 \\ + \tau)' , -1' (5 - 2\tau + \tau^2)' (-1 + \tau)' , -1' (-5 - \tau - 3\tau^2 + \tau^3)' \end{bmatrix}$$

For  $\tau=1/2$ , [75, 43, 67, 33, 41, 27, 17, 49] . FixedPtCheck, [75, 43, 67, 33, 41, 27, 17, 49]

$$\det(A + \tau \Delta) = 1' (\tau)' ^2 (1 + \tau)' ^2 (-1 + \tau)' ^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	3 vs 6	3 vs 6

Omega Rank for R : cycles: {{1, 3}, {2, 5, 8}}, net cycles: 1 . order: 6

$$\$ [ [2, 1, 1, 2, 1, 0, 0, 1] , [3, 1, 2, 0, 1, 0, 0, 1] , [2, 1, 3, 0, 1, 0, 0, 1] , [3, 1, 2, 0, 1, 0, 0, 1] , [2, 1, 3, 0, 1, 0, 0, 1] , [3, 1, 2, 0, 1, 0, 0, 1] ] \$$$

$$[5 y_3 - y_1 - y_2, y_3, y_1, y_2, y_3, 0, 0, y_3]$$

$$p = -s^2 + s^4 \quad p' = -s^2 + s^4 \quad p = -s^2 + s^6$$

Omega Rank for B : cycles: {{5, 7}, {2, 3, 8}}, net cycles: 1 . order: 6

$$\$ [ [0, 1, 1, 0, 1, 2, 2, 1] , [0, 1, 1, 0, 2, 0, 3, 1] , [0, 1, 1, 0, 3, 0, 2, 1] , [0, 1, 1, 0, 2, 0, 3, 1] , [0, 1, 1, 0, 3, 0, 2, 1] , [0, 1, 1, 0, 2, 0, 3, 1] ] \$$$

$$[0, y_3, y_3, 0, 5 y_3 - y_1 - y_2, y_1, y_2, y_3]$$

$$p = -s^2 + s^6 \quad p' = s^2 - s^4 \quad p = -s^2 + s^4$$

Â» SYNC'D 1125/32768 , 0.03433227539

82 . Coloring, {2, 5, 6, 8}

**R:** [3, 8, 1, 1, 2, 4, 5, 2]    **B:** [6, 3, 8, 6, 7, 7, 4, 5]

‘ See graph

‘ ‘ See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ 4(5 + 2\tau^2 + \tau^4)^{(1+\tau)}, 4(1+\tau)^2(5 - \tau - \tau^2 + \tau^3), 4(1+\tau)^2(5 - 3\tau + \tau^2 + \tau^3), \\ & -4(5 - 2\tau + 2\tau^2 + 2\tau^3 + \tau^4)^{(-1+\tau)}, 4(-5 + \tau^2)^{(1+\tau)^{(-1+\tau)}}, -4(5 - \tau + 3\tau^2 + \tau^3)^{(-1+\tau)}, \\ & 4(-1+\tau)^2(5 + 2\tau + \tau^2), 4(1+\tau)^2(5 - 2\tau + \tau^2) ] \end{aligned}$$

For  $\tau=1/2$ , [267, 315, 279, 77, 114, 86, 50, 306] . FixedPtCheck, [267, 315, 279, 77, 114, 86, 50, 306]

$$\det(A + \tau \Delta) = 1(\tau)^2(1+\tau)^2(-1+\tau)^2$$

$\Delta$ -Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	2 vs 6	6 vs 6

Omega Rank for R : cycles: {{1, 3}, {2, 8}}, net cycles: 0 . order: 2

$$\begin{aligned} \$ [ [2, 2, 1, 1, 1, 0, 0, 1], [2, 2, 2, 0, 0, 0, 0, 2], [2, 2, 2, 0, 0, 0, 0, 2], [2, 2, 2, 0, 0, 0, 0, 2], [2, 2, 2, 0, 0, \\ 0, 0, 2], [2, 2, 2, 0, 0, 0, 0, 2] ] \$ \end{aligned}$$

$$[y_2 + y_1, y_2 + y_1, y_2, y_1, y_1, 0, 0, y_2]$$

$$p = -s^2 + s^6 \quad p = -s^2 + s^3 \quad p = -s^2 + s^4 \quad p = -s^2 + s^5$$

Omega Rank for B : cycles: {{4, 6, 7}}, net cycles: 0 . order: 6

$$[0, 0, y_5, y_6, y_1, y_2, y_3, y_4]$$

$$\begin{aligned} B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, \\ 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [0, 0, 0, 0, 0, 0, 0, 0] \\ , [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, \\ 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, 3/8, -5/8, 3/8], [1, -1, 0, -5/8, 11/8, \\ -5/8], [0, 1, -1, -5/8, -5/8, 11/8], [0, 0, 0, 3/8, -5/8, 3/8], [0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, 3/8, 3/8, -5/8], \\ [0, 0, 0, -5/8, 3/8, 3/8], [0, 0, 1, 3/8, -5/8, -5/8] ] \$ \times \$ [ [0, 0, 1, 1, 1, 2, 2, 1], [0, 0, 0, 2, 1, 1, 3, 1], [0, 0, \\ 0, 3, 1, 2, 2, 0], [0, 0, 0, 2, 0, 3, 3, 0], [0, 0, 0, 3, 0, 2, 3, 0], [0, 0, 0, 3, 0, 3, 2, 0] ] \$ \end{aligned}$$

Â» SYNC'D 1409/65536 , 0.02149963379

83 . Coloring, {2, 5, 7, 8}

**R:** [3, 8, 1, 1, 2, 7, 4, 2]    **B:** [6, 3, 8, 6, 7, 4, 5, 5]

‘ See graph

‘ ‘ See pair graph

Ω for A+τΔ :

$$[1^4(1+\tau)^4, 1^4(1+\tau)^4, 1^4(1+\tau)^4, -1^4(-1+\tau)^4, -1^4(-1+\tau)^4, -1^4(-1+\tau)^4, -1^4(-1+\tau)^4, 1^4(1+\tau)^4]$$

For τ=1/2, [3, 3, 3, 1, 1, 1, 1, 3] . FixedPtCheck, [3, 3, 3, 1, 1, 1, 1, 3]

$$\det(A + \tau \Delta) = 1^4(-1 + \tau)^2 (\tau)^2 (1 + \tau)^2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	4 vs 6	4 vs 6

Omega Rank for R : cycles: {{1, 3}, {2, 8}}, net cycles: 1 . order: 4

$$\$ [ [2, 2, 1, 1, 0, 0, 1, 1], [2, 1, 2, 1, 0, 0, 0, 2], [3, 2, 2, 0, 0, 0, 0, 1], [2, 1, 3, 0, 0, 0, 0, 2], [3, 2, 2, 0, 0, 0, 0, 1], [2, 1, 3, 0, 0, 0, 0, 2] ] \$$$

$$[y_3, 3y_1 + 3y_4 - 4y_2, y_1, y_4, 0, 0, -y_3 + 4y_1 + 4y_4 - 5y_2, y_2]$$

$$p = -s^3 + s^5 \quad p' = -s^3 + s^5$$

Omega Rank for B : cycles: {{4, 6}, {5, 7}}, net cycles: 1 . order: 4

$$\$ [ [0, 0, 1, 1, 2, 2, 1, 1], [0, 0, 0, 2, 2, 1, 2, 1], [0, 0, 0, 1, 3, 2, 2, 0], [0, 0, 0, 2, 2, 1, 3, 0], [0, 0, 0, 1, 3, 2, 2, 0], [0, 0, 0, 2, 2, 1, 3, 0] ] \$$$

$$[0, 0, y_1, 3y_1 + 3y_2 - 4y_3, y_2, y_3, y_4, 4y_1 + 4y_2 - 5y_3 - y_4]$$

$$p = -s^3 + s^5 \quad p' = -s^3 + s^5$$

Â» SYNC'D 99/16384 , 0.006042480469

84 . Coloring, {2, 6, 7, 8}

**R:** [3, 8, 1, 1, 7, 4, 4, 2]    **B:** [6, 3, 8, 6, 2, 7, 5, 5]

‘ See graph

‘ ‘ See pair graph

Ω for A+τΔ :

$$[4^4(1+\tau)^4(5+4\tau+6\tau^2+\tau^4)^4, 4^4(5-3\tau+10\tau^2+2\tau^3+\tau^4+\tau^5)^4, 4^4(5+3\tau+16\tau^2+4\tau^3+3\tau^4+\tau^5)^4, -4^4(-1+\tau)^4(1+\tau)^2(5+\tau^2)^4, -4^4(-1+\tau)^4(5-\tau+3\tau^2+\tau^3)^4, -4^4(-1+\tau)^4(1+\tau)^4(5+2\tau+\tau^2)^4, -4^4(-1+\tau)^4(1+\tau)^4(5-$$

$$2\tau + \tau^2, 4(5 + 10\tau^2 + \tau^4)$$

For  $\tau=1/2$ , [411, 203, 359, 189, 86, 150, 102, 242]. FixedPtCheck, [411, 203, 359, 189, 86, 150, 102, 242]

$$\det(A + \tau \Delta) = 1(\tau)^2(-1 + \tau)^2(1 + \tau)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	3 vs 6	6 vs 6

Omega Rank for R : cycles: {{1, 3}, {2, 8}}, net cycles: 1 . order: 4

$$\$ [ [2, 1, 1, 2, 0, 0, 1, 1], [3, 1, 2, 1, 0, 0, 0, 1], [3, 1, 3, 0, 0, 0, 0, 1], [3, 1, 3, 0, 0, 0, 0, 1], [3, 1, 3, 0, 0, 0, 0, 1], [3, 1, 3, 0, 0, 0, 0, 1], [3, 1, 3, 0, 0, 0, 0, 1] ] \$$$

$$[3y_2 - y_1, y_2, 3y_2 - y_3, y_3, 0, 0, y_1, y_2]$$

$$p = -s^3 + s^4 \quad p = -s^3 + s^6 \quad p = -s^3 + s^5$$

Omega Rank for B : cycles: {{2, 3, 5, 8}}, net cycles: 0 . order: 4

$$[0, y_1, y_2, 0, y_5, y_6, y_4, y_3]$$

$$\mathbf{B} = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [0, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [1/2, -1/4, -3/32, 5/32, -3/32, -3/32], [0, 0, -3/32, 5/32, 13/32, -11/32], [0, 0, -11/32, -3/32, 5/32, 13/32], [1/2, -1/4, -3/32, 5/32, -3/32, -3/32], [0, 0, 5/32, 13/32, -11/32, -3/32], [0, 1/2, -11/32, -3/32, 5/32, -3/32], [0, 0, 13/32, -11/32, -3/32, 5/32], [0, 0, 13/32, -11/32, -3/32, 5/32] ] \$ \times \$ [ [0, 1, 1, 0, 2, 2, 1, 1], [0, 2, 1, 0, 2, 0, 2, 1], [0, 2, 2, 0, 3, 0, 0, 1], [0, 3, 2, 0, 1, 0, 0, 2], [0, 1, 3, 0, 2, 0, 0, 2], [0, 2, 1, 0, 2, 0, 0, 3] ] \$$$

Â» SYNC'D 59/4096 , 0.01440429688

85 . Coloring, {3, 4, 5, 6}

**R**: [3, 3, 8, 6, 2, 4, 5, 5]    **B**: [6, 8, 1, 1, 7, 7, 4, 2]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[ -1(-1 + \tau), 1(1 + \tau), 1(1 + \tau), -1(-1 + \tau), 1(1 + \tau), -1(-1 + \tau), -1(-1 + \tau), 1(1 + \tau) ]$$

For  $\tau=1/2$ ,  $[1, 3, 3, 1, 3, 1, 1, 3]$  . FixedPtCheck,  $[1, 3, 3, 1, 3, 1, 1, 3]$

$$\det(A + \tau \Delta) = 0$$

Delta Range :  $[y_2, -y_1 - y_3 - y_5, -y_2 - y_4 - y_6, y_1, y_3, y_4, y_5, y_6]$

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

$\$ [ [0, 1, 2, 1, 2, 1, 0, 1], [1, 3, 1, 3, 1, 3, 1, 3], [2, 1, 2, 3, 2, 3, 2, 1], [3, 5, 3, 5, 3, 5, 3, 5], [4, 3, 4, 5, 4, 5, 4, 3], [7, 9, 7, 9, 7, 9, 7, 9] ] \$ [ [2, 1, 0, 1, 0, 1, 2, 1], [3, 1, 3, 1, 3, 1, 3, 1], [2, 3, 2, 1, 2, 1, 2, 3], [5, 3, 5, 3, 5, 3, 5, 3], [4, 5, 4, 3, 4, 3, 4, 5], [9, 7, 9, 7, 9, 7, 9, 7] ] \$ [ [-1, 0, 1, 0, 1, 0, -1, 0], [-1, 1, -1, 1, -1, 1, -1, 1], [0, -1, 0, 1, 0, 1, 0, -1], [-1, 1, -1, 1, -1, 1, -1, 1], [0, -1, 0, 1, 0, 1, 0, -1], [-1, 1, -1, 1, -1, 1, -1, 1] ] \$$

$$[y_2, y_3, -y_1 - y_2 - y_3, y_1, -y_1 - y_2 - y_3, y_1, y_2, y_3]$$

$$p = s^2 - 4s^6$$

$$S+ \quad \backslash ; \quad S- \quad \backslash ; \quad NM$$

$\$ [ [7, 5, 3, 5, 3, 5, 7, 5], [5, 7, 5, 3, 5, 3, 5, 7], [3, 5, 7, 5, 7, 5, 3, 5], [5, 3, 5, 7, 5, 7, 5, 3], [3, 5, 7, 5, 7, 5, 3, 5], [5, 3, 5, 7, 5, 7, 5, 3], [7, 5, 3, 5, 3, 5, 7, 5], [5, 7, 5, 3, 5, 3, 5, 7] ] \$ [ [7, 5, 3, 5, 3, 5, 7, 5], [5, 7, 5, 3, 5, 3, 5, 7], [3, 5, 7, 5, 7, 5, 3, 5], [5, 3, 5, 7, 5, 7, 5, 3], [3, 5, 7, 5, 7, 5, 3, 5], [5, 3, 5, 7, 5, 7, 5, 3], [7, 5, 3, 5, 3, 5, 7, 5], [5, 7, 5, 3, 5, 3, 5, 7] ] \$ [ [9, 6, 5, 4, 4, 5, 0, 3], [6, 9, 4, 3, 5, 6, 3, 0], [5, 4, 9, 6, 0, 3, 4, 5], [4, 3, 6, 9, 3, 0, 5, 6], [4, 5, 0, 3, 9, 6, 5, 4], [5, 6, 3, 0, 6, 9, 4, 3], [0, 3, 4, 5, 5, 4, 9, 6], [3, 0, 5, 6, 4, 3, 6, 9] ] \$$

CmmCk true, true, true

$$p' = s^2 - 2s^4 \quad p' = s^3 - 2s^5$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
3 vs 6	3 vs 6	3 vs 6	2 vs 6	2 vs 6

Omega Rank for R : cycles:  $\{\{2, 3, 5, 8\}, \{4, 6\}\}$ , net cycles: 2 . order: 4

$\$ [ [0, 1, 2, 1, 2, 1, 0, 1], [0, 2, 1, 1, 1, 1, 0, 2], [0, 1, 2, 1, 2, 1, 0, 1], [0, 2, 1, 1, 1, 1, 0, 2], [0, 1, 2, 1, 2, 1, 0, 1], [0, 2, 1, 1, 1, 1, 0, 2] ] \$$

$$[0, -y_2 + 3y_1, y_2, y_1, y_2, y_1, 0, -y_2 + 3y_1]$$

$$p = -s + s^5 \quad p' = -s + s^5 \quad p = -s + s^3 \quad p' = -s + s^3$$

Omega Rank for B : cycles:  $\{\{2, 8\}, \{1, 4, 6, 7\}\}$ , net cycles: 2 . order: 4

$\$ [ [2, 1, 0, 1, 0, 1, 2, 1], [1, 1, 0, 2, 0, 2, 1, 1], [2, 1, 0, 1, 0, 1, 2, 1], [1, 1, 0, 2, 0, 2, 1, 1], [2, 1, 0, 1, 0, 1, 2, 1], [1, 1, 0, 2, 0, 2, 1, 1] ] \$$



$$[y_2, y_1, 0, -y_2 + 3y_1, 0, -y_2 + 3y_1, y_2, y_1]$$

$$p = -s + s^3 \quad p' = -s + s^3 \quad p = -s + s^5 \quad p' = -s + s^5$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[0, x_1, x_2, x_3, -2x_2, -4x_1 - 2x_3]$$

$$\text{For } A+2\Delta: [9y_5 + 9y_1 - y_3, -3y_5 - 3y_1 - y_4, y_5, -3y_5 - 3y_1 - y_2, y_1, y_2, y_3, y_4]$$

$$\text{For } A-2\Delta: [y_1, -3y_1 - 3y_4 - y_5, 9y_1 + 9y_4 - y_2, -3y_1 - y_3 - 3y_4, y_2, y_3, y_4, y_5]$$

$$\text{Range of } \{\Omega\Delta^i\}: [\mu_2, \mu_3, -\mu_2 - \mu_3 - \mu_1, \mu_1, -\mu_2 - \mu_3 - \mu_1, \mu_1, \mu_2, \mu_3]$$

rank of M is 8 , rank of N is 5

$$M \quad \setminus ; \quad N$$

$$\begin{aligned} & \$ [ [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], \\ & [0, 0, 0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \quad \$ [ [0, 3, 4, 5, 5, 4, 9, 6], \\ & [3, 0, 5, 6, 4, 3, 6, 9], [4, 5, 0, 3, 9, 6, 5, 4], [5, 6, 3, 0, 6, 9, 4, 3], [5, 4, 9, 6, 0, 3, 4, 5], [4, 3, 6, 9, 3, 0, 5, \\ & 6], [9, 6, 5, 4, 4, 5, 0, 3], [6, 9, 4, 3, 5, 6, 3, 0] ] \$ \end{aligned}$$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [-1, 0, 1, 0, 1, 0, -1, 0]$

ker M,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range M,  $[x_8, x_7, x_6, x_5, x_4, x_3, x_2, x_1]$

$$\tau = 32, r' = 1/2$$

Ranges

Action of R on ranges,  $[[3], [3], [2], [4]]$

Action of B on ranges,  $[[4], [2], [1], [1]]$

$$\beta(\{1, 7\}) = 1/4$$

$$\beta(\{2, 8\}) = 1/4$$

$$\beta(\{3, 5\}) = 1/4$$

$$\beta(\{4, 6\}) = 1/4$$

ker N,  $[\mu_2, \mu_1, -\mu_2 - \mu_3 - \mu_1, \mu_3, -\mu_2 - \mu_3 - \mu_1, \mu_3, \mu_2, \mu_1]$

Range of N

$$[y_5 + y_2 - y_1, y_4, y_5, y_5 + y_2 - y_3, y_2, y_3, y_1, y_5 + y_2 - y_4]$$

Partitions

Action of R on partitions, [[2], [3], [4], [6], [6], [2]]  
 Action of B on partitions, [[5], [1], [6], [6], [4], [1]]

$$\alpha(\{1, 3, 4, 8\}, \{2, 5, 6, 7\}) = 2/9$$

$$\alpha(\{1, 2, 3, 6\}, \{4, 5, 7, 8\}) = 2/9$$

$$\alpha(\{3, 6, 7, 8\}, \{1, 2, 4, 5\}) = 1/9$$

$$\alpha(\{1, 2, 3, 4\}, \{5, 6, 7, 8\}) = 1/9$$

$$\alpha(\{2, 3, 4, 7\}, \{1, 5, 6, 8\}) = 1/9$$

$$\alpha(\{3, 4, 7, 8\}, \{1, 2, 5, 6\}) = 2/9$$

b1 = {3, 6, 7, 8} ‘ , ‘ b2 = {1, 2, 3, 6} ‘ , ‘ b3 = {1, 2, 3, 4} ‘ , ‘ b4 = {1, 3, 4, 8} ‘ , ‘ b5 = {2, 3, 4, 7} ‘ , ‘  
 b6 = {2, 5, 6, 7} ‘ , ‘ b7 = {4, 5, 7, 8} ‘ , ‘ b8 = {5, 6, 7, 8} ‘ , ‘ b9 = {1, 5, 6, 8} ‘ , ‘ b10 = {3, 4, 7, 8} ‘ , ‘  
 b11 = {1, 2, 5, 6} ‘ , ‘ b12 = {1, 2, 4, 5}

Action of R and B on the blocks of the partitions: \$ [ [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0] , [0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0] , [0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0] ] \$ + \$ [ [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] ] \$  
 [‘3‘, C, B, ‘2‘, B, ‘7‘, ‘1‘, A, A, ‘2‘, ‘7‘, ‘8‘], [B, ‘4‘, A, ‘5‘, ‘8‘, ‘9‘, ‘6‘, B, ‘3‘, ‘6‘, ‘4‘, A] with invariant measure [1, 2, 1, 2, 1, 2, 2, 1, 1, 2, 2, 1]

N by blocks, check: true . ‘ See partition graph.

‘ ‘ See level-2 partition graph.

‘

Sandwich	
<b>Coloring</b>	{3, 4, 5, 6}
<b>Rank</b>	2
<b>R,B</b>	[3, 3, 8, 6, 2, 4, 5, 5], [6, 8, 1, 1, 7, 7, 4, 2]
$\pi_2$	[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]
$u_2$	[3, 4, 5, 5, 4, 9, 6, 5, 6, 4, 3, 6, 9, 3, 9, 6, 5, 4, 6, 9, 4, 3, 3, 4, 5, 5, 6, 3] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4]

86 . Coloring, {3, 4, 5, 7}

**R:** [3, 3, 8, 6, 2, 7, 4, 5]   **B:** [6, 8, 1, 1, 7, 4, 5, 2]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

‘ [ ‘ -2‘ (‘ - 1 +  $\tau$  ‘ )‘ (‘ 5 +  $\tau$  ‘<sup>2</sup> ‘ )‘ , 2‘ (‘ 5 +  $\tau$  +  $\tau$  ‘<sup>2</sup> +  $\tau$  ‘<sup>3</sup> ‘ )‘ , 2‘ (‘ 1 +  $\tau$  ‘ )‘ (‘ 5 - 2 $\tau$  +  $\tau$  ‘<sup>2</sup> ‘ )‘ , -2‘ (‘ - 5 + 3 $\tau$  - 3 $\tau$  ‘<sup>2</sup> +  $\tau$  ‘<sup>3</sup> ‘ )‘ , 6‘ (‘ 5 + 3 $\tau$  ‘<sup>2</sup> ‘ )‘ , 6‘ (‘ 5 - 4 $\tau$  + 3 $\tau$  ‘<sup>2</sup> ‘ )‘ , 2‘ (‘ 5 - 2 $\tau$  +  $\tau$  ‘<sup>2</sup> ‘ )‘ , 2‘ (‘ 5 + 2 $\tau$  +  $\tau$  ‘<sup>2</sup> ‘ )‘ ]‘

For  $\tau=1/2$ , [21, 47, 51, 33, 46, 30, 34, 50] . FixedPtCheck, [21, 47, 51, 33, 46, 30, 34, 50]

$\det(A + \tau \Delta) = 0$

Delta Range : [  $y_2, -y_1 - y_3 - y_5, -y_2 - y_4 - y_6, y_1, y_3, y_4, y_5, y_6$  ]

[1, 1, 1, 1, 1, 1, 1, 1]

+ \ ; - \ ;  $\Delta$

\$ [ [0, 1, 2, 1, 1, 1, 1, 1] , [1, 2, 1, 2, 2, 3, 2, 3] , [5, 3, 3, 3, 5, 5, 5, 3] , [5, 5, 4, 4, 3, 3, 4, 4] , [8, 7, 10, 9, 8, 7, 8, 7] , [13, 17, 15, 17, 15, 17, 15, 19] ] \$ \$ [ [2, 1, 0, 1, 1, 1, 1, 1] , [3, 2, 3, 2, 2, 1, 2, 1] , [3, 5, 5, 5, 3, 3, 3, 5] , [3, 3, 4, 4, 5, 5, 4, 4] , [8, 9, 6, 7, 8, 9, 8, 9] , [19, 15, 17, 15, 17, 15, 17, 13] ] \$ \$ [ [-1, 0, 1, 0, 0, 0, 0, 0] , [-1, 0, -1, 0, 0, 1, 0, 1] , [1, -1, -1, -1, 1, 1, 1, -1] , [1, 1, 0, 0, -1, -1, 0, 0] , [0, -1, 2, 1, 0, -1, 0, -1] , [-3, 1, -1, 1, -1, 1, -1, 3] ] \$

[  $-y_1 - y_3 - y_5, -y_2 - y_3 + y_5, y_1, y_2, -y_4 + y_3 - y_5, y_3, y_4, y_5$  ]

$p = s^2 - s^3 - 4s^6$

S+ \ ; S- \ ; NM

\$ [ [9, 5, 3, 7, 5, 7, 7, 5] , [2, 5, 4, 1, 3, 2, 3, 4] , [5, 5, 7, 7, 7, 5, 5, 7] , [3, 2, 3, 4, 3, 4, 3, 2] , [2, 4, 4, 2, 4, 3, 2, 3] , [7, 3, 5, 9, 5, 9, 7, 3] , [3, 3, 3, 3, 2, 3, 4, 3] , [7, 7, 5, 5, 7, 3, 5, 9] ] \$ \$ [ [7, 7, 5, 5, 5, 7, 7, 5] , [3, 4, 3, 2, 3, 2, 3, 4] , [3, 7, 9, 5, 7, 5, 5, 7] , [4, 1, 2, 5, 3, 4, 3, 2] , [3, 3, 3, 3, 4, 3, 2, 3] , [5, 5, 7, 7, 5, 9, 7, 3] , [4, 2, 2, 4, 2, 3, 4, 3] , [5, 9, 7, 3, 7, 3, 5, 9] ] \$ \$ [ [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] ] \$

CmmCk true, true, true

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
5 vs 6	7 vs 7	7 vs 7	4 vs 7	3 vs 7

Omega Rank for R : cycles:  $\{\{2, 3, 5, 8\}, \{4, 6, 7\}\}$ , net cycles: 2 .

$\$ [ [0, 1, 2, 1, 1, 1, 1, 1], [0, 1, 1, 1, 1, 1, 1, 2], [0, 1, 1, 1, 2, 1, 1, 1], [0, 2, 1, 1, 1, 1, 1, 1], [0, 1, 2, 1, 1, 1, 1, 1], [0, 1, 1, 1, 1, 1, 1, 2], [0, 1, 1, 1, 2, 1, 1, 1] ] \$$

$$[0, -y_1 + 5y_3 - y_2 - y_4, y_1, y_3, y_2, y_3, y_3, y_4]$$

$$p = -s + s^5 \quad p' = -s + s^5 \quad p'' = -s^2 + s^6$$

Omega Rank for B : cycles:  $\{\{2, 8\}, \{1, 4, 6\}, \{5, 7\}\}$ , net cycles: 3 . order: 6

$\$ [ [2, 1, 0, 1, 1, 1, 1, 1], [1, 1, 0, 1, 1, 2, 1, 1], [1, 1, 0, 2, 1, 1, 1, 1], [2, 1, 0, 1, 1, 1, 1, 1], [1, 1, 0, 1, 1, 2, 1, 1], [1, 1, 0, 2, 1, 1, 1, 1], [2, 1, 0, 1, 1, 1, 1, 1] ] \$$

$$[4y_3 - y_1 - y_2, y_3, 0, y_1, y_3, y_2, y_3, y_3]$$

$$p = -s + s^4 \quad p' = -s + s^4 \quad p'' = -s^2 + s^5 \quad p''' = -s + s^7$$

$\hat{A} \gg \text{SYNC'D } 19/2048, 0.009277343750$

87 . Coloring,  $\{3, 4, 5, 8\}$

**R:**  $[3, 3, 8, 6, 2, 7, 5, 2]$  **B:**  $[6, 8, 1, 1, 7, 4, 4, 5]$

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ \text{' -3' ( '5 - 2}\tau + 8\tau^2 + 2\tau^3 + 3\tau^4 \text{' )' ( ' - 1 + } \tau \text{' )' , 3' ( ' 5 - 4}\tau + 3\tau^2 \text{' )' ( ' 1 + } \tau \text{' )' }^3 \text{ , 3' ( ' 5 + } \\ & 2\tau + 8\tau^2 - 2\tau^3 + 3\tau^4 \text{' )' ( ' 1 + } \tau \text{' )' , -3' ( ' 5 + 4}\tau + 3\tau^2 \text{' )' ( ' - 1 + } \tau \text{' )' }^3 \text{ , -1' ( ' 1 + } \tau \text{' )' }^2 \text{ ( ' 5 } \\ & - 2\tau + \tau^2 \text{' )' ( ' - 1 + } \tau \text{' )' , -1' ( ' - 5 - } \tau - 3\tau^2 + \tau^3 \text{' )' ( ' - 1 + } \tau \text{' )' }^2 \text{ , 1' ( ' 1 + } \tau \text{' )' ( ' 5 + 2}\tau + \tau^2 \\ & \text{' )' ( ' - 1 + } \tau \text{' )' }^2 \text{ , 1' ( ' 5 - } \tau + 3\tau^2 + \tau^3 \text{' )' ( ' 1 + } \tau \text{' )' }^2 \text{ ]' } \end{aligned}$$

For  $\tau=1/2$ ,  $[103, 405, 381, 31, 153, 49, 75, 387]$  . FixedPtCheck,  $[103, 405, 381, 31, 153, 49, 75, 387]$

$$\det(A + \tau \Delta) = 0$$

Delta Range :  $[y_2, -y_1 - y_3 - y_5, -y_2 - y_4 - y_6, y_1, y_3, y_4, y_5, y_6]$

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

+ \; - \;  $\Delta$

\$ [ [0, 2, 2, 0, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$ \$ [ [2, 0, 0, 2, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$ \$ [ [-1, 1, 1, -1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$

[y<sub>1</sub>, -y<sub>1</sub>, -y<sub>1</sub>, y<sub>1</sub>, 0, 0, 0, 0]

p' = s<sup>2</sup> p' = s<sup>3</sup> p' = s<sup>4</sup> p' = s<sup>5</sup> p = s<sup>2</sup>

S+ \; S- \; NM

\$ [ [0, 0, 0, 1, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0, 0, 1], [0, 1, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 1, 1, 0], [0, 0, 1, 1, 0, 0, 0, 0], [1, 0, 0, 0, 1, 0, 0, 0], [1, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 1, 0] ] \$ \$ [ [1, 0, 0, 0, 0, 0, 1, 0], [1, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 1, 0, 0, 0], [0, 0, 1, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 1], [0, 0, 0, 1, 0, 1, 0, 0], [0, 0, 0, 0, 0, 1, 1, 0], [0, 1, 0, 0, 0, 0, 0, 1] ] \$ \$ [ [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$

CmmCk true, true, true

$\Delta$ -Rank	A+(1/2) $\Delta$	A-(1/2) $\Delta$	R	B
1 vs 6	6 vs 6	6 vs 6	6 vs 6	6 vs 6

Omega Rank for R : cycles: {{2, 3, 8}}, net cycles: 0 . order: 6

[0, y<sub>1</sub>, y<sub>2</sub>, 0, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \$ x \$ [ [0, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, -5/8, 3/8, 3/8], [1, -1, 0, -5/8, 3/8, 3/8], [0, 0, 0, 3/8, -5/8, 3/8], [0, 1, -1, 3/8, -5/8, 3/8], [0, 0, 1, -5/8, 3/8, -5/8], [0, 0, 0, 3/8, -5/8, 3/8] ] \$ x \$ [ [0, 2, 2, 0, 1, 1, 1, 1], [0, 2, 2, 0, 1, 0, 1, 2], [0, 3, 2, 0, 1, 0, 0, 2], [0, 3, 3, 0, 0, 0, 0, 2], [0, 2, 3, 0, 0, 0, 0, 3], [0, 3, 2, 0, 0, 0, 0, 3] ] \$

Omega Rank for B : cycles: {{1, 4, 6}}, net cycles: 0 . order: 6

[y<sub>1</sub>, 0, 0, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, -5/8, 3/8, 3/8], [1, -1, 0, -5/8, 3/8, 3/8], [0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 1, -5/8, 3/8, -5/8], [0, 0, 0, 3/8, -5/8, 3/8], [0, 0, 0, 3/8, -5/8, 3/8], [0, 1, -1, 3/8, -5/8, 3/8] ] \$ x \$ [ [2, 0, 0, 2, 1, 1, 1, 1], [2, 0, 0, 2, 1, 2, 1, 0], [2, 0, 0, 3, 0, 2, 1, 0], [3, 0, 0, 3, 0, 2, 0, 0], [3, 0, 0, 2, 0, 3, 0, 0], [2, 0, 0, 3, 0, 3, 0, 0] ] \$

Â» SYNC'D 15/512 , 0.02929687500

88 . Coloring, {3, 4, 6, 7}

**R:** [3, 3, 8, 6, 7, 4, 4, 5]    **B:** [6, 8, 1, 1, 2, 7, 5, 2]

' See graph

' ' See pair graph

Ω for A+τΔ :

' [ '-1' (' - 1 + τ ' )'' (' 5 - τ + 3τ <sup>2</sup> + τ <sup>3</sup> ' )'' (' 1 + τ ' )' , 1' (' - 1 + τ ' )' <sup>2</sup> ' (' 5 + 3τ + 7τ <sup>2</sup> + τ <sup>3</sup> ' )' , 1' (' - 1 + τ ' )'' (' - 5 - τ - 3τ <sup>2</sup> + τ <sup>3</sup> ' )'' (' 1 + τ ' )' , -1' (' 1 + τ ' )' <sup>2</sup> ' (' - 5 + 3τ - 7τ <sup>2</sup> + τ <sup>3</sup> ' )' , 3' (' - 1 + τ ' )'' (' 1 + τ ' )'' (' - 5 + τ - 7τ <sup>2</sup> + 3τ <sup>3</sup> ' )' , 3' (' 5 - 2τ + 8τ <sup>2</sup> + 2τ <sup>3</sup> + 3τ <sup>4</sup> ' )'' (' 1 + τ ' )' , -3' (' - 1 + τ ' )'' (' 5 + τ + 7τ <sup>2</sup> + 3τ <sup>3</sup> ' )'' (' 1 + τ ' )' , -3' (' - 1 + τ ' )'' (' 5 + 2τ + 8τ <sup>2</sup> - 2τ <sup>3</sup> + 3τ <sup>4</sup> ' )'' ]'

For τ=1/2, [129, 67, 147, 369, 141, 309, 183, 127] . FixedPtCheck, [129, 67, 147, 369, 141, 309, 183, 127]

det(A + τ Δ) = 0

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

[1, 1, 1, 1, 1, 1, 1, 1]

+ \ ; - \ ; Δ

\$ [ [0, 0, 2, 2, 1, 1, 1, 1] , [0, 1, 0, 1, 1, 2, 1, 2] , [3, 1, 1, 3, 3, 3, 1, 1] , [1, 1, 1, 1, 1, 1, 1, 1] , [1, 1, 1, 1, 1, 1, 1, 1] , [1, 1, 1, 1, 1, 1, 1, 1] ] \$ \$ [ [2, 2, 0, 0, 1, 1, 1, 1] , [2, 1, 2, 1, 1, 0, 1, 0] , [1, 3, 3, 1, 1, 1, 3, 3] , [1, 1, 1, 1, 1, 1, 1, 1] , [1, 1, 1, 1, 1, 1, 1, 1] , [1, 1, 1, 1, 1, 1, 1, 1] ] \$ \$ [ [-1, -1, 1, 1, 0, 0, 0, 0] , [-1, 0, -1, 0, 0, 1, 0, 1] , [1, -1, -1, 1, 1, 1, -1, -1] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] ] \$

[y<sub>3</sub>, y<sub>2</sub>, y<sub>3</sub> - 2y<sub>2</sub> - 4y<sub>1</sub>, -y<sub>2</sub>, y<sub>1</sub>, y<sub>2</sub> - y<sub>3</sub> + 3y<sub>1</sub>, -y<sub>1</sub>, y<sub>2</sub> - y<sub>3</sub> + y<sub>1</sub>]

p = s <sup>4</sup>

S+ \ ; S- \ ; NM

\$ [ [1, 1, 1, 1, 0, 1, 2, 1] , [1, 1, 1, 1, 1, 0, 1, 2] , [1, 1, 1, 1, 2, 1, 0, 1] , [1, 1, 1, 1, 1, 2, 1, 0] , [0, 1, 2, 1, 1, 1, 1, 1] , [1, 0, 1, 2, 1, 1, 1, 1] , [2, 1, 0, 1, 1, 1, 1, 1] , [1, 2, 1, 0, 1, 1, 1, 1] ] \$ \$ [ [1, 1, 1, 1, 0, 1, 2, 1] , [1, 1, 1, 1, 1, 0, 1, 2] , [1, 1, 1, 1, 2, 1, 0, 1] , [1, 1, 1, 1, 1, 2, 1, 0] , [0, 1, 2, 1, 1, 1, 1, 1] , [1, 0, 1, 2, 1, 1, 1, 1] , [2, 1, 0, 1, 1, 1, 1, 1] , [1, 2, 1, 0, 1, 1, 1, 1] ] \$ \$ [ [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] ] \$

CmmCk true, true, true

p' = s <sup>5</sup>    p' = s <sup>4</sup>

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
3 vs 6	6 vs 6	6 vs 6	6 vs 6	6 vs 6

Omega Rank for R : cycles:  $\{\{4, 6\}\}$ , net cycles: 0 . order: 6

$$[0, 0, y_1, y_2, y_3, y_4, y_5, y_6]$$

$$\begin{aligned} \mathbf{R} = \$ [ & [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, \\ & 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [0, 0, 0, 0, 0, 0, 0, 0] \\ & , [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, \\ & 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [1/2, -1/4, -1/8, -1/16, -1/16, 1/8], [1/2, -1/4, \\ & -1/8, -1/16, -1/16, 1/8], [0, 1/2, -1/4, -1/8, 1/16, -1/16], [0, 0, 0, 0, -3/16, 5/16], [0, 0, 0, 1/2, -3/16, -3/16] \\ & , [0, 0, 0, 0, 5/16, -3/16], [0, 0, 0, 0, 5/16, -3/16], [0, 0, 1/2, -1/4, -3/16, 1/16] ] \$ \times \$ [ [0, 0, 2, 2, 1, 1, 1, \\ & 1], [0, 0, 0, 2, 1, 2, 1, 2], [0, 0, 0, 3, 2, 2, 1, 0], [0, 0, 0, 3, 0, 3, 2, 0], [0, 0, 0, 5, 0, 3, 0, 0], [0, 0, 0, 3, 0, \\ & 5, 0, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles:  $\{\{2, 8\}\}$ , net cycles: 0 . order: 6

$$[y_1, y_2, 0, 0, y_3, y_4, y_6, y_5]$$

$$\begin{aligned} \mathbf{B} = \$ [ & [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, \\ & 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0] \\ & , [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, \\ & 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 1/2, -1/4, -1/8, 1/16, -1/16], [0, 0, 0, 0, \\ & -3/16, 5/16], [1/2, -1/4, -1/8, -1/16, -1/16, 1/8], [1/2, -1/4, -1/8, -1/16, -1/16, 1/8], [0, 0, 0, 0, 5/16, -3/16] \\ & , [0, 0, 1/2, -1/4, -3/16, 1/16], [0, 0, 0, 1/2, -3/16, -3/16], [0, 0, 0, 0, 5/16, -3/16] ] \$ \times \$ [ [2, 2, 0, 0, 1, 1, \\ & 1, 1], [0, 2, 0, 0, 1, 2, 1, 2], [0, 3, 0, 0, 1, 0, 2, 2], [0, 3, 0, 0, 2, 0, 0, 3], [0, 5, 0, 0, 0, 0, 0, 3], [0, 3, 0, 0, \\ & 0, 0, 0, 5] ] \$ \end{aligned}$$

$\hat{A}$ » SYNC'D 77/1024 , 0.07519531250

89 . Coloring,  $\{3, 4, 6, 8\}$

**R:**  $[3, 3, 8, 6, 7, 4, 5, 2]$  **B:**  $[6, 8, 1, 1, 2, 7, 4, 5]$

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} [ & -2^{\cdot} (-1 + \tau^{\cdot})^{\cdot} (5 + 2\tau + \tau^2)^{\cdot}, 2^{\cdot} (5 + 3\tau + 3\tau^2 + \tau^3)^{\cdot}, 2^{\cdot} (1 + \tau^{\cdot})^{\cdot} (5 + \tau^2)^{\cdot}, \\ & -2^{\cdot} (-5 + \tau - \tau^2 + \tau^3)^{\cdot}, 2^{\cdot} (5 + 2\tau + \tau^2)^{\cdot}, 2^{\cdot} (5 - 2\tau + \tau^2)^{\cdot}, 6^{\cdot} (5 + 3\tau^2)^{\cdot}, 6^{\cdot} (5 + 4\tau \\ & + 3\tau^2)^{\cdot} ]^{\cdot} \end{aligned}$$

For  $\tau=1/2$ , [25, 59, 63, 37, 50, 34, 46, 62] . FixedPtCheck, [25, 59, 63, 37, 50, 34, 46, 62]

$$\det(A + \tau \Delta) = 0$$

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

\$ [ [0, 1, 2, 1, 1, 1, 1, 1], [1, 2, 1, 2, 2, 3, 2, 3], [5, 5, 3, 5, 3, 5, 3, 3], [4, 4, 5, 5, 4, 4, 3, 3], [6, 7, 8, 9, 8, 9, 8, 9], [15, 17, 13, 17, 15, 19, 15, 17] ] \$ \$ [ [2, 1, 0, 1, 1, 1, 1, 1], [3, 2, 3, 2, 2, 1, 2, 1], [3, 3, 5, 3, 5, 3, 5, 5], [4, 4, 3, 3, 4, 4, 5, 5], [10, 9, 8, 7, 8, 7, 8, 7], [17, 15, 19, 15, 17, 13, 17, 15] ] \$ \$ [ [-1, 0, 1, 0, 0, 0, 0, 0], [-1, 0, -1, 0, 0, 1, 0, 1], [1, 1, -1, 1, -1, 1, -1, -1], [0, 0, 1, 1, 0, 0, -1, -1], [-2, -1, 0, 1, 0, 1, 0, 1], [-1, 1, -3, 1, -1, 3, -1, 1] ] \$

$$[-y_1 - 2y_3 - y_5 - y_4, -y_5 - y_4 - y_2, y_1, y_2, y_5, y_3, y_4, y_3 + y_5 + y_4]$$

$$p = s^2 + s^3 - 4s^6$$

$$S+ \quad \backslash ; \quad S- \quad \backslash ; \quad NM$$

\$ [ [7, 7, 5, 5, 5, 7, 7, 5], [3, 4, 3, 2, 3, 2, 3, 4], [3, 7, 9, 5, 7, 5, 5, 7], [4, 1, 2, 5, 3, 4, 3, 2], [3, 3, 3, 3, 4, 3, 2, 3], [5, 5, 7, 7, 5, 9, 7, 3], [4, 2, 2, 4, 2, 3, 4, 3], [5, 9, 7, 3, 7, 3, 5, 9] ] \$ \$ [ [9, 5, 3, 7, 5, 7, 7, 5], [2, 5, 4, 1, 3, 2, 3, 4], [5, 5, 7, 7, 7, 5, 5, 7], [3, 2, 3, 4, 3, 4, 3, 2], [2, 4, 4, 2, 4, 3, 2, 3], [7, 3, 5, 9, 5, 9, 7, 3], [3, 3, 3, 3, 2, 3, 4, 3], [7, 7, 5, 5, 7, 3, 5, 9] ] \$ \$ [ [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$

CmmCk true, true, true

$\Delta$ -Rank	A+(1/2) $\Delta$	A-(1/2) $\Delta$	R	B
5 vs 6	7 vs 7	7 vs 7	3 vs 7	4 vs 7

Omega Rank for R : cycles: {{4, 6}, {2, 3, 8}, {5, 7}}, net cycles: 3 . order: 6

\$ [ [0, 1, 2, 1, 1, 1, 1, 1], [0, 1, 1, 1, 1, 1, 1, 2], [0, 2, 1, 1, 1, 1, 1, 1], [0, 1, 2, 1, 1, 1, 1, 1], [0, 1, 1, 1, 1, 1, 1, 2], [0, 2, 1, 1, 1, 1, 1, 1], [0, 1, 2, 1, 1, 1, 1, 1] ] \$

$$[0, y_1, -y_1 + 4y_3 - y_2, y_3, y_3, y_3, y_3, y_2]$$

$$p' = s^2 - s^5 \quad p' = s^3 - s^6 \quad p' = s - s^4 \quad p = s - s^7$$

Omega Rank for B : cycles: {{1, 4, 6, 7}, {2, 5, 8}}, net cycles: 2 .

\$ [ [2, 1, 0, 1, 1, 1, 1, 1], [1, 1, 0, 1, 1, 2, 1, 1], [1, 1, 0, 1, 1, 1, 2, 1], [1, 1, 0, 2, 1, 1, 1, 1], [2, 1, 0, 1, 1, 1, 1, 1], [1, 1, 0, 1, 1, 2, 1, 1], [1, 1, 0, 1, 1, 1, 2, 1] ] \$

$$[5y_4 - y_1 - y_2 - y_3, y_4, 0, y_1, y_4, y_2, y_3, y_4]$$



$$p = -s + s^5 \quad p' = -s + s^5 \quad p'' = s^2 - s^6$$

Â» SYNC'D 19/2048 , 0.009277343750

90 . Coloring, {3, 4, 7, 8}

**R:** [3, 3, 8, 6, 7, 7, 4, 2]    **B:** [6, 8, 1, 1, 2, 4, 5, 5]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$\begin{aligned} & [ '-1' (' - 1 + \tau ')'' (' 5 + 2\tau + \tau^2 ')', 1' (' 5 - \tau + 3\tau^2 + \tau^3 ')', 1' (' 1 + \tau ')'' (' 5 - 2\tau + \tau^2 ')', \\ & )', -1' (' - 5 - \tau - 3\tau^2 + \tau^3 ')', -1' (' - 1 + \tau ')'' (' 5 + 2\tau + \tau^2 ')', 1' (' 5 - \tau + 3\tau^2 + \tau^3 ')', 1' (' 1 + \tau ')'' (' 5 - 2\tau + \tau^2 ')', \\ & -1' (' - 5 - \tau - 3\tau^2 + \tau^3 ')']' \end{aligned}$$

For τ=1/2, [25, 43, 51, 49, 25, 43, 51, 49] . FixedPtCheck, [25, 43, 51, 49, 25, 43, 51, 49]

$$\det(A + \tau \Delta) = 0$$

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

\$ [ [0, 1, 2, 1, 0, 1, 2, 1] , [1, 3, 1, 3, 1, 3, 1, 3] , [2, 3, 2, 1, 2, 3, 2, 1] , [5, 3, 5, 3, 5, 3, 5, 3] , [4, 3, 4, 5, 4, 3, 4, 5] , [7, 9, 7, 9, 7, 9, 7, 9] ] \$ \$ [ [2, 1, 0, 1, 2, 1, 0, 1] , [3, 1, 3, 1, 3, 1, 3, 1] , [2, 1, 2, 3, 2, 1, 2, 3] , [3, 5, 3, 5, 3, 5, 3, 5] , [4, 5, 4, 3, 4, 5, 4, 3] , [9, 7, 9, 7, 9, 7, 9, 7] ] \$ \$ [ [-1, 0, 1, 0, -1, 0, 1, 0] , [-1, 1, -1, 1, -1, 1, -1, 1] , [0, 1, 0, -1, 0, 1, 0, -1] , [1, -1, 1, -1, 1, -1, 1, -1] , [0, -1, 0, 1, 0, -1, 0, 1] , [-1, 1, -1, 1, -1, 1, -1, 1] ] \$

$$[-y_3 - y_1 - y_2, y_1, y_2, y_3, -y_3 - y_1 - y_2, y_1, y_2, y_3]$$

$$p' = s^2 + 2s^4 \quad p'' = s^3 + 2s^5 \quad p''' = s^2 - 4s^6$$

S+        \ ;        S-        \ ;        NM  
 \$ [ [7, 5, 3, 5, 3, 5, 7, 5] , [5, 7, 5, 3, 5, 3, 5, 7] , [3, 5, 7, 5, 7, 5, 3, 5] , [5, 3, 5, 7, 5, 7, 5, 3] , [3, 5, 7, 5, 7, 5, 3, 5] , [5, 3, 5, 7, 5, 7, 5, 3] , [7, 5, 3, 5, 3, 5, 7, 5] , [5, 7, 5, 3, 5, 3, 5, 7] ] \$ \$ [ [7, 5, 3, 5, 3, 5, 7, 5] , [5, 7, 5, 3, 5, 3, 5, 7] , [3, 5, 7, 5, 7, 5, 3, 5] , [5, 3, 5, 7, 5, 7, 5, 3] , [3, 5, 7, 5, 7, 5, 3, 5] , [5, 3, 5, 7, 5, 7, 5, 3] , [7, 5, 3, 5, 3, 5, 7, 5] , [5, 7, 5, 3, 5, 3, 5, 7] ] \$ \$ [ [5, 4, 1, 2, 0, 1, 4, 3] , [4, 5, 2, 3, 1, 0, 3, 2] , [1, 2, 5, 4, 4, 3, 0, 1] , [2, 3, 4, 5, 3, 2, 1, 0] , [0, 1, 4, 3, 5, 4, 1, 2] , [1, 0, 3, 2, 4, 5, 2, 3] , [4, 3, 0, 1, 1, 2, 5, 4] , [3, 2, 1, 0, 2, 3, 4, 5] ] \$

CmmCk true, true, true

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
3 vs 6	3 vs 6	3 vs 6	3 vs 6	3 vs 6

Omega Rank for R : cycles:  $\{\{4, 6, 7\}, \{2, 3, 8\}\}$ , net cycles: 2 . order: 3

$\$ [ [0, 1, 2, 1, 0, 1, 2, 1], [0, 1, 1, 2, 0, 1, 1, 2], [0, 2, 1, 1, 0, 2, 1, 1], [0, 1, 2, 1, 0, 1, 2, 1], [0, 1, 1, 2, 0, 1, 1, 2], [0, 2, 1, 1, 0, 2, 1, 1] ] \$$

$[0, y_1, y_2, y_3, 0, y_1, y_2, y_3]$

$$p' = -s^2 + s^5 \quad p = -s + s^4 \quad p' = -s + s^4$$

Omega Rank for B : cycles:  $\{\{2, 5, 8\}, \{1, 4, 6\}\}$ , net cycles: 2 . order: 3

$\$ [ [2, 1, 0, 1, 2, 1, 0, 1], [1, 2, 0, 1, 1, 2, 0, 1], [1, 1, 0, 2, 1, 1, 0, 2], [2, 1, 0, 1, 2, 1, 0, 1], [1, 2, 0, 1, 1, 2, 0, 1], [1, 1, 0, 2, 1, 1, 0, 2] ] \$$

$[y_1, y_2, 0, y_3, y_1, y_2, 0, y_3]$

$$p' = -s^2 + s^5 \quad p = s - s^4 \quad p' = s - s^4$$

$\hat{A} \ll \text{NOT SYNC'D } \hat{A} \gg$

Nullspace of  $\{\Omega\Delta^i\}$  :

$[0, x_1, x_2, x_3, 2x_2, -4x_1 + 2x_3]$

For  $A+2\Delta$  :  $[9y_1 + 9y_4 - y_2, -3y_1 - y_3 - 3y_4, y_1, -3y_1 - 3y_4 - y_5, y_2, y_3, y_4, y_5]$

For  $A-2\Delta$  :  $[y_1, -3y_1 - 3y_3 - y_4, 9y_1 + 9y_3 - y_5, y_2, y_3, y_4, y_5, -3y_1 - y_2 - 3y_3]$

Range of  $\{\Omega\Delta^i\}$  :  $[-\mu_1 - \mu_2 - \mu_3, \mu_1, \mu_2, \mu_3, -\mu_1 - \mu_2 - \mu_3, \mu_1, \mu_2, \mu_3]$

rank of M is 8 , rank of N is 5

$M \setminus ; N$

$\$ [ [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0] ] \$ \quad \$ [ [0, 1, 4, 3, 5, 4, 1, 2], [1, 0, 3, 2, 4, 5, 2, 3], [4, 3, 0, 1, 1, 2, 5, 4], [3, 2, 1, 0, 2, 3, 4, 5], [5, 4, 1, 2, 0, 1, 4, 3], [4, 5, 2, 3, 1, 0, 3, 2], [1, 2, 5, 4, 4, 3, 0, 1], [2, 3, 4, 5, 3, 2, 1, 0] ] \$$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [-1, 0, 1, 0, -1, 0, 1, 0]$

ker M,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range M,  $[x_1, x_8, x_7, x_6, x_5, x_4, x_3, x_2]$

$$\tau = 32, r' = 1/2$$

### Ranges

Action of R on ranges, [[3], [3], [4], [2]]

Action of B on ranges, [[2], [4], [1], [1]]

$$\beta(\{1, 5\}) = 1/4$$

$$\beta(\{2, 6\}) = 1/4$$

$$\beta(\{3, 7\}) = 1/4$$

$$\beta(\{4, 8\}) = 1/4$$

ker N,  $[-\mu_1 - \mu_3 - \mu_2, \mu_2, \mu_1, \mu_3, -\mu_1 - \mu_3 - \mu_2, \mu_2, \mu_1, \mu_3]$

Range of N

$$[y_4 - y_3 + y_5, y_4 + y_5 - y_1, y_4 + y_5 - y_2, y_4, y_3, y_1, y_2, y_5]$$

### Partitions

Action of R on partitions, [[4], [1], [2], [2]]

Action of B on partitions, [[2], [3], [4], [2]]

$$\alpha(\{\{3, 5, 6, 8\}, \{1, 2, 4, 7\}\}) = 1/5$$

$$\alpha(\{\{3, 4, 5, 6\}, \{1, 2, 7, 8\}\}) = 2/5$$

$$\alpha(\{\{2, 3, 4, 5\}, \{1, 6, 7, 8\}\}) = 1/5$$

$$\alpha(\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}) = 1/5$$

$$b_1 = \{3, 4, 5, 6\}, b_2 = \{1, 2, 3, 4\}, b_3 = \{2, 3, 4, 5\}, b_4 = \{1, 6, 7, 8\}, b_5 = \{1, 2, 7, 8\}, b_6 = \{5, 6, 7, 8\}, b_7 = \{3, 5, 6, 8\}, b_8 = \{1, 2, 4, 7\}$$

Action of R and B on the blocks of the partitions: \$ [ [0, 0, 0, 1, 0, 0, 0, 1], [1, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 1, 0, 0], [1, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 1, 0], [1, 0, 0, 0, 1, 0, 0, 0], [0, 1, 0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 1, 0, 0] ] \$ = \$ [ [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0] ] \$ + \$ [ [0, 0, 0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0] ] \$  
 ['8', '5', '5', '1', '7', '1', '2', '6'], ['4', '1', '6', '2', '3', '5', '5', '1'] with invariant measure [2, 1, 1, 1, 2, 1, 1, 1]

N by blocks, check: true . ' See partition graph.

' ' See level-2 partition graph.

'

<b>Sandwich</b>	
<b>Coloring</b>	{3, 4, 7, 8}
<b>Rank</b>	2
<b>R,B</b>	[3, 3, 8, 6, 7, 7, 4, 2], [6, 8, 1, 1, 2, 4, 5, 5]
$\pi_2$	[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0]
$u_2$	[1, 4, 3, 5, 4, 1, 2, 3, 2, 4, 5, 2, 3, 1, 1, 2, 5, 4, 2, 3, 4, 5, 1, 4, 3, 3, 2, 1] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

91 . Coloring, {3, 5, 6, 7}

**R:** [3, 3, 8, 1, 2, 4, 4, 5]    **B:** [6, 8, 1, 6, 7, 7, 5, 2]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

‘ [ ‘4‘ (‘ - 1 +  $\tau$  ‘)‘ (‘ 1 +  $\tau$  ‘)‘ (‘ 5 + 2 $\tau$  +  $\tau^2$  ‘)‘ , -4‘ (‘ 5 + 10 $\tau^2$  +  $\tau^4$  ‘)‘ , 4‘ (‘ 1 +  $\tau$  ‘)‘ (‘ - 5 -  $\tau$  - 3 $\tau^2$  +  $\tau^3$  ‘)‘ , 4‘ (‘ - 1 +  $\tau$  ‘)‘ (‘ 1 +  $\tau$  ‘)‘ (‘ 5 - 2 $\tau$  +  $\tau^2$  ‘)‘ , 4‘ (‘ - 5 + 3 $\tau$  - 16 $\tau^2$  + 4 $\tau^3$  - 3 $\tau^4$  +  $\tau^5$  ‘)‘ , -4‘ (‘ - 1 +  $\tau$  ‘)‘<sup>2</sup> ‘ (‘ 1 +  $\tau$  ‘)‘ (‘ 5 +  $\tau^2$  ‘)‘ , 4‘ (‘ - 1 +  $\tau$  ‘)‘ (‘ 5 - 4 $\tau$  + 6 $\tau^2$  +  $\tau^4$  ‘)‘ , 4‘ (‘ - 5 - 3 $\tau$  - 10 $\tau^2$  + 2 $\tau^3$  -  $\tau^4$  +  $\tau^5$  ‘)‘ ]‘

For  $\tau=1/2$ , [-150, -242, -294, -102, -229, -63, -73, -281] . FixedPtCheck, [150, 242, 294, 102, 229, 63, 73, 281]

$$\det(A + \tau \Delta) = 1 \cdot (‘ - 1 + \tau ‘)^2 \cdot (‘ \tau ‘)^2 \cdot (‘ 1 + \tau ‘)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	6 vs 6	3 vs 6

Omega Rank for R : cycles: {{2, 3, 5, 8}}, net cycles: 0 . order: 4

$$[y_1, y_2, y_3, y_4, y_5, 0, 0, y_6]$$

$R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ]$

$[0, 0], [0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1]$  ] \$ = \$ [  $[0, 0, 13/32, -11/32, -3/32, 5/32], [0, 0, 13/32, -11/32, -3/32, 5/32], [0, 0, 5/32, 13/32, -11/32, -3/32], [0, 1/2, -11/32, -3/32, 5/32, -3/32], [0, 0, -11/32, -3/32, 5/32, 13/32], [1/2, -1/4, -3/32, 5/32, -3/32, -3/32], [1/2, -1/4, -3/32, 5/32, -3/32, -3/32], [0, 0, -3/32, 5/32, 13/32, -11/32]$  ] \$ x \$ [  $[1, 1, 2, 2, 1, 0, 0, 1], [2, 1, 2, 0, 1, 0, 0, 2], [0, 1, 3, 0, 2, 0, 0, 2], [0, 2, 1, 0, 2, 0, 0, 3], [0, 2, 2, 0, 3, 0, 0, 1], [0, 3, 2, 0, 1, 0, 0, 2]$  ] \$

Omega Rank for B : cycles:  $\{\{2, 8\}, \{5, 7\}\}$ , net cycles: 1 . order: 4

$[ [1, 1, 0, 0, 1, 2, 2, 1], [0, 1, 0, 0, 2, 1, 3, 1], [0, 1, 0, 0, 3, 0, 3, 1], [0, 1, 0, 0, 3, 0, 3, 1], [0, 1, 0, 0, 3, 0, 3, 1], [0, 1, 0, 0, 3, 0, 3, 1], [0, 1, 0, 0, 3, 0, 3, 1] ]$  \$

$[3 y_3 - y_2, y_3, 0, 0, 3 y_3 - y_1, y_1, y_2, y_3]$

$$p' = -s^3 + s^4 \quad p = s^3 - s^4 \quad p' = -s^3 + s^5$$

Â» SYNC'D 59/4096 , 0.01440429688

92 . Coloring,  $\{3, 5, 6, 8\}$

**R**:  $[3, 3, 8, 1, 2, 4, 5, 2]$  **B**:  $[6, 8, 1, 6, 7, 7, 4, 5]$

' See graph

' ' See pair graph

,

$\Omega$  for  $A + \tau \Delta$  :

$[ -2^{(-1 + \tau)^2} (1 + \tau)^2, 2^{(1 + \tau)^2}, 2^{(1 + \tau)^2}, 2^{(-1 + \tau)^2}, -2^{(-1 + \tau)^2}, 2^{(1 + \tau)^2}, 2^{(-1 + \tau)^2}, 2^{(-1 + \tau)^2}, 2^{(1 + \tau)^2} ]$

For  $\tau=1/2$ ,  $[3, 9, 9, 1, 3, 1, 1, 9]$  . FixedPtCheck,  $[3, 9, 9, 1, 3, 1, 1, 9]$

$$\det(A + \tau \Delta) = 1^{(-1 + \tau)^2} (-1 + \tau)^2 (1 + \tau)^2 (\tau)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	5 vs 6	5 vs 6

Omega Rank for R : cycles:  $\{\{2, 3, 8\}\}$ , net cycles: -1 . order: 3

$[ [1, 2, 2, 1, 1, 0, 0, 1], [1, 2, 3, 0, 0, 0, 0, 2], [0, 2, 3, 0, 0, 0, 0, 3], [0, 3, 2, 0, 0, 0, 0, 3], [0, 3, 3, 0, 0, 0, 0, 2], [0, 2, 3, 0, 0, 0, 0, 3] ]$  \$

$[y_1, y_3, y_2, y_4, y_4, 0, 0, y_5]$

$$p = s^3 - s^6$$

Omega Rank for B : cycles: {{4, 6, 7}}, net cycles: -1 . order: 3

$$\$ [ [1, 0, 0, 1, 1, 2, 2, 1], [0, 0, 0, 2, 1, 2, 3, 0], [0, 0, 0, 3, 0, 2, 3, 0], [0, 0, 0, 3, 0, 3, 2, 0], [0, 0, 0, 2, 0, 3, 3, 0], [0, 0, 0, 3, 0, 2, 3, 0] ] \$$$

$$[y_4, 0, 0, y_5, y_1, y_2, y_3, y_4]$$

$$p = -s^3 + s^6$$

Â» SYNC'D 1269/32768 , 0.03872680664

93 . Coloring, {3, 5, 7, 8}

**R**: [3, 3, 8, 1, 2, 7, 4, 2] **B**: [6, 8, 1, 6, 7, 4, 5, 5]

' See graph

' ' See pair graph

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Ω for A+τΔ :

$$\begin{aligned} & [ -4' (-1 + \tau')^{2'} (-5 + \tau^2)' (-1 + \tau')', 4' (-5 - \tau - 3\tau^2 + \tau^3)' (-1 + \tau')', -4' (1 + \tau')^2 (-5 - 2\tau + \tau^2)' (-1 + \tau')^2 (5 + 2\tau + \tau^2)', 4' (5 + 2\tau^2 + \tau^4)' (-1 + \tau')^2 (5 - \tau + \tau^2 + \tau^3)' (-1 + \tau')^2, 4' (-5 - 3\tau - \tau^2 + \tau^3)' (-1 + \tau')^2, -4' (5 + 2\tau^2 - 2\tau^3 + \tau^4)' (1 + \tau')^2 ] \end{aligned}$$

For τ=1/2, [-114, -294, -306, -50, -89, -41, -53, -303] . FixedPtCheck, [114, 294, 306, 50, 89, 41, 53, 303]

$$\det(A + \tau \Delta) = 1' (\tau')^2 (-1 + \tau')^2 (1 + \tau')^2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	6 vs 6	2 vs 6

Omega Rank for R : cycles: {{2, 3, 8}}, net cycles: 0 . order: 6

$$[y_1, y_2, y_3, y_4, 0, 0, y_5, y_6]$$

$$\begin{aligned} R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] = \$ [ [0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, -5/8, 3/8, 3/8], [0, 0, 1, 3/8, -5/8, -5/8], [0, 0, 0, 3/8, -5/8, 3/8], [1, -1, 0, -5/8, 11/8, -5/8], [0, 1, -1, -5/8, -5/8, 11/8], [0, 0, 0, 3/8, -5/8, 3/8] ] \times \$ [ [1, 2, 2, 1, 0, 0, 1, 1], [1, 1, 3, 1, 0, 0, 0, 2], [1, 2, 2, 0, 0, 0, 0, 3], [0, 3, 3, 0, 0, 0, 0, 2], [0, 2, 3, 0, 0, 0, 0, 3], [0, 3, 2, 0, 0, 0, 0, 3] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{4, 6}, {5, 7}}, net cycles: 0 . order: 2

\$ [ [1, 0, 0, 1, 2, 2, 1, 1] , [0, 0, 0, 2, 2, 2, 2, 0] , [0, 0, 0, 2, 2, 2, 2, 0] , [0, 0, 0, 2, 2, 2, 2, 0] , [0, 0, 0, 2, 2, 2, 2, 0] , [0, 0, 0, 2, 2, 2, 2, 0] ] \$

[y<sub>2</sub>, 0, 0, y<sub>1</sub>, y<sub>1</sub> + y<sub>2</sub>, y<sub>1</sub> + y<sub>2</sub>, y<sub>1</sub>, y<sub>2</sub>]

$$p' = s^4 - s^5 \quad p' = s^3 - s^5 \quad p' = s^2 - s^5 \quad p = s^2 - s^6$$

Â» SYNC'D 1409/65536 , 0.02149963379

94 . Coloring, {3, 6, 7, 8}

**R**: [3, 3, 8, 1, 7, 4, 4, 2]    **B**: [6, 8, 1, 6, 2, 7, 5, 5]

' See graph

' ' See pair graph

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Ω for A+τΔ :

$$\begin{aligned} & [ \text{' -2' ( ' 1 + \tau \text{' ) ' ( ' 5 + 4\tau + 6\tau^2 + \tau^4 \text{' ) ' ( ' - 1 + \tau \text{' ) ' , 2' ( ' 5 - 2\tau + 19\tau^2 + 7\tau^4 + 2\tau^5 + \tau^6 } \\ & \text{' ) ' , 2' ( ' 1 + \tau \text{' ) ' ( ' 5 + \tau + 10\tau^2 - 2\tau^3 + \tau^4 + \tau^5 \text{' ) ' , 2' ( ' 1 + \tau \text{' ) ' ^2 ( ' - 5 + 3\tau - 3\tau^2 + \tau^3 \text{' ) ' ,} \\ & \text{( ' - 1 + \tau \text{' ) ' , 2' ( ' - 5 + \tau - 10\tau^2 - 2\tau^3 - \tau^4 + \tau^5 \text{' ) ' ( ' - 1 + \tau \text{' ) ' , 2' ( ' 1 + \tau \text{' ) ' ( ' 5 + 3\tau + 3\tau^2 + } \\ & \text{\tau^3 \text{' ) ' ( ' - 1 + \tau \text{' ) ' ^2 , -2' ( ' 1 + \tau \text{' ) ' ( ' 5 - 4\tau + 6\tau^2 + \tau^4 \text{' ) ' ( ' - 1 + \tau \text{' ) ' , 2' ( ' 5 + 2\tau + 19\tau^2 + } \\ & \text{7\tau^4 - 2\tau^5 + \tau^6 \text{' ) ' } \text{' } \end{aligned}$$

For τ=1/2, [411, 593, 753, 297, 233, 177, 219, 713] . FixedPtCheck, [411, 593, 753, 297, 233, 177, 219, 713]

$$\det(A + \tau \Delta) = 1 \text{' ( ' 1 + \tau \text{' ) ' ^2 ( ' \tau \text{' ) ' ^2 ( ' - 1 + \tau \text{' ) ' ^2}$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	6 vs 6	6 vs 6

Omega Rank for R : cycles: {{2, 3, 8}}, net cycles: 0 . order: 6

[y<sub>6</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>3</sub>, 0, 0, y<sub>2</sub>, y<sub>1</sub>]

R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 1] , [1, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, 3/8, -5/8, 3/8] , [0, 0, 0, 3/8, -5/8, 3/8] , [0, 0, 0, 3/8, 3/8, -5/8] , [0, 0, 1, -5/8, 3/8, -5/8] , [1, -2, 3, -5/8, 11/8, -21/8] , [0, 1, -2, 3/8, -5/8, 11/8] , [0, 1, -2, 3/8, -5/8, 11/8] , [0, 0, 0, -5/8, 3/8, 3/8] ] \$ x \$ [ [1, 1, 2, 2, 0, 0, 1, 1] , [2, 1, 2, 1, 0, 0, 0, 2] , [1, 2, 3, 0, 0, 0, 0, 2] , [0, 2, 3, 0, 0, 0, 0, 3] , [0, 3, 2, 0, 0, 0, 0, 3] , [0, 3, 3, 0, 0, 0, 0, 2] ] \$

Omega Rank for B : cycles: {{2, 5, 8}}, net cycles: 0 . order: 6

$$[y_1, y_2, 0, 0, y_3, y_4, y_5, y_6]$$

$$\begin{aligned} B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, \\ 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0] \\ , [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, \\ 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 1, -2, 3/8, -5/8, 11/8], [0, 0, 0, -5/8, 3/8, 3/8] \\ , [1, -2, 3, -5/8, 11/8, -21/8], [0, 1, -2, 3/8, -5/8, 11/8], [0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 1, -5/8, 3/8, -5/8], [0, \\ 0, 0, 3/8, -5/8, 3/8], [0, 0, 0, 3/8, -5/8, 3/8] ] \$ \times \$ [ [1, 1, 0, 0, 2, 2, 1, 1], [0, 2, 0, 0, 2, 1, 2, 1], [0, 2, 0, 0, \\ 3, 0, 1, 2], [0, 3, 0, 0, 3, 0, 0, 2], [0, 3, 0, 0, 2, 0, 0, 3], [0, 2, 0, 0, 3, 0, 0, 3] ] \$ \end{aligned}$$

Â» SYNC'D 665/16384 , 0.04058837891

95 . Coloring, {4, 5, 6, 7}

**R:** [3, 3, 1, 6, 2, 4, 4, 5] **B:** [6, 8, 8, 1, 7, 7, 5, 2]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[ '1' ('1 + \tau')', -1' ('-1 + \tau')', 1' ('1 + \tau')', 1' ('1 + \tau')', -1' ('-1 + \tau')', 1' ('1 + \tau')', -1' ('-1 + \tau')', -1' ('-1 + \tau')' ]'$$

For τ=1/2, [3, 1, 3, 3, 1, 3, 1, 1] . FixedPtCheck, [3, 1, 3, 3, 1, 3, 1, 1]

$$\det(A + \tau \Delta) = 1' (' \tau ')'^2 ('1 + \tau')'^2 ('-1 + \tau')'^2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	4 vs 6	4 vs 6

Omega Rank for R : cycles: {{1, 3}, {4, 6}}, net cycles: 1 . order: 4

$$\begin{aligned} \$ [ [1, 1, 2, 2, 1, 1, 0, 0], [2, 1, 2, 1, 0, 2, 0, 0], [2, 0, 3, 2, 0, 1, 0, 0], [3, 0, 2, 1, 0, 2, 0, 0], [2, 0, 3, 2, 0, \\ 1, 0, 0], [3, 0, 2, 1, 0, 2, 0, 0] ] \$ \end{aligned}$$

$$[y_1, y_2, 4y_1 + 4y_2 - y_3 - 5y_4, 3y_1 + 3y_2 - 4y_4, y_3, y_4, 0, 0]$$

$$p = s^3 - s^5 \quad p' = -s^3 + s^5$$

Omega Rank for B : cycles: {{2, 8}, {5, 7}}, net cycles: 1 . order: 4



\$ [ [1, 1, 0, 0, 1, 1, 2, 2] , [0, 2, 0, 0, 2, 1, 2, 1] , [0, 1, 0, 0, 2, 0, 3, 2] , [0, 2, 0, 0, 3, 0, 2, 1] , [0, 1, 0, 0, 2, 0, 3, 2] , [0, 2, 0, 0, 3, 0, 2, 1] ] \$

$$[-5 y_4 + 4 y_3 + 4 y_2 - y_1, y_4, 0, 0, y_3, y_2, y_1, -4 y_4 + 3 y_3 + 3 y_2]$$

$$p = s^3 - s^5 \quad p' = s^3 - s^5$$

Â» SYNC'D 99/16384 , 0.006042480469

96 . Coloring, {4, 5, 6, 8}

**R:** [3, 3, 1, 6, 2, 4, 5, 2]    **B:** [6, 8, 8, 1, 7, 7, 4, 5]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$\begin{aligned} & [ ' 4 ( ' 5 + 3\tau + 16\tau^2 + 4\tau^3 + 3\tau^4 + \tau^5 ' ) ' , -4 ( ' - 1 + \tau ' ) ' ( ' 1 + \tau ' ) ' ^2 ( ' 5 + \tau^2 ' ) ' , 4 ( ' 5 + 4\tau + 6\tau^2 + \tau^4 ' ) ' ( ' 1 + \tau ' ) ' , 4 ( ' 5 - 3\tau + 10\tau^2 + 2\tau^3 + \tau^4 + \tau^5 ' ) ' , -4 ( ' - 1 + \tau ' ) ' ( ' 5 - 2\tau + \tau^2 ' ) ' ( ' 1 + \tau ' ) ' , 4 ( ' 5 + 10\tau^2 + \tau^4 ' ) ' , -4 ( ' - 1 + \tau ' ) ' ( ' 5 - \tau + 3\tau^2 + \tau^3 ' ) ' , -4 ( ' - 1 + \tau ' ) ' ( ' 1 + \tau ' ) ' ( ' 5 + 2\tau + \tau^2 ' ) ' ] ' \end{aligned}$$

For τ=1/2, [359, 189, 411, 203, 102, 242, 86, 150] . FixedPtCheck, [359, 189, 411, 203, 102, 242, 86, 150]

$$\det(A + \tau \Delta) = 1 ( ' - 1 + \tau ' ) ' ^2 ( ' \tau ' ) ' ^2 ( ' 1 + \tau ' ) ' ^2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	3 vs 6	6 vs 6

Omega Rank for R : cycles: {{1, 3}, {4, 6}}, net cycles: 1 . order: 4

\$ [ [1, 2, 2, 1, 1, 1, 0, 0] , [2, 1, 3, 1, 0, 1, 0, 0] , [3, 0, 3, 1, 0, 1, 0, 0] , [3, 0, 3, 1, 0, 1, 0, 0] , [3, 0, 3, 1, 0, 1, 0, 0] , [3, 0, 3, 1, 0, 1, 0, 0] ] \$

$$[-y_1 + 3 y_3, y_1, y_2, y_3, -y_2 + 3 y_3, y_3, 0, 0]$$

$$p = -s^3 + s^6 \quad p = -s^3 + s^4 \quad p = -s^3 + s^5$$

Omega Rank for B : cycles: {{1, 4, 6, 7}}, net cycles: 0 . order: 4

$$[y_1, 0, 0, y_2, y_3, y_4, y_5, y_6]$$

$$B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, -11/32, -3/32, 5/32, 13/32], [1/2, -1/4, -3/32, 5/32, -3/32, -3/32], [1/2, -1/4, -3/32, 5/32, -3/32, -3/32], [0, 0, -3/32, 5/32, 13/32, -11/32], [0, 0, 13/32, -11/32, -3/32, 5/32], [0, 0, 13/32, -11/32, -3/32, 5/32], [0, 0, 5/32, 13/32, -11/32, -3/32], [0, 1/2, -11/32, -3/32, 5/32, -3/32] ] \$ \times \$ [ [1, 0, 0, 1, 1, 1, 2, 2], [1, 0, 0, 2, 2, 1, 2, 0], [2, 0, 0, 2, 0, 1, 3, 0], [2, 0, 0, 3, 0, 2, 1, 0], [3, 0, 0, 1, 0, 2, 2, 0], [1, 0, 0, 2, 0, 3, 2, 0] ] \$$$

Â» SYNC'D 59/4096 , 0.01440429688

97 . Coloring, {4, 5, 7, 8}

**R:** [3, 3, 1, 6, 2, 7, 4, 2]    **B:** [6, 8, 8, 1, 7, 4, 5, 5]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$[ '1' ('5 + 3\tau + 7\tau^2 + \tau^3)', -1' ('5 + \tau')' ('1 + \tau')' ('-1 + \tau')', 1' ('1 + \tau')' ('5 + 2\tau + \tau^2)', 1' ('5 - \tau + 3\tau^2 + \tau^3)', -1' ('-1 + \tau')' ('5 - 2\tau + \tau^2)', -1' ('-5 - \tau - 3\tau^2 + \tau^3)', -1' ('-5 + 3\tau - 7\tau^2 + \tau^3)', 1' ('1 + \tau')' ('-1 + \tau')' ('-5 + \tau')' ]'$$

For τ=1/2, [67, 33, 75, 43, 17, 49, 41, 27] . FixedPtCheck, [67, 33, 75, 43, 17, 49, 41, 27]

$$\det(A + \tau \Delta) = 1' ('1 + \tau')'^2 ('\tau')'^2 ('-1 + \tau')'^2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	3 vs 6	3 vs 6

Omega Rank for R : cycles: {{1, 3}, {4, 6, 7}}, net cycles: 1 . order: 6

$$\$ [ [1, 2, 2, 1, 0, 1, 1, 0], [2, 0, 3, 1, 0, 1, 1, 0], [3, 0, 2, 1, 0, 1, 1, 0], [2, 0, 3, 1, 0, 1, 1, 0], [3, 0, 2, 1, 0, 1, 1, 0], [2, 0, 3, 1, 0, 1, 1, 0] ] \$$$

$$[-y_1 - y_2 + 5 y_3, y_1, y_2, y_3, 0, y_3, y_3, 0]$$

$$p = -s^2 + s^4 \quad p' = -s^2 + s^4 \quad p = -s^2 + s^6$$

Omega Rank for B : cycles: {{1, 4, 6}, {5, 7}}, net cycles: 1 . order: 6

$$\$ [ [1, 0, 0, 1, 2, 1, 1, 2], [1, 0, 0, 1, 3, 1, 2, 0], [1, 0, 0, 1, 2, 1, 3, 0], [1, 0, 0, 1, 3, 1, 2, 0], [1, 0, 0, 1, 2, 1, 3, 0], [1, 0, 0, 1, 3, 1, 2, 0] ] \$$$

$$[y_3, 0, 0, y_3, y_1, y_3, y_2, 5y_3 - y_1 - y_2]$$

$$p = -s^2 + s^4 \quad p' = -s^2 + s^4 \quad p = -s^2 + s^6$$

Â» SYNC'D 1125/32768 , 0.03433227539

98 . Coloring, {4, 6, 7, 8}

**R:** [3, 3, 1, 6, 7, 4, 4, 2]   **B:** [6, 8, 8, 1, 2, 7, 5, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ ' 4 ' ( ' 1 + \tau ' ) ^ 2 ( ' 5 - 3\tau + \tau ^ 2 + \tau ^ 3 ' ) , -4 ( ' - 1 + \tau ' ) ^ 2 ( ' 5 - 2\tau + 2\tau ^ 2 + 2\tau ^ 3 + \tau ^ 4 ' ) , 4 ( ' 1 + \tau ' ) ^ 2 ( ' 5 + 2\tau ^ 2 + \tau ^ 4 ' ) , 4 ( ' 1 + \tau ' ) ^ 2 ( ' 5 - \tau - \tau ^ 2 + \tau ^ 3 ' ) , 4 ( ' - 1 + \tau ' ) ^ 2 ( ' 5 + 2\tau + \tau ^ 2 ' ) , 4 ( ' 1 + \tau ' ) ^ 2 ( ' 5 - 2\tau + \tau ^ 2 ' ) , 4 ( ' 1 + \tau ' ) ^ 2 ( ' - 5 + \tau ^ 2 ' ) ( ' - 1 + \tau ' ) , -4 ( ' 5 - \tau + 3\tau ^ 2 + \tau ^ 3 ' ) ( ' - 1 + \tau ' ) ] \end{aligned}$$

For τ=1/2, [279, 77, 267, 315, 50, 306, 114, 86] . FixedPtCheck, [279, 77, 267, 315, 50, 306, 114, 86]

$$\det(A + \tau \Delta) = 1 ( ' \tau ' ) ^ 2 ( ' 1 + \tau ' ) ^ 2 ( ' - 1 + \tau ' ) ^ 2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	2 vs 6	6 vs 6

Omega Rank for R : cycles: {{1, 3}, {4, 6}}, net cycles: 0 . order: 2

$$\$ [ [1, 1, 2, 2, 0, 1, 1, 0], [2, 0, 2, 2, 0, 2, 0, 0], [2, 0, 2, 2, 0, 2, 0, 0], [2, 0, 2, 2, 0, 2, 0, 0], [2, 0, 2, 2, 0, 2, 0, 0], [2, 0, 2, 2, 0, 2, 0, 0] ] \$$$

$$[y_2, y_1, y_2 + y_1, y_2 + y_1, 0, y_2, y_1, 0]$$

$$p' = s^4 - s^5 \quad p' = s^3 - s^5 \quad p' = s^2 - s^5 \quad p = s^2 - s^6$$

Omega Rank for B : cycles: {{2, 5, 8}}, net cycles: 0 . order: 6

$$[y_2, y_1, 0, 0, y_6, y_5, y_4, y_3]$$

$$\begin{aligned} \mathbf{B} = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 1, -1, -5/8, -5/8, 11/8], [0, 0, 0, 3/8, -5/8, 3/8], [0, 0, 0, 3/8, -5/8, 3/8], [1, -1, 0, -5/8, 11/8, -5/8], [0, 0, 0, -5/8, 3/8, 3/8], [0, 0, 1, 3/8, -5/8, -5/8], \end{aligned}$$

[0, 0, 0, 3/8, 3/8, -5/8], [0, 0, 0, 3/8, 3/8, -5/8] x [ [1, 1, 0, 0, 2, 1, 1, 2], [0, 2, 0, 0, 3, 1, 1, 1], [0, 3, 0, 0, 2, 0, 1, 2], [0, 2, 0, 0, 3, 0, 0, 3], [0, 3, 0, 0, 3, 0, 0, 2], [0, 3, 0, 0, 2, 0, 0, 3] ]

Â» SYNC'D 1409/65536 , 0.02149963379

99 . Coloring, {5, 6, 7, 8}

**R:** [3, 3, 1, 1, 2, 4, 4, 2] **B:** [6, 8, 8, 6, 7, 7, 5, 5]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

' [ '1' ('1 + τ')<sup>2</sup> , -1' ('-1 + τ')' ('1 + τ')<sup>2</sup> , 1' ('1 + τ')<sup>2</sup> , -1' ('-1 + τ')' ('1 + τ')<sup>2</sup> , 1' ('-1 + τ')<sup>2</sup> , -1' ('-1 + τ')' ('1 + τ')<sup>2</sup> , -1' ('-1 + τ')' ('1 + τ')<sup>2</sup> , -1' ('-1 + τ')' ('1 + τ')<sup>2</sup> ]

For τ=1/2, [9, 3, 9, 3, 1, 3, 1, 3] . FixedPtCheck, [9, 3, 9, 3, 1, 3, 1, 3]

det(A + τ Δ) = 0

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

[1, 1, 1, 1, 1, 1, 1, 1]

+ \; - \; Δ

\$ [ [2, 2, 2, 2, 0, 0, 0, 0], [2, 0, 2, 0, 2, 0, 2, 0], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$ [ [0, 0, 0, 0, 2, 2, 2, 2], [0, 2, 0, 2, 0, 2, 0, 2], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1], [1, 1, 1, 1, 1, 1, 1, 1] ] \$ [ [1, 1, 1, 1, -1, -1, -1, -1], [1, -1, 1, -1, 1, -1, 1, -1], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$

[y<sub>1</sub>, -y<sub>2</sub>, y<sub>1</sub>, -y<sub>2</sub>, y<sub>2</sub>, -y<sub>1</sub>, y<sub>2</sub>, -y<sub>1</sub>]

p = s<sup>3</sup>

S+ \; S- \; NM  
 \$ [ [0, 0, 1, 1, 1, 1, 0, 0], [0, 1, 1, 0, 1, 0, 0, 1], [1, 1, 0, 0, 0, 0, 1, 1], [1, 0, 0, 1, 0, 1, 1, 0], [1, 1, 0, 0, 1, 1, 0, 0], [0, 1, 1, 0, 0, 1, 1, 0], [0, 0, 1, 1, 0, 0, 1, 1], [1, 0, 0, 1, 1, 0, 0, 1] ] \$ [ [1, 1, 0, 0, 1, 1, 0, 0], [0, 1, 1, 0, 0, 1, 1, 0], [0, 0, 1, 1, 0, 0, 1, 1], [1, 0, 0, 1, 1, 0, 0, 1], [1, 1, 0, 0, 0, 0, 1, 1], [1, 0, 0, 1, 0, 1, 1, 0], [0, 0, 1, 1, 1, 1, 0, 0], [0, 1, 1, 0, 1, 0, 0, 1] ] \$ [ [2, 1, 0, 1, 1, 1, 1, 1], [1, 2, 1, 0, 1, 1, 1, 1], [0, 1, 2, 1, 1, 1, 1, 1], [1, 0, 1, 2, 1, 1, 1, 1], [1, 1, 1, 1, 2, 1, 0, 1], [1, 1, 1, 1, 1, 2, 1, 0], [1, 1, 1, 1, 0, 1, 2, 1], [1, 1, 1, 1, 1, 0, 1, 2] ] \$

CmmCk true, true, true

p' = s<sup>3</sup> p' = s<sup>4</sup> p' = s<sup>5</sup>

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
2 vs 6	2 vs 6	2 vs 6	2 vs 4	2 vs 4

Omega Rank for R : cycles:  $\{\{1, 3\}\}$ , net cycles: -1 . order: 2

$$\$ [ [2, 2, 2, 2, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0], [4, 0, 4, 0, 0, 0, 0, 0] ] \$$$

$$[y_1, y_2, y_1, y_2, 0, 0, 0, 0]$$

$$p = s^2 - s^4 \quad p' = s^2 - s^3$$

Omega Rank for B : cycles:  $\{\{5, 7\}\}$ , net cycles: -1 . order: 2

$$\$ [ [0, 0, 0, 0, 2, 2, 2, 2], [0, 0, 0, 0, 4, 0, 4, 0], [0, 0, 0, 0, 4, 0, 4, 0], [0, 0, 0, 0, 4, 0, 4, 0] ] \$$$

$$[0, 0, 0, 0, y_1, y_2, y_1, y_2]$$

$$p = -s^2 + s^3 \quad p = -s^2 + s^4$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[0, 0, x_1, x_2, x_3, x_4]$$

$$\text{For } A+2\Delta: [y_1, y_2, y_3, y_4, -3y_2 - 3y_4 - y_5, -3y_1 - 3y_3 - y_6, y_5, y_6]$$

$$\text{For } A-2\Delta: [y_5, y_6, -3y_2 - 3y_4 - y_5, -3y_1 - 3y_3 - y_6, y_1, y_2, y_3, y_4]$$

Range of  $\{\Omega\Delta^i\}$  :  $[-\mu_2, -\mu_1, -\mu_2, -\mu_1, \mu_1, \mu_2, \mu_1, \mu_2]$

rank of M is 8 , rank of N is 5

M \ ; N

$$\$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0] ] \$ \quad \$ [ [0, 1, 2, 1, 1, 1, 1, 1], [1, 0, 1, 2, 1, 1, 1, 1], [2, 1, 0, 1, 1, 1, 1, 1], [1, 2, 1, 0, 1, 1, 1, 1], [1, 1, 1, 1, 0, 1, 2, 1], [1, 1, 1, 1, 1, 0, 1, 2], [1, 1, 1, 1, 2, 1, 0, 1], [1, 1, 1, 1, 1, 2, 1, 0] ] \$$$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [1, 1, 1, 1, -1, -1, -1, -1]$

ker M,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range M,  $[x_1, x_2, x_3, x_4, x_5, x_8, x_6, x_7]$

$$\tau = 32, r' = 1/2$$



0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] ] \$  
[C, '6', E, E, '6', E, '6', F, F, C, F, E, F, C, C, '6'], ['9', '5', '9', '10', '10', '5', D, D, '10', '10', '5', D, '9', '5', D, '9'] with invariant measure [0, 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1]

N by blocks, check: true . ‘ See partition graph.

‘ ‘ See level-2 partition graph.

‘

<b>Sandwich</b>	
<b>Coloring</b>	{5, 6, 7, 8}
<b>Rank</b>	2
<b>R,B</b>	[3, 3, 1, 1, 2, 4, 4, 2], [6, 8, 8, 6, 7, 7, 5, 5]
$\pi_2$	[0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0]
$u_2$	[1, 2, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 2, 1] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

100 . Coloring, {2, 3, 4, 5, 6}

**R**: [3, 8, 8, 6, 2, 4, 5, 5]    **B**: [6, 3, 1, 1, 7, 7, 4, 2]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

[ '12' (' - 1 +  $\tau$  ') ' 2 ' (' 5 + 4 $\tau$  + 3 $\tau$  ' 2 ') ' , 12' (' 5 + 3 $\tau$  ' 2 ') ' (' 1 +  $\tau$  ') ' 2 , -4' (' - 1 +  $\tau$  ') ' (' 1 +  $\tau$  ') ' (' 5 + 2 $\tau$  +  $\tau$  ' 2 ') ' , 4' (' - 5 -  $\tau$  - 3 $\tau$  ' 2 +  $\tau$  ' 3 ') ' (' - 1 +  $\tau$  ') ' , 4' (' 5 + 4 $\tau$  + 6 $\tau$  ' 2 +  $\tau$  ' 4 ') ' (' 1 +  $\tau$  ') ' , -4' (' - 1 +  $\tau$  ') ' (' 5 + 2 $\tau$  ' 2 +  $\tau$  ' 4 ') ' , -4' (' 1 +  $\tau$  ' 2 ') ' (' - 1 +  $\tau$  ') ' (' 5 + 2 $\tau$  +  $\tau$  ' 2 ') ' , 4' (' 5 +  $\tau$  +  $\tau$  ' 2 +  $\tau$  ' 3 ') ' (' 1 +  $\tau$  ') ' 2 ' ]'

For  $\tau=1/2$ , [62, 414, 150, 98, 411, 89, 125, 423] . FixedPtCheck, [62, 414, 150, 98, 411, 89, 125, 423]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	2 vs 6	6 vs 6

Omega Rank for R : cycles: {{2, 5, 8}, {4, 6}}, net cycles: 1 . order: 6

$$\$ [ [0, 1, 1, 1, 2, 1, 0, 2], [0, 2, 0, 1, 2, 1, 0, 2], [0, 2, 0, 1, 2, 1, 0, 2], [0, 2, 0, 1, 2, 1, 0, 2], [0, 2, 0, 1, 2, 1, 0, 2], [0, 2, 0, 1, 2, 1, 0, 2] ] \$$$

$$[0, -y_1 + 2 y_2, y_1, y_2, 2 y_2, y_2, 0, 2 y_2]$$

$$p = s^2 - s^6 \quad p' = s^2 - s^5 \quad p'' = s^3 - s^5 \quad p''' = s^4 - s^5$$

Omega Rank for B : cycles: {{1, 4, 6, 7}}, net cycles: 0 . order: 4

$$[y_6, y_3, y_4, y_5, 0, y_1, y_2, 0]$$

$$\begin{aligned} B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 0, 5/32, 13/32, -11/32, -3/32], [0, 1, -11/32, -3/32, 5/32, -19/32], [0, 0, 13/32, -11/32, -3/32, 5/32], [0, 0, 13/32, -11/32, -3/32, 5/32], [0, 0, -3/32, 5/32, 13/32, -11/32], [0, 0, -11/32, -3/32, 5/32, 13/32], [1, -1, -3/32, 5/32, -19/32, 21/32] ] \$ \times \$ [ [2, 1, 1, 1, 0, 1, 2, 0], [2, 0, 1, 2, 0, 2, 1, 0], [3, 0, 0, 1, 0, 2, 2, 0], [1, 0, 0, 2, 0, 3, 2, 0], [2, 0, 0, 2, 0, 1, 3, 0], [2, 0, 0, 3, 0, 2, 1, 0] ] \$ \end{aligned}$$

Â» SYNC'D 4447/262144 , 0.01696395874

101 . Coloring, {2, 3, 4, 5, 7}

**R**: [3, 8, 8, 6, 2, 7, 4, 5]    **B**: [6, 3, 1, 1, 7, 4, 5, 2]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$\begin{aligned} & [ ' 4' ( ' - 5 + \tau - \tau^2 + \tau^3 ' ) ' ( ' - 1 + \tau ' ) ' , 4' ( ' 5 + \tau + \tau^2 + \tau^3 ' ) ' ( ' 1 + \tau ' ) ' , -4' ( ' - 1 + \tau ' ) ' ( ' 5 + \tau^2 ' ) ' ( ' 1 + \tau ' ) ' , 4' ( ' 1 + \tau^2 ' ) ' ( ' 5 - 2\tau + \tau^2 ' ) ' , 4' ( ' 5 + 4\tau + 6\tau^2 + \tau^4 ' ) ' , 4' ( ' 5 - 4\tau + 6\tau^2 + \tau^4 ' ) ' , 4' ( ' 5 + 2\tau^2 + \tau^4 ' ) ' , 4' ( ' 5 - 2\tau + \tau^2 ' ) ' ( ' 1 + \tau ' ) ' ^2 ' ] ' \end{aligned}$$



For  $\tau=1/2$ , [37, 141, 63, 85, 137, 73, 89, 153] . FixedPtCheck, [37, 141, 63, 85, 137, 73, 89, 153]

$$\det(A + \tau \Delta) = 1' (' \tau ' )' 2' (' 1 + \tau ' )' (' - 1 + \tau ' )' (' 1 + \tau ' )'$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	4 vs 7	3 vs 7

Omega Rank for R : cycles: {{4, 6, 7}, {2, 5, 8}}, net cycles: 1 . order: 3

$$\$ [ [0, 1, 1, 1, 1, 1, 2], [0, 1, 0, 1, 2, 1, 1, 2], [0, 2, 0, 1, 2, 1, 1, 1], [0, 2, 0, 1, 1, 1, 1, 2], [0, 1, 0, 1, 2, 1, 1, 2], [0, 2, 0, 1, 2, 1, 1, 1], [0, 2, 0, 1, 1, 1, 1, 2] ] \$$$

$$[0, y_1, -y_1 + 5 y_3 - y_2 - y_4, y_3, y_2, y_3, y_3, y_4]$$

$$p = s^2 - s^5 \quad p' = -s^2 + s^5 \quad p'' = s^3 - s^6$$

Omega Rank for B : cycles: {{1, 4, 6}, {5, 7}}, net cycles: 1 . order: 6

$$\$ [ [2, 1, 1, 1, 1, 1, 1, 0], [2, 0, 1, 1, 1, 2, 1, 0], [2, 0, 0, 2, 1, 2, 1, 0], [2, 0, 0, 2, 1, 2, 1, 0], [2, 0, 0, 2, 1, 2, 1, 0], [2, 0, 0, 2, 1, 2, 1, 0], [2, 0, 0, 2, 1, 2, 1, 0], [2, 0, 0, 2, 1, 2, 1, 0] ] \$$$

$$[2 y_1, y_3, y_2, -y_2 + 2 y_1, y_1, -y_3 + 2 y_1, y_1, 0]$$

$$p = -s^3 + s^4 \quad p' = -s^3 + s^5 \quad p'' = -s^3 + s^6 \quad p''' = -s^3 + s^7$$

Â» SYNC'D 3999/262144 , 0.01525497437

102 . Coloring, {2, 3, 4, 5, 8}

**R**: [3, 8, 8, 6, 2, 7, 5, 2] **B**: [6, 3, 1, 1, 7, 4, 4, 5]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$$[ ' 12' (' 5 - 3\tau + 3\tau^2 + 3\tau^3 ' )' (' - 1 + \tau ' )' 2 , 12' (' 5 - 4\tau + 3\tau^2 ' )' (' 1 + \tau ' )' 3 , -4' (' 5 - \tau + 3\tau^2 + \tau^3 ' )' (' - 1 + \tau ' )' (' 1 + \tau ' )' , 4' (' - 5 + \tau^2 ' )' (' - 1 + \tau ' )' 3 , -4' (' 5 - 3\tau + \tau^2 + \tau^3 ' )' (' - 1 + \tau ' )' (' 1 + \tau ' )' 2 , -4' (' - 1 + \tau ' )' 3 (' 5 + \tau + \tau^2 + \tau^3 ' )' , 4' (' 5 - \tau - \tau^2 + \tau^3 ' )' (' - 1 + \tau ' )' 2 (' 1 + \tau ' )' , 4' (' 5 + 2\tau^2 + \tau^4 ' )' (' 1 + \tau ' )' 2 ]'$$

For  $\tau=1/2$ , [74, 810, 258, 38, 279, 47, 105, 801] . FixedPtCheck, [74, 810, 258, 38, 279, 47, 105, 801]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	4 vs 6	5 vs 6

Omega Rank for R : cycles:  $\{\{2, 8\}\}$ , net cycles: -1 . order: 4

$\$ [ [0, 2, 1, 0, 1, 1, 1, 2], [0, 3, 0, 0, 1, 0, 1, 3], [0, 4, 0, 0, 1, 0, 0, 3], [0, 4, 0, 0, 0, 0, 0, 4], [0, 4, 0, 0, 0, 0, 0, 4], [0, 4, 0, 0, 0, 0, 0, 4] ] \$$

$$[0, y_1 - y_2 + y_3, y_4, 0, y_1, y_4, y_2, y_3]$$

$$p = -s^4 + s^5 \quad p = -s^4 + s^6$$

Omega Rank for B : cycles:  $\{\{1, 4, 6\}\}$ , net cycles: -1 . order: 3

$\$ [ [2, 0, 1, 2, 1, 1, 1, 0], [3, 0, 0, 2, 0, 2, 1, 0], [2, 0, 0, 3, 0, 3, 0, 0], [3, 0, 0, 3, 0, 2, 0, 0], [3, 0, 0, 2, 0, 3, 0, 0], [2, 0, 0, 3, 0, 3, 0, 0] ] \$$

$$[y_1, 0, y_3, y_2, y_3, y_4, y_5, 0]$$

$$p = -s^3 + s^6$$

$\hat{A} \gg \text{SYNC'D } 855/65536, 0.01304626465$

103 . Coloring,  $\{2, 3, 4, 6, 7\}$

**R:**  $[3, 8, 8, 6, 7, 4, 4, 5]$     **B:**  $[6, 3, 1, 1, 2, 7, 5, 2]$

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ ' 4' ( ' - 5 + 2\tau - 12\tau^2 - 2\tau^3 + \tau^4 ' ) ' ( ' - 1 + \tau ' ) ' , 4' ( ' - 1 + \tau ' ) ' ^2 ( ' 5 + 3\tau + 7\tau^2 + \tau^3 ' ) ' \\ & , -12' ( ' 5 - 2\tau + 8\tau^2 + 2\tau^3 + 3\tau^4 ' ) ' ( ' - 1 + \tau ' ) ' , 12' ( ' 1 + \tau ' ) ' ( ' 1 + 3\tau^2 ' ) ' ( ' 5 - 2\tau + \tau^2 ' ) ' \\ & , 4' ( ' - 1 + \tau ' ) ' ( ' - 5 - 3\tau - 10\tau^2 + 2\tau^3 - \tau^4 + \tau^5 ' ) ' , 4' ( ' 5 - 2\tau + 19\tau^2 + 7\tau^4 + 2\tau^5 + \tau^6 ' ) ' , \\ & -4' ( ' - 1 + \tau ' ) ' ( ' 5 + 3\tau + 16\tau^2 + 4\tau^3 + 3\tau^4 + \tau^5 ' ) ' , -4' ( ' 1 + \tau ' ) ' ( ' - 1 + \tau ' ) ' ( ' 1 + \tau^2 ' ) ' \\ & ( ' 5 - 2\tau + \tau^2 ' ) ' ] ' \end{aligned}$$

For  $\tau=1/2$ ,  $[230, 134, 206, 714, 281, 593, 359, 255]$  . FixedPtCheck,  $[230, 134, 206, 714, 281, 593, 359, 255]$

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	5 vs 6	6 vs 6

Omega Rank for R : cycles:  $\{\{4, 6\}\}$ , net cycles: 0 . order: 6

$\$ [ [0, 0, 1, 2, 1, 1, 1, 2], [0, 0, 0, 2, 2, 2, 1, 1], [0, 0, 0, 3, 1, 2, 2, 0], [0, 0, 0, 4, 0, 3, 1, 0], [0, 0, 0, 4, 0, 4, 0, 0], [0, 0, 0, 4, 0, 4, 0, 0] ] \$$

$$[0, 0, -y_3 - y_1 + y_2 + y_5 + y_4, y_3, y_1, y_2, y_5, y_4]$$

$$p = s^5 - s^6$$

Omega Rank for B : cycles:  $\{\{1, 2, 3, 5, 6, 7\}\}$ , net cycles: 1 . order: 6

$$[y_1, y_2, y_3, 0, y_4, y_5, y_6, 0]$$

$B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 0] ] \$ = \$ [ [-1/16, 7/16, -1/16, 7/16, -1/16, -9/16], [-1/16, 7/16, -1/16, -9/16, -1/16, 7/16], [7/16, -1/16, 7/16, -1/16, -9/16, -1/16], [7/16, -1/16, 7/16, -1/16, -9/16, -1/16], [7/16, -1/16, -9/16, -1/16, 7/16, -1/16], [-9/16, -1/16, 7/16, -1/16, 7/16, -1/16], [-1/16, -9/16, -1/16, 7/16, -1/16, 7/16], [7/16, -1/16, -9/16, -1/16, 7/16, -1/16] ] \$ \times \$ [ [2, 2, 1, 0, 1, 1, 1, 0], [1, 1, 2, 0, 1, 2, 1, 0], [2, 1, 1, 0, 1, 1, 2, 0], [1, 1, 1, 0, 2, 2, 1, 0], [1, 2, 1, 0, 1, 1, 2, 0], [1, 1, 2, 0, 2, 1, 1, 0] ] \$$

$\hat{A} \gg \text{SYNC'D } 2665/65536, 0.04066467285$

104 . Coloring,  $\{2, 3, 4, 6, 8\}$

**R:**  $[3, 8, 8, 6, 7, 4, 5, 2]$  **B:**  $[6, 3, 1, 1, 2, 7, 4, 5]$

' See graph

' ' See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

'  $[ '4' ('-1+\tau')$ ''  $(' - 5 - \tau + \tau^2 + \tau^3')$ '' ,  $4' ('1+\tau')$ ''  $('5+3\tau+3\tau^2+\tau^3')$ '' ,  $-4' ('-1+\tau')$ ''  $('1+\tau')$ ''  $('5+2\tau+\tau^2')$ '' ,  $4' ('5+2\tau^2+\tau^4')$ '' ,  $4' ('5+\tau+\tau^2+\tau^3')$ ''  $('1+\tau')$ '' ,  $4' ('5-2\tau+2\tau^2+2\tau^3+\tau^4')$ '' ,  $4' ('1+\tau^2')$ ''  $('5+2\tau+\tau^2')$ '' ,  $4' ('1+\tau')$ ''  $^2 ('5+\tau^2')$ '' ]'

For  $\tau=1/2$ ,  $[41, 177, 75, 89, 141, 77, 125, 189]$  . FixedPtCheck,  $[41, 177, 75, 89, 141, 77, 125, 189]$

$$\det(A + \tau \Delta) = 1' (' \tau')$$
''  $^2 ('1+\tau^2')$ ''  $(' - 1 + \tau')$ ''  $('1+\tau')$ ''

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	2 vs 7	4 vs 7

Omega Rank for R : cycles:  $\{\{2, 8\}, \{4, 6\}, \{5, 7\}\}$ , net cycles: 2 . order: 2

$\$ [ [0, 1, 1, 1, 1, 1, 1, 2], [0, 2, 0, 1, 1, 1, 1, 2], [0, 2, 0, 1, 1, 1, 1, 2], [0, 2, 0, 1, 1, 1, 1, 2], [0, 2, 0, 1, 1, 1, 1, 2], [0, 2, 0, 1, 1, 1, 1, 2], [0, 2, 0, 1, 1, 1, 1, 2], [0, 2, 0, 1, 1, 1, 1, 2] ] \$$

$$[0, -y_1 + 2y_2, y_1, y_2, y_2, y_2, y_2, 2y_2]$$

$$p = -s^2 + s^6 \quad p = -s^2 + s^3 \quad p = -s^2 + s^4 \quad p = -s^2 + s^5 \quad p = -s^2 + s^7$$

Omega Rank for B : cycles:  $\{\{1, 4, 6, 7\}\}$ , net cycles: 0 . order: 4

$\$ [ [2, 1, 1, 1, 1, 1, 1, 0], [2, 1, 1, 1, 0, 2, 1, 0], [2, 0, 1, 1, 0, 2, 2, 0], [2, 0, 0, 2, 0, 2, 2, 0], [2, 0, 0, 2, 0, 2, 2, 0], [2, 0, 0, 2, 0, 2, 2, 0], [2, 0, 0, 2, 0, 2, 2, 0], [2, 0, 0, 2, 0, 2, 2, 0] ] \$$

$$[y_2 + y_3, y_2 + y_3 - y_4, y_2 + y_3 - y_1, y_1, y_2, y_3, y_4, 0]$$

$$p = -s^4 + s^6 \quad p = -s^4 + s^7 \quad p = -s^4 + s^5$$

$\hat{A} \gg \text{SYNC'D } 285/262144, 0.001087188721$

105 . Coloring,  $\{2, 3, 4, 7, 8\}$

**R:**  $[3, 8, 8, 6, 7, 7, 4, 2]$  **B:**  $[6, 3, 1, 1, 2, 4, 5, 5]$

' See graph

' ' See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

$$[ ' 4' ( ' - 1 + \tau ' ) ' ( ' - 5 - \tau - 3\tau^2 + \tau^3 ' ) ' , 4' ( ' 1 + \tau ' ) ' ( ' 5 - \tau + 3\tau^2 + \tau^3 ' ) ' , -12' ( ' 5 + 3\tau^2 ' ) ' ( ' - 1 + \tau ' ) ' ( ' 1 + \tau ' ) ' , 12' ( ' 5 + 2\tau + 8\tau^2 - 2\tau^3 + 3\tau^4 ' ) ' , -4' ( ' - 1 + \tau ' ) ' ( ' 5 + \tau + \tau^2 + \tau^3 ' ) ' ( ' 1 + \tau ' ) ' , 4' ( ' 1 + \tau^2 ' ) ' ( ' 5 - \tau + 3\tau^2 + \tau^3 ' ) ' , 4' ( ' 1 + \tau ' ) ' ( ' 5 + 2\tau^2 + \tau^4 ' ) ' , -4' ( ' - 5 + 3\tau - 3\tau^2 + \tau^3 ' ) ' ( ' 1 + \tau ' ) ' ^2 ' ]'$$

For  $\tau=1/2$ ,  $[98, 258, 138, 254, 141, 215, 267, 297]$  . FixedPtCheck,  $[98, 258, 138, 254, 141, 215, 267, 297]$

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	4 vs 6	6 vs 6

Omega Rank for R : cycles: {{2, 8}, {4, 6, 7}}, net cycles: 1 . order: 6

\$ [ [0, 1, 1, 1, 0, 1, 2, 2] , [0, 2, 0, 2, 0, 1, 1, 2] , [0, 2, 0, 1, 0, 2, 1, 2] , [0, 2, 0, 1, 0, 1, 2, 2] , [0, 2, 0, 2, 0, 1, 1, 2] , [0, 2, 0, 1, 0, 2, 1, 2] ] \$

[0, y<sub>4</sub>, -y<sub>4</sub> + y<sub>3</sub>, 2 y<sub>3</sub> - y<sub>1</sub> - y<sub>2</sub>, 0, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>]

$$p' = -s^2 + s^5 \quad p = -s^2 + s^5$$

Omega Rank for B : cycles: {{1, 4, 6}}, net cycles: 0 . order: 6

[y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>6</sub>, y<sub>5</sub>, 0, 0]

B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 1, 0, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, -1/8, 3/8, -1/8] , [0, 0, 1/2, -1/8, -1/8, -1/8] , [0, 0, 0, 3/8, -1/8, -1/8] , [0, 0, 0, 3/8, -1/8, -1/8] , [0, 1/2, -1/4, -1/8, -1/8, 1/8] , [0, 0, 0, -1/8, -1/8, 3/8] , [1/2, -1/4, -1/8, -1/8, 1/8, 0] , [1/2, -1/4, -1/8, -1/8, 1/8, 0] ] \$ x \$ [ [2, 1, 1, 1, 2, 1, 0, 0] , [2, 2, 1, 1, 0, 2, 0, 0] , [2, 0, 2, 2, 0, 2, 0, 0] , [4, 0, 0, 2, 0, 2, 0, 0] , [2, 0, 0, 2, 0, 4, 0, 0] , [2, 0, 0, 4, 0, 2, 0, 0] ] \$

Â» SYNC'D 1665/32768 , 0.05081176758

106 . Coloring, {2, 3, 5, 6, 7}

**R:** [3, 8, 8, 1, 2, 4, 4, 5]    **B:** [6, 3, 1, 6, 7, 7, 5, 2]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

' [-4' (' - 1 + τ ' )' (' 1 + τ ' )' (' - 5 - τ - 3τ<sup>2</sup> + τ<sup>3</sup> ' )' , -4' (' 5 + 10τ<sup>2</sup> + τ<sup>4</sup> ' )' (' 1 + τ ' )' , 4' (' - 1 + τ ' )' (' 5 + 3τ + 7τ<sup>2</sup> + τ<sup>3</sup> ' )' (' 1 + τ ' )' , 4' (' - 1 + τ ' )' (' 5 - τ + 3τ<sup>2</sup> + τ<sup>3</sup> ' )' (' 1 + τ ' )' , 12' (' 1 + 3τ<sup>2</sup> ' )' (' - 5 - τ - 3τ<sup>2</sup> + τ<sup>3</sup> ' )' , -12' (' 5 + 3τ<sup>2</sup> ' )' (' - 1 + τ ' )' <sup>2</sup> (' 1 + τ ' )' , 12' (' 5 - 2τ + 8τ<sup>2</sup> + 2τ<sup>3</sup> + 3τ<sup>4</sup> ' )' (' - 1 + τ ' )' , 12' (' - 5 + τ - 7τ<sup>2</sup> + 3τ<sup>3</sup> ' )' (' 1 + τ ' )' <sup>2</sup> ]'

For τ=1/2, [-147, -363, -201, -129, -343, -69, -103, -423] . FixedPtCheck, [147, 363, 201, 129, 343, 69, 103, 423]

det(A + τ Δ) = 0

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	6 vs 6	5 vs 6

Omega Rank for R : cycles: {{2, 5, 8}}, net cycles: 0 . order: 6

$$[y_3, y_4, y_5, y_6, y_2, 0, 0, y_1]$$

$$\begin{aligned} R = & \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, \\ & 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0], \\ & [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, \\ & 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 1/2, -1/8, -1/8, -1/8], [0, 0, 0, 3/8, -1/8, \\ & -1/8], [0, 0, 0, 3/8, -1/8, -1/8], [0, 1/2, -1/4, -1/8, -1/8, 1/8], [0, 0, 0, -1/8, -1/8, 3/8], [1/2, -1/4, -1/8, -1/8, \\ & 1/8, 0], [1/2, -1/4, -1/8, -1/8, 1/8, 0], [0, 0, 0, -1/8, 3/8, -1/8] ] \$ \times \$ [ [1, 1, 1, 2, 1, 0, 0, 2], [2, 1, 1, 0, 2, \\ & 0, 0, 2], [0, 2, 2, 0, 2, 0, 0, 2], [0, 2, 0, 0, 2, 0, 0, 4], [0, 2, 0, 0, 4, 0, 0, 2], [0, 4, 0, 0, 2, 0, 0, 2] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{5, 7}}, net cycles: 0 . order: 6

$$\begin{aligned} \$ [ [1, 1, 1, 0, 1, 2, 2, 0], [1, 0, 1, 0, 2, 1, 3, 0], [1, 0, 0, 0, 3, 1, 3, 0], [0, 0, 0, 0, 3, 1, 4, 0], [0, 0, 0, 0, 4, \\ 0, 4, 0], [0, 0, 0, 0, 4, 0, 4, 0] ] \$ \end{aligned}$$

$$[-y_2 + y_1 + y_3 + y_4 - y_5, y_2, y_1, 0, y_3, y_4, y_5, 0]$$

$$p = -s^5 + s^6$$

Â» SYNC'D 555/8192 , 0.06774902344

107 . Coloring, {2, 3, 5, 6, 8}

**R:** [3, 8, 8, 1, 2, 4, 5, 2] **B:** [6, 3, 1, 6, 7, 7, 4, 5]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$\begin{aligned} & [ '4' ('5 - \tau - \tau^2 + \tau^3')'' ('1 + \tau')'' ('-1 + \tau')'^2, 4' ('1 + \tau')'^2 ('5 + 2\tau^2 + \tau^4')', \\ & -4' ('5 - 3\tau + \tau^2 + \tau^3')'' ('1 + \tau')'^2 ('-1 + \tau')', -4' ('5 + \tau + \tau^2 + \tau^3')'' ('-1 + \tau')'^3, -4' \\ & ('1 + \tau')'' ('-1 + \tau')'' ('5 - \tau + 3\tau^2 + \tau^3')', 4' ('-5 + \tau^2')'' ('-1 + \tau')'^3, 12' ('-1 + \tau')'^2 \\ & ('5 - 3\tau + 3\tau^2 + 3\tau^3')', 12' ('1 + \tau')'^3 ('5 - 4\tau + 3\tau^2')' ]' \end{aligned}$$

For τ=1/2, [105, 801, 279, 47, 258, 38, 74, 810] . FixedPtCheck, [105, 801, 279, 47, 258, 38, 74, 810]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	4 vs 6	5 vs 6

Omega Rank for R : cycles: {{2, 8}}, net cycles: -1 . order: 4

\$ [ [1, 2, 1, 1, 1, 0, 0, 2] , [1, 3, 1, 0, 0, 0, 0, 3] , [0, 3, 1, 0, 0, 0, 0, 4] , [0, 4, 0, 0, 0, 0, 0, 4] , [0, 4, 0, 0, 0, 0, 0, 4] , [0, 4, 0, 0, 0, 0, 0, 4] ] \$

$$[y_1 + y_2 - y_3, y_1, y_2, y_4, y_4, 0, 0, y_3]$$

$$p = -s^4 + s^6 \quad p = -s^4 + s^5$$

Omega Rank for B : cycles: {{4, 6, 7}}, net cycles: -1 . order: 3

\$ [ [1, 0, 1, 1, 1, 2, 2, 0] , [1, 0, 0, 2, 0, 2, 3, 0] , [0, 0, 0, 3, 0, 3, 2, 0] , [0, 0, 0, 2, 0, 3, 3, 0] , [0, 0, 0, 3, 0, 2, 3, 0] , [0, 0, 0, 3, 0, 3, 2, 0] ] \$

$$[y_1, 0, y_5, y_2, y_5, y_3, y_4, 0]$$

$$p = -s^3 + s^6$$

Â» SYNC'D 855/65536 , 0.01304626465

108 . Coloring, {2, 3, 5, 7, 8}

**R:** [3, 8, 8, 1, 2, 7, 4, 2] **B:** [6, 3, 1, 6, 7, 4, 5, 5]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$[ '4' ('-5 + \tau')'' ('-1 + \tau')'^2 ' ('1 + \tau')' , 4' ('-5 - \tau - 3\tau^2 + \tau^3')'' ('1 + \tau')' , -4' ('-5 + \tau^2')'' ('-1 + \tau')'' ('1 + \tau')' , -4' ('-1 + \tau')'^2 ' ('5 - 2\tau + \tau^2')' , 4' ('5 - \tau + 3\tau^2 + \tau^3')'' ('-1 + \tau')' , 4' ('5 + \tau')'' ('-1 + \tau')'^3 , 4' ('-5 + \tau^2')'' ('-1 + \tau')'^2 , -4' ('5 - 2\tau + \tau^2')'' ('1 + \tau')'^2 ' ]'$$

For  $\tau=1/2$ , [-27, -147, -57, -17, -43, -11, -19, -153] . FixedPtCheck, [27, 147, 57, 17, 43, 11, 19, 153]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	5 vs 6	4 vs 6

Omega Rank for R : cycles: {{2, 8}}, net cycles: 0 . order: 6

\$ [ [1, 2, 1, 1, 0, 0, 1, 2] , [1, 2, 1, 1, 0, 0, 0, 3] , [1, 3, 1, 0, 0, 0, 0, 3] , [0, 3, 1, 0, 0, 0, 0, 4] , [0, 4, 0, 0, 0, 0, 0, 4] , [0, 4, 0, 0, 0, 0, 0, 4] ] \$

$$[y_1 + y_2 + y_3 - y_4 - y_5, y_1, y_2, y_3, 0, 0, y_4, y_5]$$

$$p = -s^5 + s^6$$

Omega Rank for B : cycles: {{4, 6}, {5, 7}}, net cycles: 1 . order: 4

$$\$ [ [1, 0, 1, 1, 2, 2, 1, 0], [1, 0, 0, 2, 1, 2, 2, 0], [0, 0, 0, 2, 2, 3, 1, 0], [0, 0, 0, 3, 1, 2, 2, 0], [0, 0, 0, 2, 2, 3, 1, 0], [0, 0, 0, 3, 1, 2, 2, 0] ] \$$$

$$[y_1, 0, 4y_1 + 4y_2 - y_3 - 5y_4, y_2, 3y_1 + 3y_2 - 4y_4, y_3, y_4, 0]$$

$$p = -s^3 + s^5 \quad p' = -s^3 + s^5$$

Â» SYNC'D 463/65536 , 0.007064819336

109 . Coloring, {2, 3, 6, 7, 8}

**R:** [3, 8, 8, 1, 7, 4, 4, 2]    **B:** [6, 3, 1, 6, 2, 7, 5, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ '4' ('-1 + \tau')'' ('-5 - 3\tau - 10\tau^2 + 2\tau^3 - \tau^4 + \tau^5')', 4' ('5 - 2\tau + 19\tau^2 + 7\tau^4 + 2\tau^5 + \tau^6')', \\ & -4' ('-1 + \tau')'' ('5 + 3\tau + 16\tau^2 + 4\tau^3 + 3\tau^4 + \tau^5')', -4' ('1 + \tau^2')'' ('1 + \tau')'' ('-1 + \tau')'' \\ & ('5 - 2\tau + \tau^2')', 4' ('-1 + \tau')'' ('-5 + 2\tau - 12\tau^2 - 2\tau^3 + \tau^4')', 4' ('-1 + \tau')''^2 ('5 + 3\tau + 7\tau^2 + \tau^3')', \\ & -12' ('-1 + \tau')'' ('5 - 2\tau + 8\tau^2 + 2\tau^3 + 3\tau^4')', 12' ('1 + \tau')'' ('5 - 2\tau + \tau^2')'' ('1 + 3\tau^2')'' ]' \end{aligned}$$

For τ=1/2, [281, 593, 359, 255, 230, 134, 206, 714] . FixedPtCheck, [281, 593, 359, 255, 230, 134, 206, 714]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	5 vs 6	6 vs 6

Omega Rank for R : cycles: {{2, 8}}, net cycles: 0 . order: 6

$$\$ [ [1, 1, 1, 2, 0, 0, 1, 2], [2, 2, 1, 1, 0, 0, 0, 2], [1, 2, 2, 0, 0, 0, 0, 3], [0, 3, 1, 0, 0, 0, 0, 4], [0, 4, 0, 0, 0, 0, 0, 4], [0, 4, 0, 0, 0, 0, 0, 4] ] \$$$



$$[y_1 + y_2 + y_3 - y_4 - y_5, y_1, y_2, y_3, 0, 0, y_4, y_5]$$

$$p = -s^5 + s^6$$

Omega Rank for B : cycles: {{1, 2, 3, 5, 6, 7}}, net cycles: 1 . order: 6

$$[y_6, y_5, y_4, 0, y_3, y_2, y_1, 0]$$

$$\begin{aligned} \mathbf{B} = & \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 1, 0, 0, \\ & 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0] \\ & , [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, \\ & 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$ = \$ [ [7/16, -1/16, -9/16, -1/16, 7/16, -1/16], [-9/16, \\ & -1/16, 7/16, -1/16, 7/16, -1/16], [-1/16, -9/16, -1/16, 7/16, -1/16, 7/16], [7/16, -1/16, -9/16, -1/16, 7/16, \\ & -1/16], [-1/16, 7/16, -1/16, 7/16, -1/16, -9/16], [-1/16, 7/16, -1/16, -9/16, -1/16, 7/16], [7/16, -1/16, 7/16, \\ & -1/16, -9/16, -1/16], [7/16, -1/16, 7/16, -1/16, -9/16, -1/16] ] \$ \times \$ [ [1, 1, 1, 0, 2, 2, 1, 0], [1, 2, 1, 0, 1, 1, \\ & 2, 0], [1, 1, 2, 0, 2, 1, 1, 0], [2, 2, 1, 0, 1, 1, 1, 0], [1, 1, 2, 0, 1, 2, 1, 0], [2, 1, 1, 0, 1, 1, 2, 0] ] \$ \end{aligned}$$

Â» SYNC'D 2665/65536 , 0.04066467285

110 . Coloring, {2, 4, 5, 6, 7}

**R:** [3, 8, 1, 6, 2, 4, 4, 5] **B:** [6, 3, 8, 1, 7, 7, 5, 2]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$\begin{aligned} & [ '4' ( ' - 5 + \tau ' ) ' ( ' 1 + \tau ' ) ' , 4' ( ' - 5 + \tau ^ 2 ' ) ' , -4' ( ' 5 + 2\tau + \tau ^ 2 ' ) ' , -4' ( ' 5 + \tau ' ) ' ( ' 1 + \tau ' \\ & ) ' , 4' ( ' - 5 + \tau - \tau ^ 2 + \tau ^ 3 ' ) ' , -4' ( ' 1 + \tau ' ) ' ( ' 5 + \tau ^ 2 ' ) ' , 4' ( ' - 1 + \tau ' ) ' ( ' 5 + 2\tau + \tau ^ 2 ' ) ' , 4' ( ' - \\ & 5 - \tau + \tau ^ 2 + \tau ^ 3 ' ) ' , ' ] \end{aligned}$$

For τ=1/2, [-54, -38, -50, -66, -37, -63, -25, -41] . FixedPtCheck, [54, 38, 50, 66, 37, 63, 25, 41]

$$\det(\mathbf{A} + \tau \Delta) = 1' ( ' \tau ' ) ' ^ 2 ( ' 1 + \tau ^ 2 ' ) ' ( ' - 1 + \tau ' ) ' ( ' 1 + \tau ' ) '$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	2 vs 7	4 vs 7

Omega Rank for R : cycles: {{1, 3}, {4, 6}, {2, 5, 8}}, net cycles: 3 . order: 6

$$\begin{aligned} \$ [ [1, 1, 1, 2, 1, 1, 0, 1], [1, 1, 1, 1, 1, 2, 0, 1], [1, 1, 1, 2, 1, 1, 0, 1], [1, 1, 1, 1, 1, 2, 0, 1], [1, 1, 1, 2, 1, \\ & 1, 0, 1], [1, 1, 1, 1, 1, 2, 0, 1], [1, 1, 1, 2, 1, 1, 0, 1] ] \$ \end{aligned}$$

$$[y_2, y_2, y_2, 3y_2 - y_1, y_2, y_1, 0, y_2]$$

$$p' = -s + s^5 \quad p' = -s^2 + s^6 \quad p = s - s^3 \quad p' = -s + s^3 \quad p' = -s^2 + s^4$$

Omega Rank for B : cycles: {{2, 3, 8}, {5, 7}}, net cycles: 1 . order: 6

$$\$ [ [1, 1, 1, 0, 1, 1, 2, 1], [0, 1, 1, 0, 2, 1, 2, 1], [0, 1, 1, 0, 2, 0, 3, 1], [0, 1, 1, 0, 3, 0, 2, 1], [0, 1, 1, 0, 2, 0, 3, 1], [0, 1, 1, 0, 3, 0, 2, 1], [0, 1, 1, 0, 2, 0, 3, 1] ] \$$$

$$[-y_4 - y_3 - y_1 + 5y_2, y_2, y_2, 0, y_4, y_3, y_1, y_2]$$

$$p = -s^3 + s^7 \quad p' = -s^3 + s^5 \quad p = -s^3 + s^5$$

Â» SYNC'D 2469/262144 , 0.009418487549

111 . Coloring, {2, 4, 5, 6, 8}

**R:** [3, 8, 1, 6, 2, 4, 5, 2]    **B:** [6, 3, 8, 1, 7, 7, 4, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ '4' ('1 + \tau^2')'' ('5 + 2\tau + \tau^2')', 4' ('1 + \tau')^2 ('5 + \tau^2')', 4' ('5 + \tau + \tau^2 + \tau^3')'', \\ & ('1 + \tau')', 4' ('5 - 2\tau + 2\tau^2 + 2\tau^3 + \tau^4')', -4' ('1 + \tau')'' ('-1 + \tau')'' ('5 + 2\tau + \tau^2')', 4' ('5 \\ & + 2\tau^2 + \tau^4')', 4' ('-5 - \tau + \tau^2 + \tau^3')'' ('-1 + \tau')', 4' ('1 + \tau')'' ('5 + 3\tau + 3\tau^2 + \tau^3')'' ]' \end{aligned}$$

For τ=1/2, [125, 189, 141, 77, 75, 89, 41, 177] . FixedPtCheck, [125, 189, 141, 77, 75, 89, 41, 177]

$$\det(A + \tau \Delta) = 1' ('\tau')^2 ('1 + \tau^2')'' ('1 + \tau')'' ('-1 + \tau')$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	2 vs 7	4 vs 7

Omega Rank for R : cycles: {{1, 3}, {2, 8}, {4, 6}}, net cycles: 2 . order: 2

$$\$ [ [1, 2, 1, 1, 1, 0, 1], [1, 2, 1, 1, 0, 1, 0, 2], [1, 2, 1, 1, 0, 1, 0, 2], [1, 2, 1, 1, 0, 1, 0, 2], [1, 2, 1, 1, 0, 1, 0, 2], [1, 2, 1, 1, 0, 1, 0, 2], [1, 2, 1, 1, 0, 1, 0, 2] ] \$$$

$$[y_2, 2y_2, y_2, y_2, y_1, y_2, 0, 2y_2 - y_1]$$

$$p' = -s^4 + s^6 \quad p' = -s^4 + s^5 \quad p' = s^3 - s^4 \quad p' = s^2 - s^4 \quad p = s^2 - s^5$$

Omega Rank for B : cycles: {{1, 4, 6, 7}}, net cycles: 0 . order: 4

\$ [ [1, 0, 1, 1, 1, 1, 2, 1] , [1, 0, 0, 2, 1, 1, 2, 1] , [2, 0, 0, 2, 1, 1, 2, 0] , [2, 0, 0, 2, 0, 2, 2, 0] , [2, 0, 0, 2, 0, 2, 2, 0] , [2, 0, 0, 2, 0, 2, 2, 0] ] \$

$$[y_4, 0, y_3, y_2, y_1, y_3 + y_2 - y_1, y_3 + y_2, -y_4 + y_3 + y_2]$$

$$p' = -s^4 + s^6 \quad p' = -s^4 + s^5 \quad p = s^4 - s^5$$

Â» SYNC'D 285/262144 , 0.001087188721

112 . Coloring, {2, 4, 5, 7, 8}

**R:** [3, 8, 1, 6, 2, 7, 4, 2]    **B:** [6, 3, 8, 1, 7, 4, 5, 5]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$[ -4' ( ' 5 + 2\tau + \tau^2 ' )' , -4' ( ' 5 + \tau ' )' ( ' 1 + \tau ' )' , 4' ( ' 1 + \tau ' )' ( ' - 5 + \tau ' )' , 4' ( ' - 5 + \tau^2 ' )' , 4' ( ' - 1 + \tau ' )' ( ' 5 + 2\tau + \tau^2 ' )' , 4' ( ' - 5 - \tau + \tau^2 + \tau^3 ' )' , 4' ( ' - 5 + \tau - \tau^2 + \tau^3 ' )' , -4' ( ' 1 + \tau ' )' ( ' 5 + \tau^2 ' )' ]'$$

For τ=1/2, [-50, -66, -54, -38, -25, -41, -37, -63] . FixedPtCheck, [50, 66, 54, 38, 25, 41, 37, 63]

$$\det(A + \tau \Delta) = 1' ( ' \tau ' )' ^2 ( ' 1 + \tau^2 ' )' ( ' 1 + \tau ' )' ( ' - 1 + \tau ' )'$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	2 vs 7	4 vs 7

Omega Rank for R : cycles: {{1, 3}, {2, 8}, {4, 6, 7}}, net cycles: 3 . order: 6

\$ [ [1, 2, 1, 1, 0, 1, 1, 1] , [1, 1, 1, 1, 0, 1, 1, 2] , [1, 2, 1, 1, 0, 1, 1, 1] , [1, 1, 1, 1, 0, 1, 1, 2] , [1, 2, 1, 1, 0, 1, 1, 1] , [1, 1, 1, 1, 0, 1, 1, 2] , [1, 2, 1, 1, 0, 1, 1, 1] ] \$

$$[y_1, 3y_1 - y_2, y_1, y_1, 0, y_1, y_1, y_2]$$

$$p = -s + s^3 \quad p' = -s + s^3 \quad p = -s + s^5 \quad p' = -s + s^5 \quad p = -s + s^7$$

Omega Rank for B : cycles: {{1, 4, 6}, {5, 7}}, net cycles: 1 . order: 6

\$ [ [1, 0, 1, 1, 2, 1, 1, 1] , [1, 0, 0, 1, 2, 1, 2, 1] , [1, 0, 0, 1, 3, 1, 2, 0] , [1, 0, 0, 1, 2, 1, 3, 0] , [1, 0, 0, 1, 3, 1, 2, 0] , [1, 0, 0, 1, 2, 1, 3, 0] , [1, 0, 0, 1, 3, 1, 2, 0] ] \$

$$[y_3, 0, y_2, y_3, y_1, y_3, -y_2 + 5y_3 - y_1 - y_4, y_4]$$

$$p = -s^3 + s^7 \quad p' = -s^3 + s^5 \quad p = -s^3 + s^5$$

Â» SYNC'D 2469/262144 , 0.009418487549

113 . Coloring, {2, 4, 6, 7, 8}

**R:** [3, 8, 1, 6, 7, 4, 4, 2]    **B:** [6, 3, 8, 1, 2, 7, 5, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ ' 4 ' ( ' 1 + \tau ' ) ' ( ' 5 + \tau + \tau^2 + \tau^3 ' ) ' , 4 ' ( ' 5 - 2\tau + 2\tau^2 + 2\tau^3 + \tau^4 ' ) ' , 4 ' ( ' 1 + \tau^2 ' ) ' ( ' 5 \\ & + 2\tau + \tau^2 ' ) ' , 4 ' ( ' 1 + \tau ' ) ^2 ' ( ' 5 + \tau^2 ' ) ' , 4 ' ( ' - 1 + \tau ' ) ' ( ' - 5 - \tau + \tau^2 + \tau^3 ' ) ' , 4 ' ( ' 1 + \tau ' \\ & ) ' ( ' 5 + 3\tau + 3\tau^2 + \tau^3 ' ) ' , -4 ' ( ' - 1 + \tau ' ) ' ( ' 1 + \tau ' ) ' ( ' 5 + 2\tau + \tau^2 ' ) ' , 4 ' ( ' 5 + 2\tau^2 + \tau^4 ' \\ & ) ' ] ' \end{aligned}$$

For τ=1/2, [141, 77, 125, 189, 41, 177, 75, 89] . FixedPtCheck, [141, 77, 125, 189, 41, 177, 75, 89]

$$\det(A + \tau \Delta) = 1 ' ( ' - 1 + \tau ' ) ' ( ' \tau ' ) ^2 ' ( ' 1 + \tau^2 ' ) ' ( ' 1 + \tau ' ) '$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	8 vs 8	8 vs 8	2 vs 7	4 vs 7

Omega Rank for R : cycles: {{1, 3}, {2, 8}, {4, 6}}, net cycles: 2 . order: 2

$$\begin{aligned} \$ [ [1, 1, 1, 2, 0, 1, 1, 1], [1, 1, 1, 2, 0, 2, 0, 1], [1, 1, 1, 2, 0, 2, 0, 1], [1, 1, 1, 2, 0, 2, 0, 1], [1, 1, 1, 2, 0, \\ 2, 0, 1], [1, 1, 1, 2, 0, 2, 0, 1], [1, 1, 1, 2, 0, 2, 0, 1] ] \$ \end{aligned}$$

$$[y_2, y_2, y_2, 2y_2, 0, y_1, 2y_2 - y_1, y_2]$$

$$p' = s^5 - s^6 \quad p' = s^4 - s^6 \quad p' = s^3 - s^6 \quad p' = s^2 - s^6 \quad p = s^2 - s^7$$

Omega Rank for B : cycles: {{2, 3, 5, 8}}, net cycles: 0 . order: 4

$$\begin{aligned} \$ [ [1, 1, 1, 0, 2, 1, 1, 1], [0, 2, 1, 0, 2, 1, 1, 1], [0, 2, 2, 0, 2, 0, 1, 1], [0, 2, 2, 0, 2, 0, 0, 2], [0, 2, 2, 0, 2, \\ 0, 0, 2], [0, 2, 2, 0, 2, 0, 0, 2], [0, 2, 2, 0, 2, 0, 0, 2] ] \$ \end{aligned}$$

$$[y_1, -y_1 + y_3, y_3 - y_4, 0, y_3, y_4, y_2, y_3 - y_2]$$

$$p = -s^4 + s^5 \quad p = -s^4 + s^7 \quad p = -s^4 + s^6$$

Â» SYNC'D 285/262144 , 0.001087188721

114 . Coloring, {2, 5, 6, 7, 8}

**R:** [3, 8, 1, 1, 2, 4, 4, 2]    **B:** [6, 3, 8, 6, 7, 7, 5, 5]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$\begin{aligned} & \left[ 4 \left( -5 + \tau - \tau^2 + \tau^3 \right) \left( 1 + \tau \right)^2, 4 \left( 1 + \tau^2 \right) \left( -5 + \tau^2 \right) \left( 1 + \tau \right), 4 \left( 1 + \tau \right) \left( -5 - 2\tau - 4\tau^2 + 2\tau^3 + \tau^4 \right), \right. \\ & \left. -4 \left( -5 - \tau + \tau^2 + \tau^3 \right) \left( 1 + \tau \right) \left( -1 + \tau \right), 4 \left( 5 - \tau + 3\tau^2 + \tau^3 \right) \left( -1 + \tau \right), \right. \\ & \left. -4 \left( -5 + \tau^2 \right) \left( 1 + \tau \right) \left( -1 + \tau \right), -4 \left( 5 + 2\tau + \tau^2 \right) \left( -1 + \tau \right)^2, \right. \\ & \left. 4 \left( -5 - \tau - 3\tau^2 + \tau^3 \right) \left( 1 + \tau \right) \right] \end{aligned}$$

For  $\tau=1/2$ , [-333, -285, -321, -123, -86, -114, -50, -294] . FixedPtCheck, [333, 285, 321, 123, 86, 114, 50, 294]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	3 vs 5	4 vs 5

Omega Rank for R : cycles: {{1, 3}, {2, 8}}, net cycles: 1 . order: 2

$$\$ [ [2, 2, 1, 2, 0, 0, 0, 1], [3, 1, 2, 0, 0, 0, 0, 2], [2, 2, 3, 0, 0, 0, 0, 1], [3, 1, 2, 0, 0, 0, 0, 2], [2, 2, 3, 0, 0, 0, 0, 1] ] \$$$

$$[y_1, 3y_1 - 4y_3, 4y_1 - y_2 - 5y_3, y_2, 0, 0, 0, y_3]$$

$$p' = -s^2 + s^4 \quad p = -s^2 + s^4$$

Omega Rank for B : cycles: {{5, 7}}, net cycles: -1 . order: 4

$$\$ [ [0, 0, 1, 0, 2, 2, 2, 1], [0, 0, 0, 0, 3, 0, 4, 1], [0, 0, 0, 0, 5, 0, 3, 0], [0, 0, 0, 0, 3, 0, 5, 0], [0, 0, 0, 0, 5, 0, 3, 0] ] \$$$

$$[0, 0, y_1, 0, y_4, 2y_1, y_2, y_3]$$

$$p = -s^3 + s^5$$

Â» SYNC'D 9/256 , 0.03515625000

115 . Coloring, {3, 4, 5, 6, 7}

**R:** [3, 3, 8, 6, 2, 4, 4, 5]    **B:** [6, 8, 1, 1, 7, 7, 5, 2]

' See graph

' ' See pair graph

Ω for A+τΔ :

' [-4' (' - 1 + τ ' )' (' - 5 + τ ' )' (' 1 + τ ' )' , 4' (' - 5 - τ - 3τ<sup>2</sup> + τ<sup>3</sup> ' )' , 4' (' - 5 + τ<sup>2</sup> ' )' (' 1 + τ ' )' , -4' (' 1 + τ ' )' (' 5 - 2τ + τ<sup>2</sup> ' )' , 12' (' - 5 + τ - 7τ<sup>2</sup> + 3τ<sup>3</sup> ' )' , -12' (' 5 - 4τ + 3τ<sup>2</sup> ' )' (' 1 + τ ' )' , 12' (' - 1 + τ ' )' (' 5 + 3τ<sup>2</sup> ' )' , 12' (' - 5 - 3τ - 3τ<sup>2</sup> + 3τ<sup>3</sup> ' )' ]'

For τ=1/2, [-27, -49, -57, -51, -47, -45, -23, -55] . FixedPtCheck, [27, 49, 57, 51, 47, 45, 23, 55]

det(A + τ Δ) = 0

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	4 vs 6	4 vs 6

Omega Rank for R : cycles: {{2, 3, 5, 8}, {4, 6}}, net cycles: 2 . order: 4

\$ [ [0, 1, 2, 2, 1, 1, 0, 1] , [0, 1, 1, 1, 1, 2, 0, 2] , [0, 1, 1, 2, 2, 1, 0, 1] , [0, 2, 1, 1, 1, 2, 0, 1] , [0, 1, 2, 2, 1, 1, 0, 1] , [0, 1, 1, 1, 1, 2, 0, 2] ] \$

[0, 4 y<sub>1</sub> - 5 y<sub>2</sub> + 4 y<sub>3</sub> - y<sub>4</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, 3 y<sub>1</sub> - 4 y<sub>2</sub> + 3 y<sub>3</sub>, 0, y<sub>4</sub>]

p = s - s<sup>5</sup>    p' = - s + s<sup>5</sup>

Omega Rank for B : cycles: {{2, 8}, {5, 7}}, net cycles: 1 . order: 4

\$ [ [2, 1, 0, 0, 1, 1, 2, 1] , [0, 1, 0, 0, 2, 2, 2, 1] , [0, 1, 0, 0, 2, 0, 4, 1] , [0, 1, 0, 0, 4, 0, 2, 1] , [0, 1, 0, 0, 2, 0, 4, 1] , [0, 1, 0, 0, 4, 0, 2, 1] ] \$

[6 y<sub>4</sub> - y<sub>1</sub> - y<sub>2</sub> - y<sub>3</sub>, y<sub>4</sub>, 0, 0, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>]

p = - s<sup>3</sup> + s<sup>5</sup>    p' = - s<sup>3</sup> + s<sup>5</sup>

Â» SYNC'D 179/16384 , 0.01092529297

116 . Coloring, {3, 4, 5, 6, 8}

**R:** [3, 3, 8, 6, 2, 4, 5, 2]    **B:** [6, 8, 1, 1, 7, 7, 4, 5]

' See graph

' ' See pair graph

$\Omega$  for  $A+\tau\Delta$  :

' [ '-4' (' - 1 +  $\tau$  ')'' (' 1 +  $\tau^2$  ')'' (' 5 + 2 $\tau$  +  $\tau^2$  ')', 4' (' 5 +  $\tau$  +  $\tau^2$  +  $\tau^3$  ')'' (' 1 +  $\tau$  ')''<sup>2</sup>, 4' (' 5 + 4 $\tau$  + 6 $\tau^2$  +  $\tau^4$  ')'' (' 1 +  $\tau$  ')', -4' (' - 1 +  $\tau$  ')'' (' 5 + 2 $\tau^2$  +  $\tau^4$  ')', -4' (' - 1 +  $\tau$  ')'' (' 1 +  $\tau$  ')'' (' 5 + 2 $\tau$  +  $\tau^2$  ')', 4' (' - 1 +  $\tau$  ')'' (' - 5 -  $\tau$  - 3 $\tau^2$  +  $\tau^3$  ')', 12' (' - 1 +  $\tau$  ')''<sup>2</sup> (' 5 + 4 $\tau$  + 3 $\tau^2$  ')', 12' (' 5 + 3 $\tau^2$  ')'' (' 1 +  $\tau$  ')''<sup>2</sup> ]'

For  $\tau=1/2$ , [125, 423, 411, 89, 150, 98, 62, 414] . FixedPtCheck, [125, 423, 411, 89, 150, 98, 62, 414]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	2 vs 6	6 vs 6

Omega Rank for **R** : cycles: {{4, 6}, {2, 3, 8}}, net cycles: 1 . order: 6

\$ [ [0, 2, 2, 1, 1, 1, 0, 1] , [0, 2, 2, 1, 0, 1, 0, 2] , [0, 2, 2, 1, 0, 1, 0, 2] , [0, 2, 2, 1, 0, 1, 0, 2] , [0, 2, 2, 1, 0, 1, 0, 2] , [0, 2, 2, 1, 0, 1, 0, 2] ] \$

[0, 2 y<sub>1</sub>, 2 y<sub>1</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>1</sub>, 0, 2 y<sub>1</sub> - y<sub>2</sub>]

$p = s^2 - s^3$      $p' = -s^2 + s^3$      $p'' = -s^2 + s^5$      $p''' = -s^2 + s^4$

Omega Rank for **B** : cycles: {{1, 4, 6, 7}}, net cycles: 0 . order: 4

[y<sub>1</sub>, 0, 0, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

$B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 1] , [1, 0, 0, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, -11/32, -3/32, 5/32, 13/32] , [1, -1, -3/32, 5/32, -19/32, 21/32] , [0, 0, -3/32, 5/32, 13/32, -11/32] , [0, 0, -3/32, 5/32, 13/32, -11/32] , [0, 0, 13/32, -11/32, -3/32, 5/32] , [0, 0, 13/32, -11/32, -3/32, 5/32] , [0, 0, 5/32, 13/32, -11/32, -3/32] , [0, 1, -11/32, -3/32, 5/32, -19/32] ] \$ \times \$ [ [2, 0, 0, 1, 1, 1, 2, 1] , [1, 0, 0, 2, 1, 2, 2, 0] , [2, 0, 0, 2, 0, 1, 3, 0] , [2, 0, 0, 3, 0, 2, 1, 0] , [3, 0, 0, 1, 0, 2, 2, 0] , [1, 0, 0, 2, 0, 3, 2, 0] ] \$$

$\hat{A}$ » SYNC'D 4447/262144 , 0.01696395874

117 . Coloring, {3, 4, 5, 7, 8}

**R:** [3, 3, 8, 6, 2, 7, 4, 2] **B:** [6, 8, 1, 1, 7, 4, 5, 5]

' See graph

' ' See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

' [ '4' (' - 1 +  $\tau$  ')', -4' (' 1 +  $\tau$  ')', -4' (' 1 +  $\tau$  ')', 4' (' - 1 +  $\tau$  ')', 4' (' - 1 +  $\tau$  ')', 4' (' - 1 +  $\tau$  ')', 4' (' - 1 +  $\tau$  ')', -4' (' 1 +  $\tau$  ')']'

For  $\tau=1/2$ , [-1, -3, -3, -1, -1, -1, -1, -3] . FixedPtCheck, [1, 3, 3, 1, 1, 1, 1, 3]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	3 vs 6	4 vs 6

Omega Rank for R : cycles: {{4, 6, 7}, {2, 3, 8}}, net cycles: 2 . order: 3

\$ [ [0, 2, 2, 1, 0, 1, 1, 1], [0, 1, 2, 1, 0, 1, 1, 2], [0, 2, 1, 1, 0, 1, 1, 2], [0, 2, 2, 1, 0, 1, 1, 1], [0, 1, 2, 1, 0, 1, 1, 2], [0, 2, 1, 1, 0, 1, 1, 2] ] \$

[0,  $y_3, y_2, y_1, 0, y_1, y_1, -y_3 - y_2 + 5 y_1$ ]

$p' = s - s^4$   $p' = s^2 - s^5$   $p = s - s^4$

Omega Rank for B : cycles: {{5, 7}, {1, 4, 6}}, net cycles: 1 . order: 6

\$ [ [2, 0, 0, 1, 2, 1, 1, 1], [1, 0, 0, 1, 2, 2, 2, 0], [1, 0, 0, 2, 2, 1, 2, 0], [2, 0, 0, 1, 2, 1, 2, 0], [1, 0, 0, 1, 2, 2, 2, 0], [1, 0, 0, 2, 2, 1, 2, 0] ] \$

[ $y_1, 0, 0, -y_1 + 2 y_3 + 2 y_2 - y_4, y_3 + y_2, y_4, y_3, y_2$ ]

$p = s^2 - s^5$   $p' = s^2 - s^5$

Â» SYNC'D 525/32768 , 0.01602172852

118 . Coloring, {3, 4, 6, 7, 8}

**R:** [3, 3, 8, 6, 7, 4, 4, 2] **B:** [6, 8, 1, 1, 2, 7, 5, 5]



' See graph

' ' See pair graph

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ -4(1+\tau)^2(5+\tau+\tau^2+\tau^3)^2(-1+\tau)^2, 4(1+\tau^2)^2(5-\tau+3\tau^2+\tau^3)^2, \\ & 4(1+\tau)^2(5+2\tau^2+\tau^4)^2, -4(1+\tau)^2(-5+3\tau-3\tau^2+\tau^3)^2, 4(-5-\tau-3\tau^2+\tau^3)^2(-1+\tau)^2, \\ & 4(1+\tau)^2(5-\tau+3\tau^2+\tau^3)^2, -12(1+\tau)^2(5+3\tau^2)^2(-1+\tau)^2, \\ & 12(5+2\tau+8\tau^2-2\tau^3+3\tau^4)^2 ] \end{aligned}$$

For  $\tau=1/2$ , [141, 215, 267, 297, 98, 258, 138, 254] . FixedPtCheck, [141, 215, 267, 297, 98, 258, 138, 254]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	4 vs 6	6 vs 6

Omega Rank for R : cycles: {{4, 6}, {2, 3, 8}}, net cycles: 1 . order: 6

$$\begin{aligned} \$ [ [0, 1, 2, 2, 0, 1, 1, 1], [0, 1, 1, 2, 0, 2, 0, 2], [0, 2, 1, 2, 0, 2, 0, 1], [0, 1, 2, 2, 0, 2, 0, 1], [0, 1, 1, 2, 0, \\ 2, 0, 2], [0, 2, 1, 2, 0, 2, 0, 1] ] \$ \end{aligned}$$

$$[0, -y_1 + 2y_2 + 2y_3 - y_4, y_1, y_2 + y_3, 0, y_2, y_3, y_4]$$

$$p = -s^2 + s^5 \quad p' = -s^2 + s^5$$

Omega Rank for B : cycles: {{2, 5, 8}}, net cycles: 0 . order: 6

$$[y_1, y_2, 0, 0, y_3, y_4, y_5, y_6]$$

$$\begin{aligned} B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, \\ 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0] \\ , [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, \\ 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 1/2, -1/4, -1/8, -1/8, 1/8], [0, 0, 0, -1/8, -1/8, \\ 3/8], [1/2, -1/4, -1/8, -1/8, 1/8, 0], [1/2, -1/4, -1/8, -1/8, 1/8, 0], [0, 0, 0, -1/8, 3/8, -1/8], [0, 0, 1/2, -1/8, \\ -1/8, -1/8], [0, 0, 0, 3/8, -1/8, -1/8], [0, 0, 0, 3/8, -1/8, -1/8] ] \$ \times \$ [ [2, 1, 0, 0, 2, 1, 1, 1], [0, 2, 0, 0, 2, 2, \\ 1, 1], [0, 2, 0, 0, 2, 0, 2, 2], [0, 2, 0, 0, 4, 0, 0, 2], [0, 4, 0, 0, 2, 0, 0, 2], [0, 2, 0, 0, 2, 0, 0, 4] ] \$ \end{aligned}$$

Â» SYNC'D 1665/32768 , 0.05081176758

119 . Coloring, {3, 5, 6, 7, 8}

**R:** [3, 3, 8, 1, 2, 4, 4, 2]    **B:** [6, 8, 1, 6, 7, 7, 5, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ '-4' (' - 1 + \tau ')'' (' - 5 + \tau - \tau^2 + \tau^3 ')'' (' 1 + \tau ')'^2, 4' (' - 5 - \tau - 12\tau^2 + \tau^4 + \tau^5 ')'' (' 1 \\ & + \tau ')', 4' (' 1 + \tau^2 ')'' (' - 5 + \tau^2 ')'' (' 1 + \tau ')'^2, 4' (' - 1 + \tau ')'^2 (' - 5 - 3\tau - \tau^2 + \tau^3 ')'' (' \\ & 1 + \tau ')', -4' (' - 1 + \tau ')'' (' - 5 + \tau - 10\tau^2 - 2\tau^3 - \tau^4 + \tau^5 ')', 4' (' - 1 + \tau ')'^2 (' - 5 - \tau + \tau^2 + \\ & \tau^3 ')'' (' 1 + \tau ')', -4' (' - 1 + \tau ')'^2 (' 5 + 2\tau^2 + \tau^4 ')', 4' (' - 5 - 3\tau - 10\tau^2 + 2\tau^3 - \tau^4 + \tau^5 ')'' (' 1 + \tau ')'' ]' \end{aligned}$$

For  $\tau=1/2$ , [-333, -807, -855, -159, -233, -123, -89, -843] . FixedPtCheck, [333, 807, 855, 159, 233, 123, 89, 843]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	5 vs 5	3 vs 5

Omega Rank for R : cycles: {{2, 3, 8}}, net cycles: 0 . order: 3

$$[y_1, y_2, y_3, y_5, 0, 0, 0, y_4]$$

$$\begin{aligned} R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, \\ 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0] \\ , [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, \\ 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 3/8, -5/8, 3/8], [0, 0, 3/8, -5/8, 3/8], [0, 0, \\ 3/8, 3/8, -5/8], [0, 1/2, -5/8, 3/8, -1/8], [0, 0, -5/8, 3/8, 3/8], [1/2, -1/4, 3/8, -1/8, -3/8], [1/2, -1/4, 3/8, \\ -1/8, -3/8], [0, 0, -5/8, 3/8, 3/8] ] \$ \times \$ [ [1, 2, 2, 2, 0, 0, 0, 1], [2, 1, 3, 0, 0, 0, 0, 2], [0, 2, 3, 0, 0, 0, 0, 3], \\ [0, 3, 2, 0, 0, 0, 0, 3], [0, 3, 3, 0, 0, 0, 0, 2] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{5, 7}}, net cycles: -1 . order: 4

$$\$ [ [1, 0, 0, 0, 2, 2, 2, 1], [0, 0, 0, 0, 3, 1, 4, 0], [0, 0, 0, 0, 4, 0, 4, 0], [0, 0, 0, 0, 4, 0, 4, 0], [0, 0, 0, 0, 4, 0, 4, 0] ] \$$$

$$[y_3, 0, 0, 0, 2y_3 - y_1 + y_2, y_1, y_2, y_3]$$

$$p = -s^3 + s^4 \quad p = -s^3 + s^5$$

Â» SYNC'D 3/64 , 0.04687500000

120 . Coloring, {4, 5, 6, 7, 8}

**R:** [3, 3, 1, 6, 2, 4, 4, 2]    **B:** [6, 8, 8, 1, 7, 7, 5, 5]

' See graph

' ' See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

' [ '4' (' - 5 - 2 $\tau$  - 4 $\tau^2$  + 2 $\tau^3$  +  $\tau^4$  )'' (' 1 +  $\tau$  ' )' , -4' (' 1 +  $\tau$  ' )'' (' - 1 +  $\tau$  ' )'' (' - 5 -  $\tau$  +  $\tau^2$  +  $\tau^3$  ' )' , 4' (' - 5 +  $\tau$  -  $\tau^2$  +  $\tau^3$  ' )'' (' 1 +  $\tau$  ' )'^2 , 4' (' 1 +  $\tau^2$  ' )'' (' - 5 +  $\tau^2$  ' )'' (' 1 +  $\tau$  ' )' , -4' (' - 1 +  $\tau$  ' )'^2 ' (' 5 + 2 $\tau$  +  $\tau^2$  ' )' , 4' (' - 5 -  $\tau$  - 3 $\tau^2$  +  $\tau^3$  ' )'' (' 1 +  $\tau$  ' )' , 4' (' - 1 +  $\tau$  ' )'' (' 5 -  $\tau$  + 3 $\tau^2$  +  $\tau^3$  ' )' , -4' (' - 5 +  $\tau^2$  ' )'' (' 1 +  $\tau$  ' )'' (' - 1 +  $\tau$  ' )'' ]'

For  $\tau=1/2$ , [-321, -123, -333, -285, -50, -294, -86, -114] . FixedPtCheck, [321, 123, 333, 285, 50, 294, 86, 114]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	7 vs 7	7 vs 7	3 vs 5	4 vs 5

Omega Rank for R : cycles: {{1, 3}, {4, 6}}, net cycles: 1 . order: 2

\$ [ [1, 2, 2, 2, 0, 1, 0, 0] , [2, 0, 3, 1, 0, 2, 0, 0] , [3, 0, 2, 2, 0, 1, 0, 0] , [2, 0, 3, 1, 0, 2, 0, 0] , [3, 0, 2, 2, 0, 1, 0, 0] ] \$

$[-y_3 + 4y_2 - 5y_1, y_3, y_2, 3y_2 - 4y_1, 0, y_1, 0, 0]$

$p = -s^2 + s^4$      $p' = -s^2 + s^4$

Omega Rank for B : cycles: {{5, 7}}, net cycles: -1 . order: 4

\$ [ [1, 0, 0, 0, 2, 1, 2, 2] , [0, 0, 0, 0, 4, 1, 3, 0] , [0, 0, 0, 0, 3, 0, 5, 0] , [0, 0, 0, 0, 5, 0, 3, 0] , [0, 0, 0, 0, 3, 0, 5, 0] ] \$

$[y_1, 0, 0, 0, y_2, y_3, y_4, 2y_1]$

$p = -s^3 + s^5$

$\hat{A} \gg \text{SYNC'D } 9/256 , 0.03515625000$

121 . Coloring, {2, 3, 4, 5, 6, 7}

**R:** [3, 8, 8, 6, 2, 4, 4, 5]    **B:** [6, 3, 1, 1, 7, 7, 5, 2]

' See graph

' ' See pair graph

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ ' 4' ( ' - 1 + \tau ' )'' ( ' 5 - 2\tau + \tau^2 ' )'' ( ' 1 + \tau ' )' , 4' ( ' - 5 - \tau - 3\tau^2 + \tau^3 ' )'' ( ' 1 + \tau ' )' , 4' ( ' - 1 \\ & + \tau ' )'' ( ' 1 + \tau ' )'' ( ' 5 + 2\tau + \tau^2 ' )' , -4' ( ' 5 - \tau + 3\tau^2 + \tau^3 ' )'' ( ' 1 + \tau ' )' , 4' ( ' - 5 - 3\tau - 10\tau^2 + \\ & 2\tau^3 - \tau^4 + \tau^5 ' )' , -4' ( ' 5 - 4\tau + 6\tau^2 + \tau^4 ' )'' ( ' 1 + \tau ' )' , 4' ( ' - 1 + \tau ' )'' ( ' 1 + \tau^2 ' )'' ( ' 5 + 2\tau \\ & + \tau^2 ' )' , 4' ( ' - 5 + \tau - \tau^2 + \tau^3 ' )'' ( ' 1 + \tau ' )'^2 ' ]' \end{aligned}$$

For  $\tau=1/2$ , [-102, -294, -150, -258, -281, -219, -125, -333] . FixedPtCheck, [102, 294, 150, 258, 281, 219, 125, 333]

$$\det(A + \tau \Delta) = 1' ( ' - 1 + \tau ' )'^2 ( ' \tau ' )'^2 ( ' 1 + \tau ' )'^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	5 vs 6	6 vs 6

Omega Rank for R : cycles: {{4, 6}, {2, 5, 8}}, net cycles: 1 . order: 6

$$\$ [ [0, 1, 1, 2, 1, 1, 0, 2] , [0, 1, 0, 1, 2, 2, 0, 2] , [0, 2, 0, 2, 2, 1, 0, 1] , [0, 2, 0, 1, 1, 2, 0, 2] , [0, 1, 0, 2, 2, 1, 0, 2] , [0, 2, 0, 1, 2, 2, 0, 1] ] \$$$

$$[0, -3 y_1 + 5 y_2 - 3 y_3 + 5 y_4 - 3 y_5, 3 y_1, 3 y_2, 3 y_3, 3 y_4, 0, 3 y_5]$$

$$p = -s^2 - s^3 + s^5 + s^6$$

Omega Rank for B : cycles: {{5, 7}}, net cycles: 0 . order: 6

$$[y_1, y_2, y_3, 0, y_4, y_5, y_6, 0]$$

$$\begin{aligned} \mathbf{B} = \$ [ [0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 1, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0, 0, 0] \\ , [0, 1, 0, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 0, 1, -3/16, -11/16] , [0, 1, -1, -1, 13/16, 5/16] , [0, 0, 1, -1, -11/16, 13/16] , [0, 0, 1, -1, -11/16, 13/16] , [0, 0, 0, 5/16, -3/16] , [0, 0, 0, 5/16, -3/16] , [0, 0, 0, 0, -3/16, 5/16] , [1, -1, -1, 2, 5/16, -19/16] ] \$ \times \$ [ [2, 1, 1, 0, 1, 1, 2, 0] , [1, 0, 1, 0, 2, 2, 2, 0] , [1, 0, 0, 0, 2, 1, 4, 0] , [0, 0, 0, 0, 4, 1, 3, 0] , [0, 0, 0, 0, 3, 0, 5, 0] , [0, 0, 0, 0, 5, 0, 3, 0] ] \$ \end{aligned}$$

Â» SYNC'D 3891/65536 , 0.05937194824

122 . Coloring, {2, 3, 4, 5, 6, 8}

**R:** [3, 8, 8, 6, 2, 4, 5, 2]    **B:** [6, 3, 1, 1, 7, 7, 4, 5]

' See graph

' ' See pair graph

$\Omega$  for  $A+\tau\Delta$  :

' [ '2' (' - 1 +  $\tau$  ') ' 2 , 2' (' 1 +  $\tau$  ') ' 2 , -2' (' 1 +  $\tau$  ') ' (' - 1 +  $\tau$  ') ' , 2' (' - 1 +  $\tau$  ') ' 2 , -2' (' 1 +  $\tau$  ') ' (' - 1 +  $\tau$  ') ' , 2' (' - 1 +  $\tau$  ') ' 2 , 2' (' - 1 +  $\tau$  ') ' 2 , 2' (' 1 +  $\tau$  ') ' 2 ' ]'

For  $\tau=1/2$ , [1, 9, 3, 1, 3, 1, 1, 9] . FixedPtCheck, [1, 9, 3, 1, 3, 1, 1, 9]

$\det(A + \tau \Delta) = 1' (' \tau ' ) ' 2 ' (' 1 + \tau ' ) ' 2 ' (' - 1 + \tau ' ) ' 2$

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

[1, 1, 1, 1, 1, 1, 1, 1]

+ \ ; - \ ;  $\Delta$

\$ [ [0, 2, 1, 1, 1, 1, 0, 2] , [2, 3, 0, 3, 0, 3, 2, 3] , [5, 3, 3, 5, 3, 5, 5, 3] , [4, 3, 5, 4, 5, 4, 4, 3] , [7, 8, 9, 8, 9, 8, 7, 8] , [15, 17, 15, 17, 15, 17, 15, 17] ] \$ \$ [ [2, 0, 1, 1, 1, 1, 2, 0] , [2, 1, 4, 1, 4, 1, 2, 1] , [3, 5, 5, 3, 5, 3, 3, 5] , [4, 5, 3, 4, 3, 4, 4, 5] , [9, 8, 7, 8, 7, 8, 9, 8] , [17, 15, 17, 15, 17, 15, 17, 15] ] \$ \$ [ [-1, 1, 0, 0, 0, 0, -1, 1] , [0, 1, -2, 1, -2, 1, 0, 1] , [1, -1, -1, 1, -1, 1, 1, -1] , [0, -1, 1, 0, 1, 0, 0, -1] , [-1, 0, 1, 0, 1, 0, -1, 0] , [-1, 1, -1, 1, -1, 1, -1, 1] ] \$

[-y<sub>3</sub> - y<sub>2</sub> - y<sub>1</sub>, y<sub>3</sub>, y<sub>2</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>1</sub>, -y<sub>3</sub> - y<sub>2</sub> - y<sub>1</sub>, y<sub>3</sub>]

$p = s + 4s^4 - 8s^5$

S+ \ ; S- \ ; NM

\$ [ [4, 2, 4, 5, 3, 3, 2, 3] , [3, 6, 5, 1, 3, 3, 2, 3] , [4, 5, 4, 2, 2, 3, 3, 3] , [3, 1, 3, 4, 2, 5, 5, 3] , [3, 3, 2, 3, 4, 2, 4, 5] , [5, 3, 2, 5, 3, 4, 3, 1] , [2, 3, 3, 3, 4, 5, 4, 2] , [2, 3, 3, 3, 5, 1, 3, 6] ] \$ \$ [ [2, 2, 4, 3, 3, 5, 4, 3] , [3, 2, 3, 3, 5, 1, 2, 7] , [4, 3, 2, 2, 4, 3, 3, 5] , [5, 3, 3, 4, 2, 5, 3, 1] , [3, 5, 4, 3, 2, 2, 4, 3] , [3, 1, 2, 5, 3, 4, 5, 3] , [4, 3, 3, 5, 4, 3, 2, 2] , [2, 7, 5, 1, 3, 3, 3, 2] ] \$ \$ [ [7, 2, 3, 3, 4, 4, 0, 5] , [2, 7, 5, 3, 2, 4, 5, 0] , [3, 5, 7, 5, 0, 2, 4, 2] , [3, 3, 5, 7, 2, 0, 4, 4] , [4, 2, 0, 2, 7, 5, 3, 5] , [4, 4, 2, 0, 5, 7, 3, 3] , [0, 5, 4, 4, 3, 3, 7, 2] , [5, 0, 2, 4, 5, 3, 2, 7] ] \$

CmmCk true, true, true

$p' = s^2 - 2s^3 + 4s^4 - 4s^5$      $p' = s + 4s^4 - 8s^5$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
3 vs 6	4 vs 8	4 vs 8	2 vs 6	2 vs 6

Omega Rank for R : cycles: {{2, 8}, {4, 6}}, net cycles: 0 . order: 2

\$ [ [0, 2, 1, 1, 1, 1, 0, 2] , [0, 3, 0, 1, 0, 1, 0, 3] , [0, 3, 0, 1, 0, 1, 0, 3] , [0, 3, 0, 1, 0, 1, 0, 3] , [0, 3, 0, 1, 0, 1, 0, 3] , [0, 3, 0, 1, 0, 1, 0, 3] ] \$

[0, y<sub>2</sub>, 3 y<sub>1</sub> - y<sub>2</sub>, y<sub>1</sub>, 3 y<sub>1</sub> - y<sub>2</sub>, y<sub>1</sub>, 0, y<sub>2</sub>]

p = -s<sup>2</sup> + s<sup>3</sup> p = -s<sup>2</sup> + s<sup>4</sup> p = -s<sup>2</sup> + s<sup>5</sup> p = -s<sup>2</sup> + s<sup>6</sup>

Omega Rank for B : cycles: {{1, 4, 6, 7}}, net cycles: -1 . order: 4

\$ [ [2, 0, 1, 1, 1, 1, 2, 0] , [2, 0, 0, 2, 0, 2, 2, 0] , [2, 0, 0, 2, 0, 2, 2, 0] , [2, 0, 0, 2, 0, 2, 2, 0] , [2, 0, 0, 2, 0, 2, 2, 0] , [2, 0, 0, 2, 0, 2, 2, 0] ] \$

[y<sub>2</sub>, 0, y<sub>1</sub>, y<sub>2</sub> - y<sub>1</sub>, y<sub>1</sub>, y<sub>2</sub> - y<sub>1</sub>, y<sub>2</sub>, 0]

p = s<sup>2</sup> - s<sup>6</sup> p' = s<sup>2</sup> - s<sup>5</sup> p' = s<sup>3</sup> - s<sup>5</sup> p' = s<sup>4</sup> - s<sup>5</sup>

Â« NOT SYNC'D Â»

Nullspace of {ΩΔ<sup>i</sup>} :

[x<sub>1</sub>, x<sub>2</sub>, x<sub>3</sub>, 4 x<sub>1</sub> - 2 x<sub>3</sub>, 4 x<sub>2</sub> - 8 x<sub>1</sub> + 4 x<sub>3</sub>, -8 x<sub>2</sub> - 4 x<sub>3</sub>]

For A+2Δ : [-y<sub>3</sub>, -y<sub>4</sub>, -y<sub>1</sub>, -y<sub>2</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>]

For A-2Δ : [-y<sub>3</sub>, -y<sub>4</sub>, -y<sub>1</sub>, -y<sub>2</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>]

Range of {ΩΔ<sup>i</sup>} : [μ<sub>2</sub>, μ<sub>3</sub>, -μ<sub>2</sub> - μ<sub>3</sub> - μ<sub>1</sub>, μ<sub>1</sub>, -μ<sub>2</sub> - μ<sub>3</sub> - μ<sub>1</sub>, μ<sub>1</sub>, μ<sub>2</sub>, μ<sub>3</sub>]

rank of M is 8 , rank of N is 5

M \ ; N

\$ [ [0, 0, 0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 1, 0, 0] , [0, 0, 1, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0, 0, 0] ] \$ \$ [ [0, 5, 4, 4, 3, 3, 7, 2] , [5, 0, 2, 4, 5, 3, 2, 7] , [4, 2, 0, 2, 7, 5, 3, 5] , [4, 4, 2, 0, 5, 7, 3, 3] , [3, 5, 7, 5, 0, 2, 4, 2] , [3, 3, 5, 7, 2, 0, 4, 4] , [7, 2, 3, 3, 4, 4, 0, 5] , [2, 7, 5, 3, 2, 4, 5, 0] ] \$

Check is ΩΔN zero? true, πΔ = [-1, 1, 0, 0, 0, 0, -1, 1]

ker M, [0, 0, 0, 0, 0, 0, 0, 0]

Range M, [x<sub>6</sub>, x<sub>7</sub>, x<sub>5</sub>, x<sub>4</sub>, x<sub>1</sub>, x<sub>2</sub>, x<sub>3</sub>, x<sub>8</sub>]

τ = 32 , r' = 1/2

Ranges



<b>Sandwich</b>	
<b>Coloring</b>	{2, 3, 4, 5, 6, 8}
<b>Rank</b>	2
<b>R,B</b>	[3, 8, 8, 6, 2, 4, 5, 2], [6, 3, 1, 1, 7, 7, 4, 5]
$\pi_2$	[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0]
$u_2$	[5, 4, 4, 3, 3, 7, 2, 2, 4, 5, 3, 2, 7, 2, 7, 5, 3, 5, 5, 7, 3, 3, 2, 4, 2, 4, 4, 5] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

123 . Coloring, {2, 3, 4, 5, 7, 8}

**R:** [3, 8, 8, 6, 2, 7, 4, 2]    **B:** [6, 3, 1, 1, 7, 4, 5, 5]

‘ See graph

‘ ‘ See pair graph

$\Omega$  for  $A+\tau\Delta$  :

$$\left[ \begin{array}{c} -4(\tau^5 + \tau^4) (\tau - 1 + \tau^2)^2, -4(\tau + 1) (\tau^5 + 2\tau + \tau^2), -4(\tau + 1) (\tau - 1 + \tau^2) (\tau - 5 + \tau^4), \\ 4(\tau - 1 + \tau^2) (\tau^5 - 2\tau + \tau^2), 4(\tau^5 + \tau + \tau^2 + \tau^3) (\tau - 1 + \tau^2), 4(\tau - 1 + \tau^2) (\tau^5 - 3\tau + \tau^2 + \tau^3), \\ 4(\tau^5 - \tau - \tau^2 + \tau^3) (\tau - 1 + \tau^2), -4(\tau + 1) (\tau^5 - 2\tau + \tau^2) \end{array} \right]$$

For  $\tau=1/2$ , [-22, -150, -54, -34, -47, -31, -35, -153] . FixedPtCheck, [22, 150, 54, 34, 47, 31, 35, 153]

$$\det(A + \tau \Delta) = (\tau^2)^2 (\tau + 1)^2 (\tau - 1)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	8 vs 8	8 vs 8	3 vs 6	5 vs 6

Omega Rank for R : cycles: {{2, 8}, {4, 6, 7}}, net cycles: 1 . order: 6

\$ [ [0, 2, 1, 1, 0, 1, 1, 2] , [0, 2, 0, 1, 0, 1, 1, 3] , [0, 3, 0, 1, 0, 1, 1, 2] , [0, 2, 0, 1, 0, 1, 1, 3] , [0, 3, 0, 1, 0, 1, 1, 2] , [0, 2, 0, 1, 0, 1, 1, 3] ] \$



$$[0, -y_1 + 5y_2 - y_3, y_1, y_2, 0, y_2, y_2, y_3]$$

$$p = s^2 - s^4 \quad p' = -s^2 + s^4 \quad p'' = -s^3 + s^5$$

Omega Rank for B : cycles: {{1, 4, 6}, {5, 7}}, net cycles: 1 . order: 6

$$\$ [ [2, 0, 1, 1, 2, 1, 1, 0], [2, 0, 0, 1, 1, 2, 2, 0], [1, 0, 0, 2, 2, 2, 1, 0], [2, 0, 0, 2, 1, 1, 2, 0], [2, 0, 0, 1, 2, 2, 1, 0], [1, 0, 0, 2, 1, 2, 2, 0] ] \$$$

$$[-3y_1 - 3y_2 + 5y_3 - 3y_4 + 5y_5, 0, 3y_1, 3y_2, 3y_3, 3y_4, 3y_5, 0]$$

$$p = -s^2 - s^3 + s^5 + s^6$$

Â» SYNC'D 2641/131072 , 0.02014923096

124 . Coloring, {2, 3, 4, 6, 7, 8}

**R:** [3, 8, 8, 6, 7, 4, 4, 2]    **B:** [6, 3, 1, 1, 2, 7, 5, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ '2' ('-5 + \tau^2') '' ('-1 + \tau')', 2' ('5 - \tau + 3\tau^2 + \tau^3')', -2' ('-1 + \tau')'' ('5 + 2\tau + \tau^2')', \\ & 2' ('5 - 2\tau + \tau^2')'' ('1 + \tau')', 2' ('-5 + \tau^2')'' ('-1 + \tau')', 2' ('5 - \tau + 3\tau^2 + \tau^3')', -2' ('- \\ & 1 + \tau')'' ('5 + 2\tau + \tau^2')', 2' ('5 - 2\tau + \tau^2')'' ('1 + \tau')' ]' \end{aligned}$$

For τ=1/2, [19, 43, 25, 51, 19, 43, 25, 51] . FixedPtCheck, [19, 43, 25, 51, 19, 43, 25, 51]

$$\det(A + \tau \Delta) = 1' ('1 + \tau')'^2 ('\tau')'^2 ('-1 + \tau')'^2$$

Delta Range : [y<sub>2</sub>, -y<sub>1</sub> - y<sub>3</sub> - y<sub>5</sub>, -y<sub>2</sub> - y<sub>4</sub> - y<sub>6</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, y<sub>5</sub>, y<sub>6</sub>]

$$[1, 1, 1, 1, 1, 1, 1, 1]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

$$\begin{aligned} \$ [ [0, 1, 1, 2, 0, 1, 1, 2], [1, 4, 1, 2, 1, 4, 1, 2], [5, 5, 1, 5, 5, 5, 1, 5], [5, 4, 4, 3, 5, 4, 4, 3], [9, 6, 9, 8, 9, \\ 6, 9, 8], [15, 15, 19, 15, 15, 19, 15] ] \$ \quad \$ [ [2, 1, 1, 0, 2, 1, 1, 0], [3, 0, 3, 2, 3, 0, 3, 2], [3, 3, 7, 3, 3, \\ 3, 7, 3], [3, 4, 4, 5, 3, 4, 4, 5], [7, 10, 7, 8, 7, 10, 7, 8], [17, 17, 13, 17, 17, 17, 13, 17] ] \$ \quad \$ [ [-1, 0, 0, 1, \\ -1, 0, 0, 1], [-1, 2, -1, 0, -1, 2, -1, 0], [1, 1, -3, 1, 1, 1, -3, 1], [1, 0, 0, -1, 1, 0, 0, -1], [1, -2, 1, 0, 1, -2, 1, \\ 0], [-1, -1, 3, -1, -1, -1, 3, -1] ] \$ \end{aligned}$$

$$[y_1, y_2, y_3, -y_1 - y_3 - y_2, y_1, y_2, y_3, -y_1 - y_3 - y_2]$$

$$p' = s + 4s^4 \quad p = s + 4s^4$$

S+ \; S- \; NM

\$ [ [1, 2, 3, 1, 0, 1, 1, 1], [1, 1, 1, 0, 1, 0, 2, 4], [1, 1, 1, 0, 3, 1, 0, 3], [1, 0, 1, 1, 2, 4, 1, 0], [0, 1, 1, 1, 1, 2, 3, 1], [1, 0, 2, 4, 1, 1, 1, 0], [3, 1, 0, 3, 1, 1, 1, 0], [2, 4, 1, 0, 1, 0, 1, 1] ] \$ \$ [ [1, 0, 1, 1, 0, 3, 3, 1], [1, 1, 3, 2, 1, 0, 0, 2], [3, 1, 1, 2, 1, 1, 0, 1], [3, 2, 1, 1, 0, 2, 1, 0], [0, 3, 3, 1, 1, 0, 1, 1], [1, 0, 0, 2, 1, 1, 3, 2], [1, 1, 0, 1, 3, 1, 1, 2], [0, 2, 1, 0, 3, 2, 1, 1] ] \$ \$ [ [4, 1, 2, 2, 0, 3, 2, 2], [1, 4, 3, 2, 3, 0, 1, 2], [2, 3, 4, 3, 2, 1, 0, 1], [2, 2, 3, 4, 2, 2, 1, 0], [0, 3, 2, 2, 4, 1, 2, 2], [3, 0, 1, 2, 1, 4, 3, 2], [2, 1, 0, 1, 2, 3, 4, 3], [2, 2, 1, 0, 2, 2, 3, 4] ] \$

CmmCk true, true, true

$$p' = s^2 + 4s^5$$

$\Delta$ -Rank	A+(1/2) $\Delta$	A-(1/2) $\Delta$	R	B
3 vs 6	4 vs 8	4 vs 8	2 vs 6	3 vs 6

Omega Rank for R : cycles: {{2, 8}, {4, 6}}, net cycles: 0 . order: 2

\$ [ [0, 1, 1, 2, 0, 1, 1, 2], [0, 2, 0, 2, 0, 2, 0, 2], [0, 2, 0, 2, 0, 2, 0, 2], [0, 2, 0, 2, 0, 2, 0, 2], [0, 2, 0, 2, 0, 2, 0, 2], [0, 2, 0, 2, 0, 2, 0, 2] ] \$

$$[0, -y_1 + y_2, y_1, y_2, 0, -y_1 + y_2, y_1, y_2]$$

$$p = -s^2 + s^4 \quad p = -s^2 + s^5 \quad p = -s^2 + s^6 \quad p = -s^2 + s^3$$

Omega Rank for B : cycles: {{1, 2, 3, 5, 6, 7}}, net cycles: 1 . order: 6

\$ [ [2, 1, 1, 0, 2, 1, 1, 0], [1, 2, 1, 0, 1, 2, 1, 0], [1, 1, 2, 0, 1, 1, 2, 0], [2, 1, 1, 0, 2, 1, 1, 0], [1, 2, 1, 0, 1, 2, 1, 0], [1, 1, 2, 0, 1, 1, 2, 0] ] \$

$$[y_1, y_2, y_3, 0, y_1, y_2, y_3, 0]$$

$$p' = s^2 - s^5 \quad p = -s + s^4 \quad p' = -s + s^4$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[x_1, x_2, x_3, 4x_1, 4x_2, 4x_3]$$

$$\text{For } A+2\Delta : [-y_1, -y_2, -y_3, -y_4, y_1, y_2, y_3, y_4]$$

$$\text{For } A-2\Delta : [-y_1, -y_2, -y_3, -y_4, y_1, y_2, y_3, y_4]$$

Range of  $\{\Omega\Delta^i\}$  :  $[\mu_1, \mu_2, \mu_3, -\mu_1 - \mu_2 - \mu_3, \mu_1, \mu_2, \mu_3, -\mu_1 - \mu_2 - \mu_3]$

rank of M is 8 , rank of N is 5

M \; N

$\$ [ [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0] ] \$ \quad \$ [ [0, 3, 2, 2, 4, 1, 2, 2], [3, 0, 1, 2, 1, 4, 3, 2], [2, 1, 0, 1, 2, 3, 4, 3], [2, 2, 1, 0, 2, 2, 3, 4], [4, 1, 2, 2, 0, 3, 2, 2], [1, 4, 3, 2, 3, 0, 1, 2], [2, 3, 4, 3, 2, 1, 0, 1], [2, 2, 3, 4, 2, 2, 1, 0] ] \$$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [-1, 0, 0, 1, -1, 0, 0, 1]$

ker M,  $[0, 0, 0, 0, 0, 0, 0, 0]$

Range M,  $[x_3, x_4, x_5, x_2, x_1, x_6, x_7, x_8]$

$\tau = 32$ ,  $r' = 1/2$

Ranges

Action of R on ranges,  $[[3], [4], [4], [2]]$

Action of B on ranges,  $[[2], [3], [1], [1]]$

$\beta(\{1, 5\}) = 1/4$

$\beta(\{2, 6\}) = 1/4$

$\beta(\{3, 7\}) = 1/4$

$\beta(\{4, 8\}) = 1/4$

ker N,  $[\mu_3, \mu_1, -\mu_1 - \mu_3 - \mu_2, \mu_2, \mu_3, \mu_1, -\mu_1 - \mu_3 - \mu_2, \mu_2]$

Range of N

$[y_2 + y_1 - y_4, -y_5 + y_2 + y_1, y_2, y_3, y_4, y_5, y_1, -y_3 + y_2 + y_1]$

Partitions

Action of R on partitions,  $[[3], [2], [2], [4], [4]]$

Action of B on partitions,  $[[5], [4], [1], [1], [4]]$

$\alpha(\{\{2, 5, 7, 8\}, \{1, 3, 4, 6\}\}) = 1/4$

$\alpha(\{\{4, 5, 6, 7\}, \{1, 2, 3, 8\}\}) = 1/8$

$\alpha(\{\{2, 3, 5, 8\}, \{1, 4, 6, 7\}\}) = 1/8$

$\alpha(\{\{2, 3, 4, 5\}, \{1, 6, 7, 8\}\}) = 3/8$

$\alpha(\{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}) = 1/8$

$b_1 = \{2, 3, 5, 8\}$ ,  $b_2 = \{1, 2, 3, 4\}$ ,  $b_3 = \{2, 5, 7, 8\}$ ,  $b_4 = \{2, 3, 4, 5\}$ ,  $b_5 = \{1, 3, 4, 6\}$ ,  $b_6 = \{5, 6, 7, 8\}$ ,  $b_7 = \{1, 6, 7, 8\}$ ,  $b_8 = \{4, 5, 6, 7\}$ ,  $b_9 = \{1, 2, 3, 8\}$ ,  $b_{10} = \{1, 4, 6, 7\}$

Action of R and B on the blocks of the partitions:  $\$ [ [0, 0, 1, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 1, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 1, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 1, 0, 0] ] \$ + \$ [ [0, 0, 1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0] ] \$$

$['9', '7', '1', '7', 'A', '4', '4', '8', '9', '8'], ['3', '4', '6', '3', '2', '7', '5', '7', '4', '5']$  with invariant

measure [1, 1, 2, 3, 2, 1, 3, 1, 1, 1]

N by blocks, check: true . ‘ See partition graph.

‘ ‘ See level-2 partition graph.

‘

<b>Sandwich</b>	
<b>Coloring</b>	{2, 3, 4, 6, 7, 8}
<b>Rank</b>	2
<b>R,B</b>	[3, 8, 8, 6, 7, 4, 4, 2], [6, 3, 1, 1, 2, 7, 5, 5]
$\pi_2$	[0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]
$u_2$	[3, 2, 2, 4, 1, 2, 2, 1, 2, 1, 4, 3, 2, 1, 2, 3, 4, 3, 2, 2, 3, 4, 3, 2, 2, 1, 2, 1] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

125 . Coloring, {2, 3, 5, 6, 7, 8}

**R:** [3, 8, 8, 1, 2, 4, 4, 2] **B:** [6, 3, 1, 6, 7, 7, 5, 5]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

[ ‘2‘ (‘1 +  $\tau$ ‘)‘ (‘-1 +  $\tau$ ‘)‘<sup>2</sup>‘ (‘-5 - 3 $\tau$  -  $\tau^2$  +  $\tau^3$ ‘)‘, 2‘ (‘1 +  $\tau$ ‘)‘ (‘-5 -  $\tau$  - 12 $\tau^2$  +  $\tau^4$  +  $\tau^5$ ‘)‘, -2‘ (‘-5 - 2 $\tau$  - 4 $\tau^2$  + 2 $\tau^3$  +  $\tau^4$ ‘)‘ (‘1 +  $\tau$ ‘)‘ (‘-1 +  $\tau$ ‘)‘, -2‘ (‘1 +  $\tau$ ‘)‘ (‘-1 +  $\tau$ ‘)‘<sup>2</sup>‘ (‘5 -  $\tau$  -  $\tau^2$  +  $\tau^3$ ‘)‘, -2‘ (‘-1 +  $\tau$ ‘)‘ (‘-5 + 2 $\tau$  - 12 $\tau^2$  - 2 $\tau^3$  +  $\tau^4$ ‘)‘, 2‘ (‘5 +  $\tau$ ‘)‘ (‘1 +  $\tau$ ‘)‘ (‘-1 +  $\tau$ ‘)‘<sup>3</sup>‘, -6‘ (‘-1 +  $\tau$ ‘)‘<sup>2</sup>‘ (‘5 - 3 $\tau$  + 3 $\tau^2$  + 3 $\tau^3$ ‘)‘, 6‘ (‘-5 +  $\tau$  - 7 $\tau^2$  + 3 $\tau^3$ ‘)‘ (‘1 +  $\tau$ ‘)‘<sup>2</sup>‘ ]‘

For  $\tau=1/2$ , [-159, -807, -321, -105, -230, -66, -74, -846] . FixedPtCheck, [159, 807, 321, 105, 230, 66, 74, 846]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	6 vs 6	6 vs 6	5 vs 5	5 vs 5

Omega Rank for R : cycles:  $\{\{2, 8\}\}$ , net cycles: 0 . order: 4

$$[y_1, y_2, y_3, y_4, 0, 0, 0, y_5]$$

$$\begin{aligned} R = \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, \\ 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, \\ 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 0, 1/2, -3/16, -3/16], [0, 0, 0, 5/16, -3/16], \\ [0, 0, 0, 5/16, -3/16], [0, 1/2, -1/4, -3/16, 1/16], [0, 0, 0, -3/16, 5/16], [1/2, -1/4, -1/8, 1/16, -1/16], [1/2, \\ -1/4, -1/8, 1/16, -1/16], [0, 0, 0, -3/16, 5/16] ] \$ x \$ [ [1, 2, 1, 2, 0, 0, 0, 2], [2, 2, 1, 0, 0, 0, 0, 3], [0, 3, 2, \\ 0, 0, 0, 0, 3], [0, 3, 0, 0, 0, 0, 0, 5], [0, 5, 0, 0, 0, 0, 0, 3] ] \$ \end{aligned}$$

Omega Rank for B : cycles:  $\{\{5, 7\}\}$ , net cycles: 0 . order: 4

$$[y_1, 0, y_2, 0, y_3, y_4, y_5, 0]$$

$$\begin{aligned} B = \$ [ [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, \\ 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0, 0, 0], \\ [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, \\ 0, 0], [0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 0, 1, -3/16, -11/16], [1, -1, -1, 13/16, 5/16], \\ [0, 1, -1, -11/16, 13/16], [0, 0, 1, -3/16, -11/16], [0, 0, 0, 5/16, -3/16], [0, 0, 0, 5/16, -3/16], [0, 0, 0, \\ -3/16, 5/16], [0, 0, 0, -3/16, 5/16] ] \$ x \$ [ [1, 0, 1, 0, 2, 2, 2, 0], [1, 0, 0, 0, 2, 1, 4, 0], [0, 0, 0, 0, 4, 1, 3, \\ 0], [0, 0, 0, 0, 3, 0, 5, 0], [0, 0, 0, 0, 5, 0, 3, 0] ] \$ \end{aligned}$$

Â» SYNC'D 87/2048 , 0.04248046875

126 . Coloring,  $\{2, 4, 5, 6, 7, 8\}$

**R:** [3, 8, 1, 6, 2, 4, 4, 2] **B:** [6, 3, 8, 1, 7, 7, 5, 5]

' See graph

' ' See pair graph

'

Ω for  $A + \tau \Delta$  :

$$[ ' 2' ( ' 1 + \tau ' )', 2' ( ' 1 + \tau ' )', 2' ( ' 1 + \tau ' )', 2' ( ' 1 + \tau ' )', -2' ( ' - 1 + \tau ' )', 2' ( ' 1 + \tau ' )', -2' ( ' - 1 + \tau ' )', 2' ( ' 1 + \tau ' ) ]'$$

For  $\tau=1/2$ , [3, 3, 3, 3, 1, 3, 1, 3] . FixedPtCheck, [3, 3, 3, 3, 1, 3, 1, 3]

$$\det(A + \tau \Delta) = 0$$

Delta Range :  $[y_2, -y_1 - y_3 - y_5, -y_2 - y_4 - y_6, y_1, y_3, y_4, y_5, y_6]$

[1, 1, 1, 1, 1, 1, 1]

+ \; - \; Δ

\$ [ [1, 2, 1, 2, 0, 1, 0, 1], [1, 1, 1, 1, 3, 3, 3, 3], [2, 3, 2, 3, 1, 2, 1, 2], [3, 3, 3, 3, 5, 5, 5, 5], [4, 5, 4, 5, 3, 4, 3, 4], [7, 7, 7, 7, 9, 9, 9, 9] ] \$ \$ [ [1, 0, 1, 0, 2, 1, 2, 1], [3, 3, 3, 3, 1, 1, 1, 1], [2, 1, 2, 1, 3, 2, 3, 2], [5, 5, 5, 5, 3, 3, 3, 3], [4, 3, 4, 3, 5, 4, 5, 4], [9, 9, 9, 9, 7, 7, 7, 7] ] \$ \$ [ [0, 1, 0, 1, -1, 0, -1, 0], [-1, -1, -1, -1, 1, 1, 1, 1], [0, 1, 0, 1, -1, 0, -1, 0], [-1, -1, -1, -1, 1, 1, 1, 1] ] \$

[-y<sub>2</sub>, -y<sub>1</sub>, -y<sub>2</sub>, -y<sub>1</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>1</sub>, y<sub>2</sub>]

$$p = s - 4s^5$$

S+ \; S- \; NM

\$ [ [0, 2, 6, 2, 3, 3, 1, 3], [3, 3, 3, 1, 4, 0, 0, 6], [6, 2, 0, 2, 1, 3, 3, 3], [3, 1, 3, 3, 0, 6, 4, 0], [3, 5, 1, 1, 4, 2, 2, 2], [2, 0, 2, 6, 3, 3, 3, 1], [1, 1, 3, 5, 2, 2, 4, 2], [2, 6, 2, 0, 3, 1, 3, 3] ] \$ \$ [ [4, 0, 2, 4, 3, 5, 1, 1], [1, 3, 5, 1, 2, 4, 2, 2], [2, 4, 4, 0, 1, 1, 3, 5], [5, 1, 1, 3, 2, 2, 2, 4], [3, 3, 1, 3, 0, 0, 6, 4], [4, 4, 0, 2, 1, 3, 5, 1], [1, 3, 3, 3, 6, 4, 0, 0], [0, 2, 4, 4, 5, 1, 1, 3] ] \$ \$ [ [2, 1, 0, 1, 1, 1, 1, 1], [1, 2, 1, 0, 1, 1, 1, 1], [0, 1, 2, 1, 1, 1, 1, 1], [1, 0, 1, 2, 1, 1, 1, 1], [1, 1, 1, 1, 2, 1, 0, 1], [1, 1, 1, 1, 1, 2, 1, 0], [1, 1, 1, 1, 0, 1, 2, 1], [1, 1, 1, 1, 1, 0, 1, 2] ] \$

CmmCk true, true, true

$$p' = s - 4s^5 \quad p' = s^2 - 2s^4 \quad p' = s^3 - 2s^5$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
2 vs 6	3 vs 7	3 vs 7	2 vs 6	3 vs 6

Omega Rank for R : cycles: {{1, 3}, {2, 8}, {4, 6}}, net cycles: 3 . order: 2

\$ [ [1, 2, 1, 2, 0, 1, 0, 1], [1, 1, 1, 1, 0, 2, 0, 2], [1, 2, 1, 2, 0, 1, 0, 1], [1, 1, 1, 1, 0, 2, 0, 2], [1, 2, 1, 2, 0, 1, 0, 1], [1, 1, 1, 1, 0, 2, 0, 2] ] \$

[y<sub>1</sub>, 3y<sub>1</sub> - y<sub>2</sub>, y<sub>1</sub>, 3y<sub>1</sub> - y<sub>2</sub>, 0, y<sub>2</sub>, 0, y<sub>2</sub>]

$$p = -s + s^3 \quad p' = -s + s^3 \quad p = -s + s^5 \quad p' = -s + s^5$$

Omega Rank for B : cycles: {{5, 7}}, net cycles: -1 . order: 4

\$ [ [1, 0, 1, 0, 2, 1, 2, 1], [0, 0, 0, 0, 3, 1, 3, 1], [0, 0, 0, 0, 4, 0, 4, 0], [0, 0, 0, 0, 4, 0, 4, 0], [0, 0, 0, 0, 4, 0, 4, 0], [0, 0, 0, 0, 4, 0, 4, 0] ] \$

[y<sub>1</sub>, 0, y<sub>1</sub>, 0, y<sub>2</sub>, y<sub>3</sub>, y<sub>2</sub>, y<sub>3</sub>]

$$p = -s^3 + s^4 \quad p = -s^3 + s^6 \quad p = -s^3 + s^5$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[x_2, x_3, x_4, x_1, -4x_2 - 2x_4, -4x_3 - 2x_1]$$

$$\text{For } A+2\Delta : [-y_2, y_1, y_2, y_3, -3y_1 - 3y_3 - y_4, -y_5, y_4, y_5]$$

$$\text{For } A-2\Delta : [-y_1, -y_2 - 3y_3 - 3y_4, y_1, y_2, y_3, -y_5, y_4, y_5]$$

$$\text{Range of } \{\Omega\Delta^i\} : [-\mu_2, \mu_1, -\mu_2, \mu_1, -\mu_1, \mu_2, -\mu_1, \mu_2]$$

rank of M is 8 , rank of N is 5

$$M \setminus ; N$$

$$\begin{aligned} & \$ [ [0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 1], \\ & [0, 0, 0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1, 0, 0] ] \$ \quad \$ [ [0, 1, 2, 1, 1, 1, 1, 1], \\ & [1, 0, 1, 2, 1, 1, 1, 1], [2, 1, 0, 1, 1, 1, 1, 1], [1, 2, 1, 0, 1, 1, 1, 1], [1, 1, 1, 1, 0, 1, 2, 1], [1, 1, 1, 1, 1, 0, 1, \\ & 2], [1, 1, 1, 1, 2, 1, 0, 1], [1, 1, 1, 1, 1, 2, 1, 0] ] \$ \end{aligned}$$

$$\text{Check is } \Omega\Delta N \text{ zero? } \text{true}, \pi\Delta = [0, 1, 0, 1, -1, 0, -1, 0]$$

$$\ker M, [0, 0, 0, 0, 0, 0, 0, 0]$$

$$\text{Range M}, [x_7, x_1, x_2, x_6, x_3, x_4, x_8, x_5]$$

$$\tau = 32, r' = 1/2$$

Ranges

$$\text{Action of R on ranges}, [[1], [4], [2], [2]]$$

$$\text{Action of B on ranges}, [[4], [1], [3], [3]]$$

$$\beta(\{1, 3\}) = 1/4$$

$$\beta(\{2, 4\}) = 1/4$$

$$\beta(\{5, 7\}) = 1/4$$

$$\beta(\{6, 8\}) = 1/4$$

$$\ker N, [-\mu_1 - \mu_2 - \mu_3, \mu_1, -\mu_1 - \mu_2 - \mu_3, \mu_1, \mu_2, \mu_3, \mu_2, \mu_3]$$

Range of N

$$[-y_1 + y_3 + y_5, -y_2 + y_3 + y_5, y_1, y_2, y_3 - y_4 + y_5, y_3, y_4, y_5]$$

Partitions

$$\text{Action of R on partitions}, [[4], [8], [1], [1], [5], [4], [5], [8]]$$

$$\text{Action of B on partitions}, [[6], [7], [3], [2], [2], [7], [3], [6]]$$

$$\alpha(\{\{2, 3, 5, 8\}, \{1, 4, 6, 7\}\}) = 1/8$$

$$\alpha(\{\{1, 2, 5, 6\}, \{3, 4, 7, 8\}\}) = 1/8$$

$$\alpha(\{\{3, 4, 5, 6\}, \{1, 2, 7, 8\}\}) = 1/8$$

$$\alpha(\{\{3, 4, 6, 7\}, \{1, 2, 5, 8\}\}) = 1/8$$

$$\alpha(\{\{2, 3, 6, 7\}, \{1, 4, 5, 8\}\}) = 1/8$$

$$\alpha(\{\{2, 3, 7, 8\}, \{1, 4, 5, 6\}\}) = 1/8$$





<b>Sandwich</b>	
<b>Coloring</b>	{2, 4, 5, 6, 7, 8}
<b>Rank</b>	2
<b>R,B</b>	[3, 8, 1, 6, 2, 4, 4, 2], [6, 3, 8, 1, 7, 7, 5, 5]
$\pi_2$	[0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 1, 0]
$u_2$	[1, 2, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 2, 1] (dim 1)
<b>wpp</b>	[4, 4, 4, 4, 4, 4, 4, 4]

127 . Coloring, {3, 4, 5, 6, 7, 8}

**R:** [3, 3, 8, 6, 2, 4, 4, 2]    **B:** [6, 8, 1, 1, 7, 7, 5, 5]

‘ See graph

‘ ‘ See pair graph

$\Omega$  for  $A+\tau\Delta$  :

‘ [ ‘ -2‘ (‘ 1 +  $\tau$  ‘ )‘ ‘ (‘ - 5 -  $\tau$  +  $\tau^2$  +  $\tau^3$  ‘ )‘ ‘ (‘ - 1 +  $\tau$  ‘ )‘ , 2‘ (‘ 1 +  $\tau$  ‘ )‘ ‘ (‘ - 5 - 2 $\tau$  - 4 $\tau^2$  + 2 $\tau^3$  +  $\tau^4$  ‘ )‘ , 2‘ (‘ 1 +  $\tau$  ‘ )‘ <sup>2</sup> ‘ (‘ - 5 +  $\tau$  -  $\tau^2$  +  $\tau^3$  ‘ )‘ , -2‘ (‘ 1 +  $\tau$  ‘ )‘ ‘ (‘ - 5 - 3 $\tau$  -  $\tau^2$  +  $\tau^3$  ‘ )‘ ‘ (‘ - 1 +  $\tau$  ‘ )‘ , -2‘ (‘ - 5 -  $\tau$  - 3 $\tau^2$  +  $\tau^3$  ‘ )‘ ‘ (‘ - 1 +  $\tau$  ‘ )‘ , 2‘ (‘ 1 +  $\tau$  ‘ )‘ ‘ (‘ 5 + 2 $\tau$  +  $\tau^2$  ‘ )‘ ‘ (‘ - 1 +  $\tau$  ‘ )‘ , -6‘ (‘ 5 + 4 $\tau$  + 3 $\tau^2$  ‘ )‘ ‘ (‘ - 1 +  $\tau$  ‘ )‘ <sup>2</sup> , 6‘ (‘ 1 +  $\tau$  ‘ )‘ ‘ (‘ - 5 - 3 $\tau$  - 3 $\tau^2$  + 3 $\tau^3$  ‘ )‘ ‘ ]‘

For  $\tau=1/2$ , [-123, -321, -333, -159, -98, -150, -62, -330] . FixedPtCheck, [123, 321, 333, 159, 98, 150, 62, 330]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
6 vs 6	6 vs 6	6 vs 6	4 vs 5	4 vs 5

Omega Rank for R : cycles: {{4, 6}, {2, 3, 8}}, net cycles: 2 . order: 6

\$ [ [0, 2, 2, 2, 0, 1, 0, 1] , [0, 1, 2, 1, 0, 2, 0, 2] , [0, 2, 1, 2, 0, 1, 0, 2] , [0, 2, 2, 1, 0, 2, 0, 1] , [0, 1, 2, 2, 0, 1, 0, 2] ] \$

[0, 3  $y_2$ , 3  $y_3$ , 3  $y_4$ , 0, 3  $y_1$ , 0, -3  $y_2$  - 3  $y_3$  + 5  $y_4$  + 5  $y_1$ ]

$$p = -s - s^2 + s^4 + s^5$$

Omega Rank for B : cycles: {{5, 7}}, net cycles: -1 . order: 4

$$\$ [ [2, 0, 0, 0, 2, 1, 2, 1], [0, 0, 0, 0, 3, 2, 3, 0], [0, 0, 0, 0, 3, 0, 5, 0], [0, 0, 0, 0, 5, 0, 3, 0], [0, 0, 0, 0, 3, 0, 5, 0] ] \$$$

$$[2 y_4, 0, 0, 0, y_1, y_2, y_3, y_4]$$

$$p = -s^3 + s^5$$

Â» SYNC'D 1/16 , 0.06250000000

128 . Coloring, {2, 3, 4, 5, 6, 7, 8}

**R**: [3, 8, 8, 6, 2, 4, 4, 2] **B**: [6, 3, 1, 1, 7, 7, 5, 5]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ ' 4 ( ' - 5 + \tau^2 ' ) ' ( ' 1 + \tau ' ) ' ( ' - 1 + \tau ' ) ' ^2 , 4 ( ' - 5 - 2\tau - 4\tau^2 + 2\tau^3 + \tau^4 ' ) ' ( ' 1 + \tau ' ) ' , \\ & -4 ( ' 1 + \tau ' ) ' ( ' - 1 + \tau ' ) ' ( ' - 5 - \tau + \tau^2 + \tau^3 ' ) ' , 4 ( ' 1 + \tau ' ) ' ( ' - 1 + \tau ' ) ' ( ' 5 - \tau - \tau^2 + \tau^3 ' ) ' , \\ & -4 ( ' 1 + \tau^2 ' ) ' ( ' - 5 + \tau^2 ' ) ' ( ' - 1 + \tau ' ) ' , 4 ( ' 5 - 3\tau + \tau^2 + \tau^3 ' ) ' ( ' 1 + \tau ' ) ' ( ' - 1 + \tau ' ) ' , \\ & -4 ( ' - 1 + \tau ' ) ' ^2 ( ' 5 + \tau + \tau^2 + \tau^3 ' ) ' , 4 ( ' 1 + \tau ' ) ' ^2 ( ' - 5 + \tau - \tau^2 + \tau^3 ' ) ' ] ' \end{aligned}$$

For τ=1/2, [-57, -321, -123, -105, -95, -93, -47, -333] . FixedPtCheck, [57, 321, 123, 105, 95, 93, 47, 333]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
6 vs 6	7 vs 7	7 vs 7	3 vs 5	4 vs 5

Omega Rank for R : cycles: {{2, 8}, {4, 6}}, net cycles: 1 . order: 2

$$\$ [ [0, 2, 1, 2, 0, 1, 0, 2], [0, 2, 0, 1, 0, 2, 0, 3], [0, 3, 0, 2, 0, 1, 0, 2], [0, 2, 0, 1, 0, 2, 0, 3], [0, 3, 0, 2, 0, 1, 0, 2] ] \$$$

$$[0, -y_1 - 5 y_2 + 4 y_3, y_1, -4 y_2 + 3 y_3, 0, y_2, 0, y_3]$$

$$p' = -s^2 + s^4 \quad p = -s^2 + s^4$$

Omega Rank for B : cycles: {{5, 7}}, net cycles: 0 . order: 4

\$ [ [2, 0, 1, 0, 2, 1, 2, 0] , [1, 0, 0, 0, 2, 2, 3, 0] , [0, 0, 0, 0, 3, 1, 4, 0] , [0, 0, 0, 0, 4, 0, 4, 0] , [0, 0, 0, 0, 4, 0, 4, 0] ] \$

$$[y_4, 0, y_2, 0, y_3, y_1, -y_4 + y_2 + y_3 + y_1, 0]$$

$$p = -s^4 + s^5$$

Â» SYNC'D 5/512 , 0.009765625000

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SUMMARY	
<b>Graph Type</b>	CC
$v(\mathbf{A})$	2
$v(\Delta)$	2
$\pi$	[1, 1, 1, 1, 1, 1, 1, 1]
<b>Dbly Stoch</b>	true

<b>SANDWICH</b>		Total 18
<b>No .</b>	<b>Coloring</b>	<b>Rank</b>
<b>1</b>	{5, 7}	2
<b>2</b>	{3, 5}	2
<b>3</b>	{3, 4, 7, 8}	2
<b>4</b>	{3, 7}	2
<b>5</b>	{2, 3, 4, 5, 6, 8}	2
<b>6</b>	{2, 3, 6, 7}	2
<b>7</b>	{2, 4}	2
<b>8</b>	{2, 8}	2
<b>9</b>	{}	4
<b>10</b>	{5, 6, 7, 8}	2
<b>11</b>	{6, 8}	2
<b>12</b>	{3, 4, 5, 6}	2
<b>13</b>	{2, 4, 5, 6, 7, 8}	2
<b>14</b>	{2, 6}	2
<b>15</b>	{4, 6}	2
<b>16</b>	{4, 8}	2
<b>17</b>	{2, 3, 5, 8}	2
<b>18</b>	{2, 3, 4, 6, 7, 8}	2

<b>RT GROUPS</b>		Total 2	
<b>No .</b>	<b>Coloring</b>	<b>Rank</b>	<b>Solv</b>
<b>1</b>	{2, 4, 5, 7}	8	["group", Not Solvable]
<b>2</b>	{2, 4, 6, 8}	8	["group", Not Solvable]

<b>CC Colorings</b>		Total 2
<b>No .</b>	<b>Coloring</b>	<b>Sandwich,Rank</b>
<b>1</b>	{}	true, 4
<b>2</b>	{5, 6, 7, 8}	true, 2

<b><math>\Delta</math>-RANK'D</b>	<b>SC'D !RK'D</b>	<b><math>\tau</math>-RANK'D</b>	<b>R/B RANK'D</b>	<b>NOT SYNC'D</b>	<b>Total Runs</b>	<b><math>2^{n-1}</math></b>
96	0	108 , 108	24 , 32	20	128	128

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