

New Graph

[3, 1, 1, 1, 2, 3], [5, 4, 4, 6, 4, 4]

$$\pi = [4, 1, 3, 4, 2, 2]$$

POSSIBLE RANKS

1 x 16

2 x 8

4 x 4

BASE DETERMINANT 2831/16384, .1727905273

NullSpace of Δ

{1, 2, 3, 4, 5, 6}

Range of Δ : $[-\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5]$

1 . Coloring, {}

R: [3, 1, 1, 1, 2, 3] **B**: [5, 4, 4, 6, 4, 4]

‘ See graph

‘ ‘ See pair graph

‘

Ω for $A + \tau \Delta$:

[‘4‘ (‘1 + τ ‘)‘, -1‘ (‘1 + τ ‘)‘²‘ (‘-1 + τ ‘)‘, 1‘ (‘1 + τ ‘)‘ (‘3 + τ ²‘)‘, -4‘ (‘-1 + τ ‘)‘, -2‘ (‘1 + τ ‘)‘ (‘-1 + τ ‘)‘, 2‘ (‘-1 + τ ‘)‘²‘]‘

For $\tau=1/2$, [48, 9, 39, 16, 12, 4] . FixedPtCheck, [48, 9, 39, 16, 12, 4]

$\det(A + \tau \Delta) = 0$

Delta Range : $[-y_5 - y_3 - y_4 - y_2 - y_1, y_5, y_3, y_4, y_2, y_1]$

[4, 1, 3, 4, 2, 2]

$$+ \quad \backslash; \quad - \quad \backslash; \quad \Delta$$

$\$ [[8, 2, 6, 0, 0, 0], [4, 0, 4, 4, 0, 4], [4, 0, 4, 4, 2, 2], [4, 1, 3, 4, 2, 2], [4, 1, 3, 4, 2, 2]] \$$ $\$ [[0, 0, 0, 8, 4, 4], [4, 2, 2, 4, 4, 0], [4, 2, 2, 4, 2, 2], [4, 1, 3, 4, 2, 2], [4, 1, 3, 4, 2, 2]] \$$ $\$ [[4, 1, 3, -4, -2, -2], [0, -1, 1, 0, -2, 2], [0, -1, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0]] \$$

$$[-y_2 - y_3, -y_2 - y_3 - y_1, y_1, y_2 + y_3, y_2, y_3]$$

$$p' = s^4 \quad p = s^4$$

$$S+ \quad \backslash; \quad S- \quad \backslash; \quad NM$$

$\$ [[1, 0, 1, 1, 1, 0], [2, 1, 1, 2, 0, 2], [2, 1, 1, 2, 0, 2], [2, 0, 2, 2, 1, 1], [2, 1, 1, 2, 1, 1], [2, 1, 1, 2, 1, 1]]$
 $\$ \quad \$ [[0, 1, 3, 2, 1, 1], [4, 0, 0, 2, 1, 1], [4, 0, 0, 2, 1, 1], [1, 0, 1, 0, 1, 1], [2, 1, 1, 4, 0, 0], [2, 1, 1, 4, 0, 0]] \$$ $\$ [[8, 0, 0, 4, 2, 2], [0, 2, 6, 4, 2, 2], [0, 2, 6, 4, 2, 2], [4, 1, 3, 8, 0, 0], [4, 1, 3, 0, 4, 4], [4, 1, 3, 0, 4, 4]] \$$

CmmCk true, true, true

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
3 vs 5	3 vs 5	3 vs 5	2 vs 3	2 vs 3

Omega Rank for R : cycles: $\{\{1, 3\}\}$, net cycles: 0 . order: 2

$\$ [[8, 2, 6, 0, 0, 0], [8, 0, 8, 0, 0, 0], [8, 0, 8, 0, 0, 0]] \$$

$$[y_2, y_1, y_2 - y_1, 0, 0, 0]$$

$$p = s^2 - s^3$$

Omega Rank for B : cycles: $\{\{4, 6\}\}$, net cycles: 0 . order: 2

$\$ [[0, 0, 0, 8, 4, 4], [0, 0, 0, 8, 0, 8], [0, 0, 0, 8, 0, 8]] \$$

$$[0, 0, 0, y_1 + y_2, y_1, y_2]$$

$$p = -s^2 + s^3$$

Â« NOT SYNC'D Â»

Nullspace of $\{\Omega\Delta^i\}$:

$$[0, 0, 0, x_2, x_1]$$

$$\text{For } A+2\Delta : [y_1, y_2, y_2, -3y_1 - 3y_2 - y_3, y_3, y_3]$$

$$\text{For } A-2\Delta : [-y_1 - 3y_2 - 3y_3, y_1, y_1, y_2, y_3, y_3]$$

Range of $\{\Omega\Delta^i\}$: $[-\mu_2 - \mu_3, -\mu_2 - \mu_3 - \mu_1, \mu_1, \mu_2 + \mu_3, \mu_2, \mu_3]$

rank of M is 4 , rank of N is 3

M N

\$ [[0, 1, 3, 0, 0, 0] , [1, 0, 0, 0, 0, 0] , [3, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 2, 2] , [0, 0, 0, 2, 0, 0] , [0, 0, 0, 2, 0, 0]]
 \$ \$ [[0, 2, 2, 1, 1, 1] , [2, 0, 0, 1, 1, 1] , [2, 0, 0, 1, 1, 1] , [1, 1, 1, 0, 2, 2] , [1, 1, 1, 2, 0, 0] , [1, 1, 1, 2, 0, 0]] \$

Check is $\Omega\Delta N$ zero? *true*, $\pi\Delta = [4, 1, 3, -4, -2, -2]$

ker M, $[0, -3\lambda_2, \lambda_2, 0, -\lambda_1, \lambda_1]$

Range M, $[x_1, x_2, 3x_2, x_4, x_3, x_3]$

$\tau = 19$, $r' = 1/2$

Ranges

Action of R on ranges, $[[2], [2], [1], [2]]$

Action of B on ranges, $[[3], [3], [4], [4]]$

$\beta(\{1, 2\}) = 1/8$

$\beta(\{1, 3\}) = 3/8$

$\beta(\{4, 5\}) = 1/4$

$\beta(\{4, 6\}) = 1/4$

ker N, $[-\mu_2 - \mu_3, -\mu_1 - \mu_2 - \mu_3, \mu_1, \mu_2 + \mu_3, \mu_2, \mu_3]$

Range of N

$[y_2, y_3, y_3, y_2 + y_3 - y_1, y_1, y_1]$

Partitions

Action of R on partitions, $[[2], [2]]$

Action of B on partitions, $[[1], [1]]$

$\alpha(\{\{2, 3, 5, 6\}, \{1, 4\}\}) = 1/2$

$\alpha(\{\{2, 3, 4\}, \{1, 5, 6\}\}) = 1/2$

$b_1 = \{2, 3, 4\}$, $b_2 = \{2, 3, 5, 6\}$, $b_3 = \{1, 5, 6\}$, $b_4 = \{1, 4\}$

Action of R and B on the blocks of the partitions: \$ [[0, 1, 1, 0] , [0, 0, 1, 1] , [1, 0, 0, 1] , [1, 1, 0, 0]] \$ =
 \$ [[0, 0, 1, 0] , [0, 0, 1, 0] , [1, 0, 0, 0] , [1, 0, 0, 0]] \$ + \$ [[0, 1, 0, 0] , [0, 0, 0, 1] , [0, 0, 0, 1] , [0, 1, 0, 0]] \$

$['3'$, $'3'$, $'1'$, $'1']$, $['2'$, $'4'$, $'4'$, $'2']$ with invariant measure $[1, 1, 1, 1]$

N by blocks, check: true . ' See partition graph.

' ' See level-2 partition graph.

Sandwich	
Coloring	{}
Rank	2
R,B	[3, 1, 1, 1, 2, 3], [5, 4, 4, 6, 4, 4]
π_2	[1, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2, 0]
u_2	[2, 2, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 2, 2, 0] (dim 1)
wpp	[5, 7, 7, 5, 7, 7]

2 . Coloring, {2}

R: [3, 4, 1, 1, 2, 3] **B:** [5, 1, 4, 6, 4, 4]

‘ See graph

‘ ‘ See pair graph

Ω for $A+\tau\Delta$:

$$\left[\begin{array}{l} -32(\tau^2 - 5\tau + 1), 8(\tau^2 - 5\tau + 1)(\tau - 1), 8(15 + 2\tau - 2\tau^3 + \tau^4)(\tau - 1), \\ -32(\tau^2 - 5\tau + 1), 16(\tau^2 - 5\tau + 1)(\tau - 1), 16(\tau - 1)^2(5 + 2\tau + \tau^2) \end{array} \right]$$

For $\tau=1/2$, [912, 171, 759, 400, 228, 100] . FixedPtCheck, [912, 171, 759, 400, 228, 100]

$$\det(A + \tau \Delta) = (\tau - 1)^2 (\tau + 1)^2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 4	3 vs 4

bi =

$$\begin{aligned} & \$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 3/4, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 1/4, 0, 3/4, 0, 0], \\ & [0, 0, 1/4, 3/4, 0, 0]] \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], \\ & [0, 0, 0, 0, 0, 1]] \$ = \end{aligned}$$

$$\begin{aligned} & \$ [[122913/5063504, 422025/1265876, -251541/316469, 1172724/316469, -594208/316469, \\ & -420384/316469], [1516449/5063504, -922679/1265876, 922147/316469, -594764/316469, \end{aligned}$$

8032/316469, -179744/316469] , [1910625/5063504, -673767/1265876, 11283/316469, -1142156/316469, 864864/316469, 334816/316469] , [-166527/5063504, 402033/1265876, -151205/316469, -166604/316469, -1045792/316469, 1293280/316469] , [-479231/5063504, 383793/1265876, 150051/316469, -652228/949407, 925024/316469, -2711648/949407] , [-525951/5063504, -559919/1265876, 177443/316469, 647356/949407, 1053664/316469, -3762784/949407]] \$ x \$ [[5/2, 1/2, 3/2, 11/2, 3, 3] , [17/8, 3/4, 11/8, 23/4, 15/8, 33/8] , [75/32, 15/32, 25/16, 183/32, 51/32, 69/16] , [139/64, 51/128, 213/128, 183/32, 225/128, 549/128] , [549/256, 225/512, 827/512, 753/128, 417/256, 549/128] , [2257/1024, 417/1024, 1647/1024, 2949/512, 1647/1024, 2259/512]] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 4

$$[y_1, y_4, y_2, y_3, 0, 0]$$

R = \$ [[0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [1, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0]] \$ x \$ [[1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0]] \$ = \$ [[0, 0, -7/32, 9/32] , [0, 1/2, -7/32, -7/32] , [0, 0, 9/32, -7/32] , [0, 0, 9/32, -7/32] , [1/2, -1/4, -7/32, 1/32] , [0, 0, -7/32, 9/32]] \$ x \$ [[7, 2, 6, 1, 0, 0] , [7, 0, 7, 2, 0, 0] , [9, 0, 7, 0, 0, 0] , [7, 0, 9, 0, 0, 0]] \$

Omega Rank for B : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$[[1, 0, 0, 7, 4, 4] , [0, 0, 0, 8, 1, 7] , [0, 0, 0, 8, 0, 8] , [0, 0, 0, 8, 0, 8]] $$$

$$[y_3, 0, 0, -y_3 + y_1 + y_2, y_1, y_2]$$

$$p = -s^3 + s^4$$

Â» SYNC'D 1/32 , 0.03125000000

3 . Coloring, {3}

R: [3, 1, 4, 1, 2, 3] **B:** [5, 4, 1, 6, 4, 4]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

' ['-32' (' 1 + τ ') ' (' 5 - 2τ + τ ² ') ' , 8' (' 1 + τ ') ' ² ' (' 5 - 2τ + τ ² ') ' (' - 1 + τ ') ' , 8' (' - 5 + τ ² ') ' (' 1 + τ ') ' (' 3 + τ ² ') ' , -32' (' 5 - τ + 3τ ² + τ ³ ') ' , 16' (' 1 + τ ') ' (' 5 - 2τ + τ ² ') ' (' - 1 + τ ') ' , 16' (' 5 - τ + 3τ ² + τ ³ ') ' (' - 1 + τ ') ']'

For $\tau=1/2$, [-816, -153, -741, -688, -204, -172] . FixedPtCheck, [816, 153, 741, 688, 204, 172]

$$\det(A + \tau \Delta) = 1' (' \tau ') ' (' 1 + \tau ') ' 2 ' (' - 1 + \tau ') ' 2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 4	3 vs 4

bi =

$$\begin{aligned} & \$ [[0, 0, 1/4, 0, 3/4, 0], [1/4, 0, 0, 3/4, 0, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 1/4, 0, 3/4, 0, 0], [0, 0, 1/4, 3/4, 0, 0]] \\ & \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ = \end{aligned}$$

$$\begin{aligned} & \$ [[45513/547024, 71601/136756, -26872/34189, 23816/34189, -10208/34189, -5344/34189], \\ & [8687895/16957744, -3070369/4239436, 1609888/1059859, 3531832/1059859, 2353760/1059859, \\ & -7204640/1059859], [5993175/16957744, -2304273/4239436, 192736/1059859, -1541960/1059859, \\ & 674144/1059859, 942816/1059859], [-3172041/16957744, 129783/4239436, -70344/1059859, \\ & -501128/1059859, -2390048/1059859, 3193568/1059859], [-2059849/16957744, 1191799/4239436, \\ & 2073488/3179577, -651944/1059859, 594016/1059859, -2208608/3179577], [727287/16957744, \\ & -899033/4239436, 64624/3179577, 724632/1059859, 2630880/1059859, -9394528/3179577]] \$ \times \$ [\\ & [7/2, 1/2, 3/2, 9/2, 3, 3], [19/8, 3/4, 13/8, 21/4, 21/8, 27/8], [87/32, 21/32, 23/16, 175/32, 57/32, 63/16], \\ & [167/64, 57/128, 213/128, 329/64, 261/128, 525/128], [677/256, 261/512, 859/512, 1371/256, 501/256, \\ & 987/256], [1395/512, 501/1024, 13/8, 5285/1024, 2031/1024, 4113/1024]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4}}, net cycles: 0 . order: 3

$$[y_4, y_1, y_2, y_3, 0, 0]$$

$$\begin{aligned} R = & \$ [[0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], \\ & 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0]] \\ & \$ = \$ [[0, -5/16, -5/16, 11/16], [0, -5/16, 11/16, -5/16], [0, 11/16, -5/16, -5/16], [0, -5/16, 11/16, \\ & -5/16], [1/2, 11/16, -5/16, -13/16], [0, -5/16, -5/16, 11/16]] \$ \times \$ [[5, 2, 6, 3, 0, 0], [5, 0, 5, 6, 0, 0], [6, \\ & 0, 5, 5, 0, 0], [5, 0, 6, 5, 0, 0]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$\begin{aligned} & \$ [[3, 0, 0, 5, 4, 4], [0, 0, 0, 8, 3, 5], [0, 0, 0, 8, 0, 8], [0, 0, 0, 8, 0, 8]] \$ \\ & [y_2 + y_3 - y_1, 0, 0, y_1, y_2, y_3] \end{aligned}$$

$$p = -s^3 + s^4$$

Â» SYNC'D 21/256 , 0.08203125000

4. Coloring, {4}

R: [3, 1, 1, 6, 2, 3] **B:** [5, 4, 4, 1, 4, 4]

‘ See graph

‘ ‘ See pair graph

‘

Ω for $A+\tau\Delta$:

$$\begin{aligned} & \left[\begin{array}{l} 8 \left(5 - \tau + 3\tau^2 + \tau^3 \right), -2 \left(5 - \tau + 3\tau^2 + \tau^3 \right) \left(-1 + \tau \right) \left(1 + \tau \right), 2 \left(15 - 2\tau + \right. \\ \left. 2\tau^3 + \tau^4 \right) \left(1 + \tau \right), 8 \left(-1 + \tau \right) \left(-5 + \tau^2 \right), -4 \left(5 - \tau + 3\tau^2 + \tau^3 \right) \left(-1 + \tau \right), \\ \left. 4 \left(-1 + \tau \right) \left(-5 + \tau^2 \right) \left(1 + \tau \right) \right] \end{array} \end{aligned}$$

For $\tau=1/2$, [688, 129, 687, 304, 172, 228] . FixedPtCheck, [688, 129, 687, 304, 172, 228]

$$\det(A + \tau \Delta) = 0$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	5 vs 5	4 vs 5	2 vs 4	3 vs 3

Omega Rank for R : cycles: {{1, 3}}, net cycles: -1 . order: 2

$$\$ [[4, 2, 6, 0, 0, 4], [8, 0, 8, 0, 0, 0], [8, 0, 8, 0, 0, 0], [8, 0, 8, 0, 0, 0]] \$$$

$$[-y_1 + y_2, y_1, y_2, 0, 0, 2y_1]$$

$$p = -s^2 + s^3 \quad p = -s^2 + s^4$$

Omega Rank for B : cycles: {{1, 4, 5}}, net cycles: 1 . order: 3

$$[y_1, 0, 0, y_2, y_3, 0]$$

$$\begin{aligned} \mathbf{B} = \$ [[0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], \\ 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0], \\ 0, 0, 0]] \$ = \$ [[-1/16, -1/16, 3/16], [3/16, -1/16, -1/16], [3/16, -1/16, -1/16], [-1/16, 3/16, -1/16], [3/16, \\ -1/16, -1/16], [3/16, -1/16, -1/16]] \$ \times \$ [[4, 0, 0, 8, 4, 0], [8, 0, 0, 4, 4, 0], [4, 0, 0, 4, 8, 0]] \$ \end{aligned}$$

$$\hat{A} \gg \text{SYNC'D } 1/4, 0.2500000000$$

5. Coloring, {5}

R: [3, 1, 1, 1, 4, 3] **B:** [5, 4, 4, 6, 2, 4]

' See graph

' ' See pair graph

Ω for $A+\tau\Delta$:

$$\begin{aligned} & [-16(1+\tau)^2(-5+\tau^2), -4(1+\tau)^2(-5+\tau^2)(-1+\tau)^2, 4(1+\tau)^2 \\ & (15+2\tau-2\tau^3+\tau^4), -16(5+2\tau+\tau^2)(-1+\tau)^2, 8(1+\tau)^2(-5+\tau^2)(-1+\tau)^2, \\ & 8(5+2\tau+\tau^2)(-1+\tau)^2] \end{aligned}$$

For $\tau=1/2$, [912, 57, 759, 400, 228, 100] . FixedPtCheck, [912, 57, 759, 400, 228, 100]

$$\det(A + \tau \Delta) = 0$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	5 vs 5	5 vs 5	2 vs 3	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 0, 1/4, 0, 3/4, 0], [1/4, 0, 0, 3/4, 0, 0], [1/4, 0, 0, 3/4, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 3/4, 0, 1/4, 0, \\ & 0], [0, 0, 1/4, 3/4, 0, 0]] \times \$ [[109/190, -9/190, -81/190, 27/190, 27/190, 27/190], [-9/190, 189/190, \\ & -9/190, 3/190, 3/190, 3/190], [-81/190, -9/190, 109/190, 27/190, 27/190, 27/190], [27/190, 3/190, 27/190, \\ & 181/190, -9/190, -9/190], [27/190, 3/190, 27/190, -9/190, 181/190, -9/190], [27/190, 3/190, 27/190, \\ & -9/190, -9/190, 181/190]] \$ = \\ & \$ [[4173/10448, -153/653, -467/653, 392/1959, 808/1959], [45/10448, -27/653, 609/653, \\ & -1160/1959, -472/1959], [45/10448, -27/653, 609/653, -1160/1959, -472/1959], [-1155/10448, 40/653, \\ & 41/653, 1912/1959, -1816/1959], [-419/10448, 382/653, -817/1959, -808/1959, 680/1959], [-483/10448, \\ & -102/653, -281/1959, -1480/1959, 760/653]] \$ \times \$ [[2, 3/2, 3/2, 5, 3, 3], [2, 9/4, 5/4, 21/4, 3/2, 15/4], \\ & [35/16, 9/8, 23/16, 93/16, 3/2, 63/16], [67/32, 9/8, 49/32, 21/4, 105/64, 279/64], [253/128, 315/256, \\ & 413/256, 363/64, 201/128, 63/16]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 2

$$\$ [[8, 0, 6, 2, 0, 0], [8, 0, 8, 0, 0, 0], [8, 0, 8, 0, 0, 0]] \$$$

$$[y_1 + y_2, 0, y_1, y_2, 0, 0]$$

$$p = -s^2 + s^3$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$[0, y_1, 0, y_2, y_3, y_4]$$

$$B = \$ [[0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 1], [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0]] \$ \times \$ [[0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ = \$ [[1/4, -1/8, -3/32, 1/32], [0, 0, 5/32, -3/32], [0, 0, 5/32, -3/32], [0, 0, -3/32, 5/32], [0, 1/4, -3/32, -3/32], [0, 0, 5/32, -3/32]] \$ \times \$ [[0, 2, 0, 6, 4, 4], [0, 4, 0, 6, 0, 6], [0, 0, 0, 10, 0, 6], [0, 0, 0, 6, 0, 10]] \$$$

Â» SYNC'D 3/32 , 0.09375000000

6 . Coloring, {6}

R: [3, 1, 1, 1, 2, 4] **B:** [5, 4, 4, 6, 4, 3]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$\begin{aligned} & ['-16' ('1 + \tau')^{'' ('5 - 2\tau + \tau^2)'} , 4' ('-1 + \tau')^{'' ('1 + \tau')'^2 ('5 - 2\tau + \tau^2)'} , -4' ('15 + \\ & \tau + 18\tau^2 - 2\tau^3 - \tau^4 + \tau^5)'} , -16' ('-1 + \tau')^{'' ('-5 + \tau^2)'} , 8' ('-1 + \tau')^{'' ('1 + \tau')^{'' ('5 - \\ & 2\tau + \tau^2)'} , 8' ('-1 + \tau')'^2 ('-5 + \tau^2)'} \end{aligned}$$

For τ=1/2, [-816, -153, -631, -304, -204, -76] . FixedPtCheck, [816, 153, 631, 304, 204, 76]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	5 vs 5	5 vs 5	2 vs 4	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 0, 1/4, 0, 3/4, 0], [1/4, 0, 0, 3/4, 0, 0], [1/4, 0, 0, 3/4, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 1/4, 0, 3/4, 0, \\ & 0], [0, 0, 3/4, 1/4, 0, 0]] \$ \times \$ [[901/1630, -729/1630, -81/1630, 243/1630, 27/1630, 243/1630], \\ & [-729/1630, 901/1630, -81/1630, 243/1630, 27/1630, 243/1630], [-81/1630, -81/1630, 1621/1630, \\ & 27/1630, 3/1630, 27/1630], [243/1630, 243/1630, 27/1630, 1549/1630, -9/1630, -81/1630], [27/1630, \\ & 27/1630, 3/1630, -9/1630, 1629/1630, -9/1630], [243/1630, 243/1630, 27/1630, -81/1630, -9/1630, \\ & 1549/1630]] \$ = \\ & \$ [[1371/3056, -47/764, 114/191, 392/573, -920/573], [-357/3056, -175/764, -242/191, -296/573, \\ & 1256/573], [-357/3056, -175/764, -242/191, -296/573, 1256/573], [171/3056, -51/764, 270/191, \\ & -680/573, -88/573], [-341/3056, 477/764, -610/573, 248/573, 104/573], [-501/3056, 69/764, -242/573, \\ & 920/573, -200/191]] \$ \times \$ [[2, 1/2, 5/2, 5, 3, 3], [2, 3/4, 11/4, 21/4, 3/2, 15/4], [35/16, 3/8, 53/16, \\ & 75/16, 3/2, 63/16], [67/32, 3/8, 7/2, 39/8, 105/64, 225/64], [35/16, 105/256, 809/256, 321/64, 201/128, \\ & 117/32]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3}}, net cycles: -1 . order: 2

$$\$ [[8, 2, 4, 2, 0, 0], [8, 0, 8, 0, 0, 0], [8, 0, 8, 0, 0, 0], [8, 0, 8, 0, 0, 0]] \$$$

$$[y_2 + 2 y_1, y_1, y_2, y_1, 0, 0]$$

$$p = s^2 - s^3 \quad p' = -s^2 + s^3$$

Omega Rank for B : cycles: {{3, 4, 6}}, net cycles: 0 . order: 3

$$[0, 0, y_3, y_1, y_2, y_4]$$

$$\begin{aligned} \mathbf{B} = \$ [[0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0], \\ 0]] \$ \times \$ [[0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ \\ = \$ [[1/4, -5/16, 3/16, -1/16], [0, 3/16, -5/16, 3/16], [0, 3/16, -5/16, 3/16], [0, 3/16, 3/16, -5/16], \\ [0, 3/16, -5/16, 3/16], [0, -5/16, 3/16, 3/16]] \$ \times \$ [[0, 0, 2, 6, 4, 4], [0, 0, 4, 6, 0, 6], [0, 0, 6, 4, 0, 6], [0, 0, 6, 6, 0, 4]] \$ \end{aligned}$$

Â» SYNC'D 5/64 , 0.07812500000

7 . Coloring, {2, 3}

R: [3, 4, 4, 1, 2, 3] **B**: [5, 1, 1, 6, 4, 4]

‘ See graph

‘ ‘ See pair graph

‘

Ω for A+τΔ :

$$\begin{aligned} [\text{' -8' (' 1 + \tau \text{')'' (' 5 - 2\tau + \tau^2 \text{')' , 2' (' 1 + \tau \text{')' }^2 \text{' (' 5 - 2\tau + \tau^2 \text{')' (' - 1 + \tau \text{')' , -2' (' 15 + } \\ 2\tau - 2\tau^3 + \tau^4 \text{')'' (' 1 + \tau \text{')' , 8' (' - 5 - \tau - 3\tau^2 + \tau^3 \text{')' , 4' (' 1 + \tau \text{')' (' 5 - 2\tau + \tau^2 \text{')' (' - 1 + \tau \text{' } \\ \text{')' , -4' (' - 5 - \tau - 3\tau^2 + \tau^3 \text{')' (' - 1 + \tau \text{')' }] \end{aligned}$$

For τ=1/2, [-816, -153, -759, -784, -204, -196] . FixedPtCheck, [816, 153, 759, 784, 204, 196]

$$\det(\mathbf{A} + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	5 vs 5	5 vs 5	4 vs 4	3 vs 4

bi =

$$\begin{aligned}
& \$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 1/4, 0, 3/4, 0, 0], [0, 0, 1/4, 3/4, 0, 0]] \$ \times \$ [[1621/1630, -81/1630, -81/1630, 27/1630, 27/1630, 3/1630], [-81/1630, 901/1630, -729/1630, 243/1630, 243/1630, 27/1630], [-81/1630, -729/1630, 901/1630, 243/1630, 243/1630, 27/1630], [27/1630, 243/1630, 243/1630, 1549/1630, -81/1630, -9/1630], [27/1630, 243/1630, 243/1630, -81/1630, 1549/1630, -9/1630], [3/1630, 27/1630, 27/1630, -9/1630, -9/1630, 1629/1630]] \$ \\
& = \\
& \$ [[3/64, 389/832, -35/104, 6/13, -15/26], [27/64, -371/832, -11/104, -10/13, 25/26], [27/64, -371/832, -11/104, -10/13, 25/26], [-9/64, -63/832, 57/104, 14/13, -35/26], [-9/64, 707/2496, 41/104, -10/13, 23/78], [-1/64, -437/2496, -63/104, -10/13, 127/78]] \$ \times \$ [[4, 1/2, 3/2, 4, 3, 3], [5/2, 3/4, 7/4, 5, 3, 3], [25/8, 3/4, 11/8, 41/8, 15/8, 15/4], [23/8, 15/32, 55/32, 19/4, 75/32, 123/32], [181/64, 75/128, 215/128, 83/16, 69/32, 57/16]] \$
\end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4}}, net cycles: 0 . order: 3

$$[y_1, y_2, y_3, y_4, 0, 0]$$

$$\begin{aligned}
R = & \$ [[0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[0, -1/16, -1/16, 3/16], [0, 3/16, -1/16, -1/16], [0, 3/16, -1/16, -1/16], [0, -1/16, 3/16, -1/16], [1/2, -1/16, -1/16, -5/16], [0, -1/16, -1/16, 3/16]] \$ \times \$ [[4, 2, 6, 4, 0, 0], [4, 0, 4, 8, 0, 0], [8, 0, 4, 4, 0, 0], [4, 0, 8, 4, 0, 0]] \$
\end{aligned}$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$\$ [[4, 0, 0, 4, 4, 4], [0, 0, 0, 8, 4, 4], [0, 0, 0, 8, 0, 8], [0, 0, 0, 8, 0, 8]] \$$$

$$[y_2 + y_3 - y_1, 0, 0, y_1, y_2, y_3]$$

$$p = -s^3 + s^4$$

Â» SYNC'D 1/4 , 0.2500000000

8 . Coloring, {2, 4}

R: [3, 4, 1, 6, 2, 3] **B:** [5, 1, 4, 1, 4, 4]

‘ See graph

‘ ‘ See pair graph

‘

Ω for A+τΔ :

$$\begin{aligned}
& [‘ -32 ‘ (‘ 5 - \tau + 3\tau^2 + \tau^3 ‘) ‘ , 8 ‘ (‘ 1 + \tau ‘) ‘ (‘ - 1 + \tau ‘) ‘ (‘ 5 - \tau + 3\tau^2 + \tau^3 ‘) ‘ , 8 ‘ (‘ - 5 + \tau^2 ‘) ‘ (‘ 3 + \tau^2 ‘) ‘ (‘ 1 + \tau ‘) ‘ , 32 ‘ (‘ 5 + 2\tau + \tau^2 ‘) ‘ (‘ - 1 + \tau ‘) ‘ , 16 ‘ (‘ - 1 + \tau ‘) ‘ (‘ 5 - \tau + 3\tau^2 + \tau^3 ‘) ‘ , 16 ‘ (‘ 5 + 2\tau + \tau^2 ‘) ‘ (‘ 1 + \tau ‘) ‘ (‘ - 1 + \tau ‘) ‘] ‘
\end{aligned}$$

For $\tau=1/2$, [-688, -129, -741, -400, -172, -300] . FixedPtCheck, [688, 129, 741, 400, 172, 300]

$$\det(A + \tau \Delta) = 1' (' - 1 + \tau ')'' (' \tau ')'' (' 1 + \tau ')' ^ 3$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	6 vs 6	6 vs 6	5 vs 5	3 vs 3

bi =

$$\begin{aligned} & \$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 3/4, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 1/4, 0, 3/4, 0, \\ & 0], [0, 0, 1/4, 3/4, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, \\ & 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ = \end{aligned}$$

$$\begin{aligned} & \$ [[-296511/14346832, -223431/3586708, -2443512/896677, -1753032/896677, 252064/896677, \\ & 4074912/896677], [1872705/14346832, -6607879/3586708, -3889984/896677, -656840/896677, \\ & 2499104/896677, 3638688/896677], [5626497/14346832, 317481/3586708, -3419712/896677, \\ & -893256/896677, -652512/896677, 4590496/896677], [-474975/14346832, 1519617/3586708, \\ & 1063784/896677, 615224/896677, 255136/896677, -2228320/896677], [-2384863/14346832, \\ & -586687/3586708, 1192768/896677, 1114648/896677, -761568/896677, -1194080/896677], \\ & [1725153/14346832, 822033/3586708, 8641248/896677, 2829272/896677, -523616/896677, \\ & -11204192/896677]] \$ \times \$ [[9/2, 1/2, 3/2, 11/2, 3, 1], [39/8, 3/4, 11/8, 17/4, 27/8, 11/8], [131/32, \\ & 27/32, 25/16, 153/32, 117/32, 17/16], [295/64, 117/128, 165/128, 315/64, 393/128, 153/128], [1203/256, \\ & 393/512, 743/512, 1125/256, 885/256, 315/256], [271/64, 885/1024, 759/512, 4911/1024, 3609/1024, \\ & 1125/1024]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 4

$$[y_1, y_2, y_3, y_4, 0, y_5]$$

$$\begin{aligned} R = & \$ [[0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, \\ & 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, \\ & 0, 0, 1]] \$ = \$ [[0, 0, 0, 9/32, -7/32], [0, 1/2, -1/4, -7/32, 1/32], [0, 0, 0, -7/32, 9/32], [0, 0, 1/2, -7/32, \\ & -7/32], [1/2, -1/4, -7/8, 1/32, 21/32], [0, 0, 0, 9/32, -7/32]] \$ \times \$ [[3, 2, 6, 1, 0, 4], [6, 0, 7, 2, 0, 1], [7, 0, \\ & 7, 0, 0, 2], [7, 0, 9, 0, 0, 0], [9, 0, 7, 0, 0, 0]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 4, 5}}, net cycles: 1 . order: 3

$$[y_3, 0, 0, y_1, y_2, 0]$$

$$\begin{aligned} B = & \$ [[0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, \\ & 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, \\ & 0, 0, 0]] \$ = \$ [[-19/112, -3/112, 29/112], [-3/112, 29/112, -19/112], [29/112, -19/112, -3/112], [-3/112, \\ & 29/112, -19/112], [29/112, -19/112, -3/112], [29/112, -19/112, -3/112]] \$ \times \$ [[5, 0, 0, 7, 4, 0], [7, 0, 0, \\ & 4, 5, 0], [4, 0, 0, 5, 7, 0]] \$ \end{aligned}$$

Â» SYNC'D 17/128 , 0.1328125000

9 . Coloring, {2, 5}

R: [3, 4, 1, 1, 4, 3] **B:** [5, 1, 4, 6, 2, 4]

' See graph

' ' See pair graph

,

Ω for $A+\tau\Delta$:

' ['32' (' - 5 + τ^2 ')' , 8' (' - 5 + τ^2 ')' (' - 1 + τ ')' ² , 8' (' 1 + τ ')' (' 5 - 2 τ + τ^2 ')' (' - 3 + τ ')' , -32' (' - 5 + τ ')' (' - 1 + τ ')' , -16' (' - 5 + τ^2 ')' (' - 1 + τ ')' , 16' (' - 5 + τ ')' (' - 1 + τ ')' ² ']'

For $\tau=1/2$, [-304, -19, -255, -144, -76, -36] . FixedPtCheck, [304, 19, 255, 144, 76, 36]

$\det(A + \tau \Delta) = 1' (' \tau ')' (' 1 + \tau ')' (' - 1 + \tau ')' ³$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	6 vs 6	6 vs 6	3 vs 3	4 vs 5

bi =

\$ [[0, 0, 1/4, 0, 3/4, 0] , [3/4, 0, 0, 1/4, 0, 0] , [1/4, 0, 0, 3/4, 0, 0] , [1/4, 0, 0, 0, 0, 3/4] , [0, 3/4, 0, 1/4, 0, 0] , [0, 0, 1/4, 3/4, 0, 0]] \$ x \$ [[1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1]] \$ =

\$ [[4899/78896, 925/4931, 2929/4931, 21148/14793, -3872/4931, -21088/14793] , [7523/78896, 2051/14793, 63985/44379, -71620/133137, -1952/4931, -90464/133137] , [35427/78896, -3807/4931, -391/4931, -67244/44379, 24608/14793, 14048/44379] , [-3389/78896, 6505/14793, -18157/14793, -4516/14793, -22112/14793, 13280/4931] , [-3901/78896, 3819/4931, 4091/14793, 6620/133137, 7264/44379, -153440/133137] , [-16573/78896, -4637/4931, 17233/44379, 10804/44379, 93152/44379, -67360/44379]] \$ x \$ [[5/2, 3/2, 3/2, 9/2, 3, 3] , [21/8, 9/4, 11/8, 9/2, 15/8, 27/8] , [101/32, 45/32, 3/2, 147/32, 63/32, 27/8] , [165/64, 189/128, 209/128, 9/2, 303/128, 441/128] , [169/64, 909/512, 771/512, 1221/256, 495/256, 27/8] , [1485/512, 1485/1024, 385/256, 2349/512, 507/256, 3663/1024]] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 2

$[y_1, 0, y_2, y_3, 0, 0]$

$R = $ [[0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [1, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 1, 0, 0, 0]] $ x $ [[1, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0]] $ = $ [[0, -7/32, 9/32] , [1/3, -7/32, -5/96] , [0, 9/32, -7/32] , [0, 9/32, -7/32] , [1/3, -7/32, -5/96] ,$

$$[0, -7/32, 9/32]] \$ x \$ [[7, 0, 6, 3, 0, 0], [9, 0, 7, 0, 0, 0], [7, 0, 9, 0, 0, 0]] \$$$

Omega Rank for B : cycles: {{1, 2, 5}, {4, 6}}, net cycles: 2 . order: 6

$$\$ [[1, 2, 0, 5, 4, 4], [2, 4, 0, 4, 1, 5], [4, 1, 0, 5, 2, 4], [1, 2, 0, 4, 4, 5], [2, 4, 0, 5, 1, 4]] \$$$

$$[7 y_1, 7 y_2, 0, 9 y_1 + 9 y_2 + 9 y_3 - 7 y_4, 7 y_3, 7 y_4]$$

$$p = s + s^2 - s^4 - s^5$$

Â» SYNC'D 7/128 , 0.05468750000

10 . Coloring, {2, 6}

R: [3, 4, 1, 1, 2, 4] **B:** [5, 1, 4, 6, 4, 3]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$\begin{aligned} & ['-32' (' 1 + \tau ')'' (' 5 - 2\tau + \tau^2 ')', 8' (' 1 + \tau ')'^2 (' 5 - 2\tau + \tau^2 ')'' (' - 1 + \tau ')', 8' (' 3 + \tau \\ & 2 ')'' (' - 5 - \tau - 3\tau^2 + \tau^3 ')', 32' (' 5 + 2\tau + \tau^2 ')'' (' - 1 + \tau ')', 16' (' 1 + \tau ')'' (' 5 - 2\tau + \tau^2 ') \\ &)'' (' - 1 + \tau ')', -16' (' 5 + 2\tau + \tau^2 ')'' (' - 1 + \tau ')'^2 ']' \end{aligned}$$

For τ=1/2, [-816, -153, -637, -400, -204, -100] . FixedPtCheck, [816, 153, 637, 400, 204, 100]

$$\det(A + \tau \Delta) = 1' (' \tau ')'' (' 1 + \tau ')'' (' - 1 + \tau ')'^3$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 4	5 vs 5

bi =

$$\$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 3/4, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 1/4, 0, 3/4, 0, 0], [0, 0, 3/4, 1/4, 0, 0]] \$ x \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ =$$

$$\begin{aligned} & \$ [[6388983/22504304, 209364/1406519, -1217352/1406519, 1016136/1406519, \\ & -1107616/1406519, 788064/1406519], [7438455/22504304, -1023788/1406519, 4529472/1406519, \\ & 144136/1406519, 1312736/1406519, -5339552/1406519], [2837751/22504304, -636660/1406519, \\ & 197952/1406519, -1138424/1406519, 2362080/1406519, -874400/1406519], [-1145961/22504304, \\ & 105210/1406519, -443096/1406519, -949176/1406519, -1440160/1406519, 2886752/1406519], \\ & [-1832489/22504304, 800046/1406519, 213168/1406519, -1366792/4219557, 830176/1406519, \\ & -3556064/4219557], [-5377257/22504304, 37690/1406519, 546064/1406519, 5871736/4219557, \\ & 65888/1406519, -6548704/4219557]] \$ x \$ [[5/2, 1/2, 5/2, 9/2, 3, 3], [17/8, 3/4, 23/8, 5, 15/8, 27/8], \end{aligned}$$

[81/32, 15/32, 49/16, 147/32, 51/32, 15/4] , [145/64, 51/128, 441/128, 291/64, 243/128, 441/128] ,
 [147/64, 243/512, 1613/512, 159/32, 435/256, 873/256] , [2443/1024, 435/1024, 3207/1024, 4719/1024,
 441/256, 477/128]] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 4

$$[y_4, y_3, y_2, y_1, 0, 0]$$

$$R = \$ [[0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[0, 0, -7/32, 9/32], [0, 1/2, -7/32, -7/32], [0, 0, 9/32, -7/32], [0, 0, 9/32, -7/32], [1/2, -3/4, -7/32, 17/32], [0, 1/2, -7/32, -7/32]] \$ \times \$ [[7, 2, 4, 3, 0, 0], [7, 0, 7, 2, 0, 0], [9, 0, 7, 0, 0, 0], [7, 0, 9, 0, 0, 0]] \$$$

Omega Rank for B : cycles: {{3, 4, 6}}, net cycles: 0 . order: 3

$$[y_1, 0, y_2, y_3, y_4, y_5]$$

$$B = \$ [[0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ = \$ [[0, 1, -5/16, 11/16, -21/16], [1, -4, 11/16, -21/16, 59/16], [0, 0, -5/16, -5/16, 11/16], [0, 0, 11/16, -5/16, -5/16], [0, 0, -5/16, -5/16, 11/16], [0, 0, -5/16, 11/16, -5/16]] \$ \times \$ [[1, 0, 2, 5, 4, 4], [0, 0, 4, 6, 1, 5], [0, 0, 5, 5, 0, 6], [0, 0, 6, 5, 0, 5], [0, 0, 5, 6, 0, 5]] \$$$

Â» SYNC'D 269/2048 , 0.1313476562

11 . Coloring, {3, 4}

R: [3, 1, 4, 6, 2, 3] **B:** [5, 4, 1, 1, 4, 4]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$['-32(' - 1 + \tau ')^2 (' 5 + 2\tau + \tau^2 ')^2 , 8(' - 1 + \tau ')^2 (' 1 + \tau ')^2 (' 5 + 2\tau + \tau^2 ')^2 , 8(' 15 - 2\tau + 2\tau^3 + \tau^4 ')^2 (' 1 + \tau ')^2 , 32(' 5 - \tau + 3\tau^2 + \tau^3 ')^2 , 16(' - 1 + \tau ')^2 (' 5 + 2\tau + \tau^2 ')^2 , 16(' 1 + \tau ')^2 (' 5 - \tau + 3\tau^2 + \tau^3 ')^2]$$

For τ=1/2, [400, 75, 687, 688, 100, 516] . FixedPtCheck, [400, 75, 687, 688, 100, 516]

$$\det(A + \tau \Delta) = 1^2 (' - 1 + \tau ')^2 (' 1 + \tau ')^3 (' \tau ')^2$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	6 vs 6	6 vs 6	5 vs 5	3 vs 3

bi =

$\$ [[0, 0, 1/4, 0, 3/4, 0], [1/4, 0, 0, 3/4, 0, 0], [3/4, 0, 0, 1/4, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 1/4, 0, 3/4, 0, 0], [0, 0, 1/4, 3/4, 0, 0]] \$ x \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ =$
 $\$ [[209385/31390672, 3959073/7847668, 3024761/1961917, 1047644/1961917, 796768/1961917, -5749408/1961917], [8612649/31390672, -3406351/7847668, 10093745/1961917, 7051228/1961917, 2949344/1961917, -19658400/1961917], [13009641/31390672, 6511809/7847668, 3272001/1961917, 2323228/1961917, -2849824/1961917, -5063840/1961917], [-2527095/31390672, -291783/7847668, -3986679/1961917, -815588/1961917, -693152/1961917, 5848928/1961917], [-5055223/31390672, -2639815/7847668, -309983/1961917, -813892/1961917, -527904/1961917, 2750304/1961917], [1565193/31390672, -12759303/7847668, -7721055/1961917, -6660676/1961917, 3120736/1961917, 14475616/1961917]] \$ x \$ [[11/2, 1/2, 3/2, 9/2, 3, 1], [37/8, 3/4, 13/8, 15/4, 33/8, 9/8], [135/32, 33/32, 23/16, 157/32, 111/32, 15/16], [321/64, 111/128, 165/128, 71/16, 405/128, 157/128], [1155/256, 405/512, 799/512, 273/64, 963/256, 71/64], [4677/1024, 963/1024, 1439/1024, 1187/256, 3465/1024, 273/256]] \$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 4, 6}}, net cycles: 0 . order: 3

$[y_5, y_4, y_3, y_2, 0, y_1]$

$R = \$ [[0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 1], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0]] \$ x \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1]] \$ = \$ [[0, 0, -5/16, 11/16, -5/16], [0, 1/2, 11/16, -5/16, -13/16], [0, 0, -5/16, -5/16, 11/16], [0, 0, 11/16, -5/16, -5/16], [1/2, -1/4, -5/16, -13/16, 15/16], [0, 0, -5/16, 11/16, -5/16]] \$ x \$ [[1, 2, 6, 3, 0, 4], [2, 0, 5, 6, 0, 3], [0, 0, 5, 5, 0, 6], [0, 0, 6, 5, 0, 5], [0, 0, 5, 6, 0, 5]] \$$

Omega Rank for B : cycles: {{1, 4, 5}}, net cycles: 1 . order: 3

$[y_1, 0, 0, y_2, y_3, 0]$

$B = \$ [[0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0]] \$ x \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[-19/112, 29/112, -3/112], [-3/112, -19/112, 29/112], [29/112, -3/112, -19/112], [29/112, -3/112, -19/112], [-3/112, -19/112, 29/112], [-3/112, -19/112, 29/112]] \$ x \$ [[7, 0, 0, 5, 4, 0], [5, 0, 0, 4, 7, 0], [4, 0, 0, 7, 5, 0]] \$$

$\hat{A} \gg \text{SYNC'D } 3/16, 0.1875000000$

12 . Coloring, {3, 5}

R: [3, 1, 4, 1, 4, 3] **B:** [5, 4, 1, 6, 2, 4]

‘ See graph

‘ ‘ See pair graph

Ω for A+τΔ :

$$\begin{aligned} & ['32' ('1 + \tau')'' ('5 - 2\tau + \tau^2')', 8' ('1 + \tau')'' ('-1 + \tau')'^2 ('5 - 2\tau + \tau^2')', 8' ('1 + \tau')'' \\ & ('15 + 2\tau - 2\tau^3 + \tau^4')', -32' ('-5 - \tau - 3\tau^2 + \tau^3')', -16' ('1 + \tau')'' ('-1 + \tau')'' ('5 - 2\tau + \tau^2')', \\ & 16' ('-5 - \tau - 3\tau^2 + \tau^3')'' ('-1 + \tau')'']' \end{aligned}$$

For τ=1/2, [816, 51, 759, 784, 204, 196] . FixedPtCheck, [816, 51, 759, 784, 204, 196]

$$\det(A + \tau \Delta) = 1' ('1 + \tau')'' ('\tau')'' ('-1 + \tau')'^3$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	3 vs 3	5 vs 5

bi =

$$\begin{aligned} & \$ [[0, 0, 1/4, 0, 3/4, 0], [1/4, 0, 0, 3/4, 0, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 3/4, 0, 1/4, 0, \\ & 0], [0, 0, 1/4, 3/4, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, \\ & 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ = \end{aligned}$$

$$\begin{aligned} & \$ [[1227957/7348048, 151748/459253, -208616/459253, 462776/459253, 13280/459253, \\ & -467232/459253], [-612043/7348048, -25196/459253, 2104256/4133277, -205336/1377759, \\ & -882336/459253, 7282144/4133277], [2233269/7348048, -147476/459253, 431680/1377759, \\ & -741640/459253, -605600/459253, 3719840/1377759], [500949/7348048, -158478/459253, \\ & 35944/459253, 427192/459253, 713440/459253, -1020704/459253], [-1270059/7348048, \\ & 553022/1377759, 1389296/4133277, -560056/1377759, 1055008/1377759, -3560480/4133277], \\ & [-1557611/7348048, 188794/1377759, -425296/1377759, -446568/459253, -1366624/1377759, \\ & 3320992/1377759]] \$ \times \$ [[7/2, 3/2, 3/2, 7/2, 3, 3], [19/8, 9/4, 13/8, 9/2, 21/8, 21/8], [93/32, 63/32, 5/4, \\ & 151/32, 57/32, 27/8], [167/64, 171/128, 201/128, 305/64, 279/128, 453/128], [173/64, 837/512, 787/512, \\ & 147/32, 501/256, 915/256], [2775/1024, 1503/1024, 1607/1024, 4895/1024, 519/256, 441/128]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4}}, net cycles: 1 . order: 3

$$[y_2, 0, y_1, y_3, 0, 0]$$

$$\begin{aligned} & R = \$ [[0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, \\ & 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, \\ & 0, 0, 0]] \$ = \$ [[11/16, -5/16, -5/16], [-5/16, -5/16, 11/16], [-5/16, 11/16, -5/16], [-5/16, -5/16, 11/16], \\ & [-5/16, 11/16, -5/16], [11/16, -5/16, -5/16]] \$ \times \$ [[5, 0, 6, 5, 0, 0], [5, 0, 5, 6, 0, 0], [6, 0, 5, 5, 0, 0]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$[y_1, y_2, 0, y_3, y_4, y_5]$$

$$B = \$ [[0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ = \$ [[0, 1/3, -4/9, -5/96, 65/288], [0, 0, 0, 9/32, -7/32], [1/3, -4/9, 10/27, 65/288, -365/864], [0, 0, 0, -7/32, 9/32], [0, 0, 1/3, -7/32, -5/96], [0, 0, 0, 9/32, -7/32]] \$ \times \$ [[3, 2, 0, 3, 4, 4], [0, 4, 0, 6, 3, 3], [0, 3, 0, 7, 0, 6], [0, 0, 0, 9, 0, 7], [0, 0, 0, 7, 0, 9]] \$$$

Â» SYNC'D 95/512 , 0.1855468750

13 . Coloring, {3, 6}

R: [3, 1, 4, 1, 2, 4] **B:** [5, 4, 1, 6, 4, 3]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$['-32' (' - 5 - \tau - 3\tau^2 + \tau^3 ')', 8' (' 1 + \tau ')'' (' - 1 + \tau ')'' (' - 5 - \tau - 3\tau^2 + \tau^3 ')', 8' (' 15 + \tau + 18\tau^2 - 2\tau^3 - \tau^4 + \tau^5 ')', 32' (' 5 - \tau + 3\tau^2 + \tau^3 ')', 16' (' - 1 + \tau ')'' (' - 5 - \tau - 3\tau^2 + \tau^3 ')', -16' (' 5 - \tau + 3\tau^2 + \tau^3 ')'' (' - 1 + \tau ')'']'$$

For τ=1/2, [784, 147, 631, 688, 196, 172] . FixedPtCheck, [784, 147, 631, 688, 196, 172]

$$\det(A + \tau \Delta) = 1' (' \tau ')'' (' 1 + \tau ')'' (' - 1 + \tau ')''^3$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 4	5 vs 5

bi =

$$\$ [[0, 0, 1/4, 0, 3/4, 0], [1/4, 0, 0, 3/4, 0, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 1/4, 0, 3/4, 0, 0], [0, 0, 3/4, 1/4, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ =$$

$$\$ [[98415/272624, 5337/17039, -20123/17039, -4700/17039, 1440/17039, 12960/17039], [8239365/2998864, -43177/187429, 701903/187429, 495564/187429, 488544/187429, -2146080/187429], [245253/2998864, -161961/187429, -94529/187429, -145012/187429, 257888/187429, 140000/187429], [-2775195/2998864, 13161/187429, 16215/187429, -30516/187429, -417568/187429, 603872/187429], [78629/2998864, 187637/187429, 329357/562287, -181460/187429, -83616/187429, -76640/562287], [318501/2998864, -66843/187429, 273997/562287, 315628/187429, 255968/187429, -1812832/562287]]$$

$$\$ \times \$ [[7/2, 1/2, 5/2, 7/2, 3, 3], [23/8, 3/4, 25/8, 4, 21/8, 21/8], [113/32, 21/32, 43/16, 127/32, 69/32, 3],$$

[203/64, 69/128, 401/128, 113/32, 339/128, 381/128] , [431/128, 339/512, 1549/512, 1003/256, 609/256, 339/128] , [437/128, 609/1024, 181/64, 947/256, 1293/512, 3009/1024]] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4}}, net cycles: 0 . order: 3

$$[y_1, y_3, y_4, y_2, 0, 0]$$

$$R = \$ [[0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[0, -3/112, 29/112, -19/112], [0, 29/112, -19/112, -3/112], [0, -19/112, -3/112, 29/112], [0, 29/112, -19/112, -3/112], [1/2, -19/112, -3/112, -27/112], [0, -19/112, -3/112, 29/112]] \$ \times \$ [[5, 2, 4, 5, 0, 0], [7, 0, 5, 4, 0, 0], [4, 0, 7, 5, 0, 0], [5, 0, 4, 7, 0, 0]] \$$$

Omega Rank for B : cycles: {{1, 3, 4, 5, 6}}, net cycles: 1 . order: 5

$$[y_3, 0, y_4, y_5, y_1, y_2]$$

$$B = \$ [[0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ = \$ [[47/176, -17/176, -65/176, 31/176, 15/176], [15/176, 47/176, -17/176, -65/176, 31/176], [-17/176, -65/176, 31/176, 15/176, 47/176], [31/176, 15/176, 47/176, -17/176, -65/176], [15/176, 47/176, -17/176, -65/176, 31/176], [-65/176, 31/176, 15/176, 47/176, -17/176]] \$ \times \$ [[3, 0, 2, 3, 4, 4], [2, 0, 4, 4, 3, 3], [4, 0, 3, 3, 2, 4], [3, 0, 4, 2, 4, 3], [4, 0, 3, 4, 3, 2]] \$$$

Â» SYNC'D 479/4096 , 0.1169433594

14 . Coloring, {4, 5}

R: [3, 1, 1, 6, 4, 3] **B:** [5, 4, 4, 1, 2, 4]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$['-16' (' 5 - \tau + 3\tau^2 + \tau^3 ') , -4' (' 5 - \tau + 3\tau^2 + \tau^3 ') ' (' - 1 + \tau ') ^ 2 , 4' (' - 5 + \tau^2 ') ' (' 3 + \tau^2 ') ' (' 1 + \tau ') , 16' (' - 1 + \tau ') ' (' 5 + 2\tau + \tau^2 ') , 8' (' 5 - \tau + 3\tau^2 + \tau^3 ') ' (' - 1 + \tau ') , 8' (' - 1 + \tau ') ' (' 5 + 2\tau + \tau^2 ') ' (' 1 + \tau ') ']$$

For τ=1/2, [-688, -43, -741, -400, -172, -300] . FixedPtCheck, [688, 43, 741, 400, 172, 300]

$$\det(A + \tau \Delta) = 0$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	5 vs 5	5 vs 5	3 vs 4	4 vs 4

bi =

$\$ [[0, 0, 1/4, 0, 3/4, 0], [1/4, 0, 0, 3/4, 0, 0], [1/4, 0, 0, 3/4, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 3/4, 0, 1/4, 0, 0], [0, 0, 1/4, 3/4, 0, 0]] \$ \times \$ [[829/910, -9/910, -81/910, 27/910, 27/910, 243/910], [-9/910, 909/910, -9/910, 3/910, 3/910, 27/910], [-81/910, -9/910, 829/910, 27/910, 27/910, 243/910], [27/910, 3/910, 27/910, 901/910, -9/910, -81/910], [27/910, 3/910, 27/910, -9/910, 901/910, -81/910], [243/910, 27/910, 243/910, -81/910, -81/910, 181/910]] \$ =$

$\$ [[-349/1008, -1/252, 2/3, -184/189, 136/189], [143/5040, -233/252, -2/3, 344/945, 1192/945], [143/5040, -233/252, -2/3, 344/945, 1192/945], [383/5040, 139/252, -2/3, -136/945, 232/945], [-23/720, 5/36, 2/3, 136/135, -232/135], [5119/5040, 155/252, 2/3, 472/945, -2584/945]] \$ \times \$ [[4, 3/2, 3/2, 5, 3, 1], [9/2, 9/4, 5/4, 15/4, 3, 5/4], [59/16, 9/4, 23/16, 69/16, 27/8, 15/16], [133/32, 81/32, 37/32, 69/16, 177/64, 69/64], [133/32, 531/256, 335/256, 273/64, 399/128, 69/64]] \$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 4

$\$ [[4, 0, 6, 2, 0, 4], [6, 0, 8, 0, 0, 2], [8, 0, 8, 0, 0, 0], [8, 0, 8, 0, 0, 0]] \$$

$[y_3, 0, y_1, y_2, 0, -y_3 + y_1 + y_2]$

$$p = -s^3 + s^4$$

Omega Rank for B : cycles: {{1, 2, 4, 5}}, net cycles: 1 . order: 4

$[y_1, y_4, 0, y_2, y_3, 0]$

$B = \$ [[0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[-3/64, 5/64, 13/64, -11/64], [13/64, -11/64, -3/64, 5/64], [13/64, -11/64, -3/64, 5/64], [5/64, 13/64, -11/64, -3/64], [-11/64, -3/64, 5/64, 13/64], [13/64, -11/64, -3/64, 5/64]] \$ \times \$ [[4, 2, 0, 6, 4, 0], [6, 4, 0, 2, 4, 0], [2, 4, 0, 4, 6, 0], [4, 6, 0, 4, 2, 0]] \$$

$\hat{A} \gg \text{SYNC'D } 11/128, 0.08593750000$

15 . Coloring, {4, 6}

R: [3, 1, 1, 6, 2, 4] **B**: [5, 4, 4, 1, 4, 3]

‘ See graph

‘ ‘ See pair graph

Ω for $A+\tau\Delta$:

$$\begin{bmatrix} -16(\tau^5 + 2\tau + \tau^2), & 4(\tau - 1 + \tau^2)(\tau^5 + 2\tau + \tau^2)(\tau + 1), & -4(\tau + 3)(\tau + 1 + \tau^2)(\tau^5 - 2\tau + \tau^2), & 16(\tau - 5 + \tau^2), & 8(\tau - 1 + \tau^2)(\tau^5 + 2\tau + \tau^2), & 8(\tau - 5 + \tau^2)(\tau + 1) \end{bmatrix}$$

For $\tau=1/2$, [-400, -75, -357, -304, -100, -228] . FixedPtCheck, [400, 75, 357, 304, 100, 228]

$$\det(A + \tau \Delta) = 0$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	5 vs 5	5 vs 5	3 vs 5	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 0, 1/4, 0, 3/4, 0], [1/4, 0, 0, 3/4, 0, 0], [1/4, 0, 0, 3/4, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 1/4, 0, 3/4, 0, 0], [0, 0, 3/4, 1/4, 0, 0]] \times \$ [[7381/8110, -729/8110, -81/8110, 243/8110, 27/8110, 2187/8110], \\ & [-729/8110, 7381/8110, -81/8110, 243/8110, 27/8110, 2187/8110], [-81/8110, -81/8110, 8101/8110, 27/8110, 3/8110, 243/8110], [243/8110, 243/8110, 27/8110, 8029/8110, -9/8110, -729/8110], [27/8110, 27/8110, 3/8110, -9/8110, 8109/8110, -81/8110], [2187/8110, 2187/8110, 243/8110, -729/8110, -81/8110, 1549/8110]] \$ = \end{aligned}$$

$$\begin{aligned} & \$ [[-11/80, 28/15, 23/15, 8/15, -56/15], [73/400, 136/75, -49/75, 56/75, -152/75], [73/400, 136/75, -49/75, 56/75, -152/75], [-7/400, -49/75, -3/25, -104/75, 56/25], [-71/400, -97/75, -77/75, 88/75, 104/75], [249/400, -119/25, -37/75, -24/25, 424/75]] \times \$ [[4, 1/2, 5/2, 5, 3, 1], [9/2, 3/4, 7/4, 19/4, 3, 5/4], [67/16, 3/4, 33/16, 71/16, 27/8, 19/16], [129/32, 27/32, 31/16, 79/16, 201/64, 71/64], [563/128, 201/256, 471/256, 151/32, 387/128, 79/64]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3}, {4, 6}}, net cycles: 1 . order: 2

$$\$ [[4, 2, 4, 2, 0, 4], [6, 0, 4, 4, 0, 2], [4, 0, 6, 2, 0, 4], [6, 0, 4, 4, 0, 2], [4, 0, 6, 2, 0, 4]] \$$$

$$[y_3, 4y_3 - y_1 - 5y_2, y_1, y_2, 0, 3y_3 - 4y_2]$$

$$p' = -s^2 + s^4 \quad p = -s^2 + s^4$$

Omega Rank for B : cycles: {{1, 4, 5}}, net cycles: 0 . order: 3

$$[y_1, 0, y_2, y_3, y_4, 0]$$

$$\begin{aligned} B = & \$ [[0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0]] \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[0, -5/16, 3/16, 3/16], [0, 3/16, -5/16, 3/16], [0, 3/16, -5/16, 3/16], [0, 3/16, 3/16, -5/16], [0, 3/16, -5/16, 3/16], [1/2, -5/16, 3/16, -5/16]] \times \$ [[4, 0, 2, 6, 4, 0], [6, 0, 0, 6, 4, 0], [6, 0, 0, 4, 6, 0], [4, 0, 0, 6, 6, 0]] \$ \end{aligned}$$

Â» SYNC'D 7/128 , 0.05468750000

16 . Coloring, {5, 6}

R: [3, 1, 1, 1, 4, 4] **B:** [5, 4, 4, 6, 2, 3]

' See graph

' ' See pair graph

,

Ω for $A+\tau\Delta$:

' ['8' (' 1 + τ ')'' (' 5 - 2 τ + τ^2 ')', 2' (' - 1 + τ ')'^ 2 ' (' 1 + τ ')'' (' 5 - 2 τ + τ^2 ')', -2' (' - 5 - τ - 3 τ^2 + τ^3 ')'' (' 3 + τ^2 ')', -8' (' 5 + 2 τ + τ^2 ')'' (' - 1 + τ ')', -4' (' - 1 + τ ')'' (' 1 + τ ')'' (' 5 - 2 τ + τ^2 ')', 4' (' 5 + 2 τ + τ^2 ')'' (' - 1 + τ ')'^ 2 ']'

For $\tau=1/2$, [816, 51, 637, 400, 204, 100] . FixedPtCheck, [816, 51, 637, 400, 204, 100]

$\det(A + \tau \Delta) = 0$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	5 vs 5	5 vs 5	2 vs 3	5 vs 5

bi =

\$ [[0, 0, 1/4, 0, 3/4, 0] , [1/4, 0, 0, 3/4, 0, 0] , [1/4, 0, 0, 3/4, 0, 0] , [1/4, 0, 0, 0, 0, 3/4] , [0, 3/4, 0, 1/4, 0, 0] , [0, 0, 3/4, 1/4, 0, 0]] \$ x \$ [[181/910, -81/910, -81/910, 243/910, 27/910, 243/910] , [-81/910, 901/910, -9/910, 27/910, 3/910, 27/910] , [-81/910, -9/910, 901/910, 27/910, 3/910, 27/910] , [243/910, 27/910, 27/910, 829/910, -9/910, -81/910] , [27/910, 3/910, 3/910, -9/910, 909/910, -9/910] , [243/910, 27/910, 27/910, -81/910, -9/910, 829/910]] \$ =

\$ [[27/64, -91/320, -3/8, -2/15, 13/30] , [3/64, -7/64, 5/8, -2/3, 1/6] , [3/64, -7/64, 5/8, -2/3, 1/6] , [-9/64, 57/320, 1/8, 14/15, -31/30] , [-1/64, 209/320, -5/24, -2/15, -7/30] , [-9/64, -71/320, -13/24, -2/15, 11/10]] \$ x \$ [[2, 3/2, 5/2, 4, 3, 3] , [2, 9/4, 11/4, 9/2, 3/2, 3] , [19/8, 9/8, 11/4, 39/8, 3/2, 27/8] , [35/16, 9/8, 25/8, 33/8, 57/32, 117/32] , [67/32, 171/128, 421/128, 291/64, 105/64, 99/32]] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 2

\$ [[8, 0, 4, 4, 0, 0] , [8, 0, 8, 0, 0, 0] , [8, 0, 8, 0, 0, 0]] \$

[$y_1 + y_2, 0, y_1, y_2, 0, 0$]

$$p = -s^2 + s^3$$

Omega Rank for B : cycles: {{3, 4, 6}}, net cycles: 0 . order: 3

$$[0, y_1, y_2, y_3, y_4, y_5]$$

$$B = \$ [[0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 1], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ = \$ [[1/4, -1/8, -1/16, -1/16, 1/16], [0, 0, 3/16, -1/16, -1/16], [0, 0, 3/16, -1/16, -1/16], [0, 0, -1/16, 3/16, -1/16], [0, 1/4, -1/16, -1/16, -1/16], [0, 0, -1/16, -1/16, 3/16]] \$ x \$ [[0, 2, 2, 4, 4, 4], [0, 4, 4, 4, 0, 4], [0, 0, 4, 8, 0, 4], [0, 0, 4, 4, 0, 8], [0, 0, 8, 4, 0, 4]] \$$$

Â» SYNC'D 1/4 , 0.2500000000

17 . Coloring, {2, 3, 4}

R: [3, 4, 4, 6, 2, 3] **B:** [5, 1, 1, 1, 4, 4]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$['-4' (' 5 + 2\tau + \tau^2 ')'' (' - 1 + \tau ')', 1' (' 5 + 2\tau + \tau^2 ')'' (' - 1 + \tau ')'^2 (' 1 + \tau ')', -1' (' - 5 + \tau^2 ')'' (' 3 + \tau^2 ')'' (' 1 + \tau ')', -4' (' - 5 - \tau - 3\tau^2 + \tau^3 ')', 2' (' 5 + 2\tau + \tau^2 ')'' (' - 1 + \tau ')'^2, -2' (' - 5 - \tau - 3\tau^2 + \tau^3 ')'' (' 1 + \tau ')'']'$$

For τ=1/2, [400, 75, 741, 784, 100, 588] . FixedPtCheck, [400, 75, 741, 784, 100, 588]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	5 vs 5	5 vs 5	4 vs 4	3 vs 3

bi =

$$\$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [3/4, 0, 0, 1/4, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 1/4, 0, 3/4, 0, 0], [0, 0, 1/4, 3/4, 0, 0]] \$ x \$ [[189/190, -9/190, -9/190, 3/190, 3/190, 3/190], [-9/190, 109/190, -81/190, 27/190, 27/190, 27/190], [-9/190, -81/190, 109/190, 27/190, 27/190, 27/190], [3/190, 27/190, 27/190, 181/190, -9/190, -9/190], [3/190, 27/190, 27/190, -9/190, 181/190, -9/190], [3/190, 27/190, 27/190, -9/190, -9/190, 181/190]] \$ = \$ [[1/16, 13/12, 1/4, -2/5, -14/15], [9/16, 9/4, 5/4, -2/5, -18/5], [9/16, 9/4, 5/4, -2/5, -18/5], [-3/16, -1, -3/4, 4/5, 6/5], [-3/16, -2/3, 1/4, 0, 2/3], [-3/16, -4, -7/4, 0, 6]] \$ x \$ [[6, 1/2, 3/2, 4, 3, 1], [9/2, 3/4, 7/4, 7/2, 9/2, 1], [9/2, 9/8, 11/8, 19/4, 27/8, 7/8], [87/16, 27/32, 43/32, 61/16, 27/8, 19/16], [9/2, 27/32, 53/32, 127/32, 261/64, 61/64]] \$$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 4, 6}}, net cycles: 0 . order: 3

$$[0, y_1, y_4, y_2, 0, y_3]$$

$$R = \$ [[0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 1], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], 0] \$ \times \$ [[0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1]] \$ = \$ [[0, -1/16, -1/16, 3/16], [0, 3/16, -1/16, -1/16], [0, 3/16, -1/16, -1/16], [0, -1/16, 3/16, -1/16], [1/2, -1/16, -1/16, -5/16], [0, -1/16, -1/16, 3/16]] \$ \times \$ [[0, 2, 6, 4, 0, 4], [0, 0, 4, 8, 0, 4], [0, 0, 4, 4, 0, 8], [0, 0, 8, 4, 0, 4]] \$$$

Omega Rank for B : cycles: {{1, 4, 5}}, net cycles: 1 . order: 3

$$[y_1, 0, 0, y_2, y_3, 0]$$

$$B = \$ [[0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], 0] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[-1/16, 3/16, -1/16], [3/16, -1/16, -1/16], [3/16, -1/16, -1/16], [3/16, -1/16, -1/16], [-1/16, -1/16, 3/16], [-1/16, -1/16, 3/16]] \$ \times \$ [[8, 0, 0, 4, 4, 0], [4, 0, 0, 4, 8, 0], [4, 0, 0, 8, 4, 0]] \$$$

Â» SYNC'D 3/8 , 0.3750000000

18 . Coloring, {2, 3, 5}

R: [3, 4, 4, 1, 4, 3] **B:** [5, 1, 1, 6, 2, 4]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$[(-16, -4)^{(-1+\tau)^2}, 4^{(1+\tau)^2} (-3+\tau)^2, -16, 8^{(-1+\tau)^2}, 8^{(-1+\tau)^2}]^2$$

For τ=1/2, [-16, -1, -15, -16, -4, -4] . FixedPtCheck, [16, 1, 15, 16, 4, 4]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	5 vs 5	5 vs 5	3 vs 3	4 vs 5

bi =

$\$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 3/4, 0, 1/4, 0, 0], [0, 0, 1/4, 3/4, 0, 0]] \$ \times \$ [[901/910, -9/910, -81/910, 27/910, 27/910, 3/910], [-9/910, 901/910, -81/910, 27/910, 27/910, 3/910], [-81/910, -81/910, 181/910, 243/910, 243/910, 27/910], [27/910, 27/910, 243/910, 829/910, -81/910, -9/910], [27/910, 27/910, 243/910, -81/910, 829/910, -9/910], [3/910, 3/910, 27/910, -9/910, -9/910, 909/910]] \$ =$

$\$ [[381/400, 127/100, -6/25, -24/25, -24/25], [369/2000, 523/500, 206/125, 24/125, -376/125], [369/2000, 523/500, 206/125, 24/125, -376/125], [-639/2000, -1113/500, -186/125, 56/125, 456/125], [417/2000, 1517/1500, 58/125, -168/125, -104/375], [-2687/2000, -1787/1500, -38/125, 248/125, 344/375]] \$ \times \$ [[4, 3/2, 3/2, 3, 3, 3], [3, 9/4, 7/4, 15/4, 3, 9/4], [63/16, 9/4, 21/16, 55/16, 9/4, 45/16], [113/32, 27/16, 27/16, 57/16, 189/64, 165/64], [219/64, 567/256, 391/256, 225/64, 339/128, 171/64]] \$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4}}, net cycles: 1 . order: 3

$$[y_1, 0, y_2, y_3, 0, 0]$$

$R = \$ [[0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[3/16, -5/16, 3/16], [3/16, 3/16, -5/16], [3/16, 3/16, -5/16], [-5/16, 3/16, 3/16], [3/16, 3/16, -5/16], [3/16, -5/16, 3/16]] \$ \times \$ [[4, 0, 6, 6, 0, 0], [6, 0, 4, 6, 0, 0], [6, 0, 6, 4, 0, 0]] \$$

Omega Rank for B : cycles: {{1, 2, 5}, {4, 6}}, net cycles: 2 . order: 6

$\$ [[4, 2, 0, 2, 4, 4], [2, 4, 0, 4, 4, 2], [4, 4, 0, 2, 2, 4], [4, 2, 0, 4, 4, 2], [2, 4, 0, 2, 4, 4]] \$$

$$[-3 y_1 + 5 y_2 - 3 y_3 + 5 y_4, 3 y_1, 0, 3 y_2, 3 y_3, 3 y_4]$$

$$p = -s - s^2 + s^4 + s^5$$

Â» SYNC'D 67/512 , 0.1308593750

19 . Coloring, {2, 3, 6}

R: [3, 4, 4, 1, 2, 4] **B:** [5, 1, 1, 6, 4, 3]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[' -16, 4(' -1 + \tau ')(' -1 + \tau '), -4(' 3 + \tau ^2 '), -16, 8(' -1 + \tau '), 8(' -1 + \tau ')']$$

For τ=1/2, [-16, -3, -13, -16, -4, -4] . FixedPtCheck, [16, 3, 13, 16, 4, 4]

$$\det(A + \tau \Delta) = 0$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	5 vs 5	5 vs 5	4 vs 4	5 vs 5

bi =

$$\begin{aligned} & \$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 1/4, 0, 3/4, 0, 0], [0, 0, 3/4, 1/4, 0, 0]] \$ \times \$ [[821/830, -81/830, -9/830, 27/830, 3/830, 3/830], [-81/830, 101/830, -81/830, 243/830, 27/830, 27/830], [-9/830, -81/830, 821/830, 27/830, 3/830, 3/830], [27/830, 243/830, 27/830, 749/830, -9/830, -9/830], [3/830, 27/830, 3/830, -9/830, 829/830, -1/830], [3/830, 27/830, 3/830, -9/830, -1/830, 829/830]] \$ = \\ & \$ [[435/688, 77/86, -15/43, -24/43, -24/43], [3141/4816, -425/602, -221/301, -216/301, 472/301], [3141/4816, -425/602, -221/301, -216/301, 472/301], [-2763/4816, -135/602, 291/301, 328/301, -360/301], [-1803/4816, 1979/1806, 335/301, -152/301, -1144/903], [-2635/4816, -1853/1806, -265/301, 264/301, 1480/903]] \$ \times \$ [[4, 1/2, 5/2, 3, 3, 3], [3, 3/4, 13/4, 15/4, 3, 9/4], [63/16, 3/4, 39/16, 61/16, 9/4, 45/16], [107/32, 9/16, 99/32, 51/16, 189/64, 183/64], [453/128, 189/256, 763/256, 123/32, 321/128, 153/64]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4}}, net cycles: 0 . order: 3

$$[y_1, y_2, y_3, y_4, 0, 0]$$

$$\begin{aligned} R = & \$ [[0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0]] \$ = \\ & \$ [[0, -5/16, 3/16, 3/16], [0, 3/16, -5/16, 3/16], [0, 3/16, -5/16, 3/16], [0, 3/16, 3/16, -5/16], [1/2, -5/16, 3/16, -5/16], [0, 3/16, -5/16, 3/16]] \$ \times \$ [[4, 2, 4, 6, 0, 0], [6, 0, 4, 6, 0, 0], [6, 0, 6, 4, 0, 0], [4, 0, 6, 6, 0, 0]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 3, 4, 5, 6}}, net cycles: 1 . order: 5

$$[y_1, 0, y_2, y_5, y_3, y_4]$$

$$\begin{aligned} B = & \$ [[0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ = \\ & \$ [[5/16, 5/16, -3/16, -3/16, -3/16], [5/16, -3/16, -3/16, -3/16, 5/16], [5/16, -3/16, -3/16, -3/16, 5/16], [-3/16, -3/16, 5/16, 5/16, -3/16], [-3/16, 5/16, 5/16, -3/16, -3/16], [-3/16, -3/16, -3/16, 5/16, 5/16]] \$ \times \$ [[4, 0, 2, 2, 4, 4], [2, 0, 4, 4, 4, 2], [4, 0, 2, 4, 2, 4], [2, 0, 4, 2, 4, 4], [4, 0, 4, 4, 2, 2]] \$ \end{aligned}$$

Â» SYNC'D 45/512 , 0.08789062500

20 . Coloring, {2, 4, 5}

R: [3, 4, 1, 6, 4, 3] **B:** [5, 1, 4, 1, 2, 4]

‘ See graph

‘ ‘ See pair graph

Ω for A+τΔ :

$$['32' ('5 - \tau + 3\tau^2 + \tau^3')', 8' ('5 - \tau + 3\tau^2 + \tau^3')'' ('-1 + \tau'')'^2, 8' ('15 + 2\tau - 2\tau^3 + \tau^4')'' ('1 + \tau'')', 32' ('-1 + \tau'')'' ('-5 + \tau'')'' ('1 + \tau'')', -16' ('5 - \tau + 3\tau^2 + \tau^3')'' ('-1 + \tau'')', 16' ('-1 + \tau'')'' ('-5 + \tau'')'' ('1 + \tau'')'^2]'$$

For τ=1/2, [688, 43, 759, 432, 172, 324] . FixedPtCheck, [688, 43, 759, 432, 172, 324]

$$\det(A + \tau \Delta) = 1' ('1 + \tau'')'^2 ('\tau'')'' ('-1 + \tau'')'^2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 4	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 3/4, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 3/4, 0, 1/4, 0, 0], [0, 0, 1/4, 3/4, 0, 0]] \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ = \\ & \$ [[407853/6515152, 16887/407197, -295272/407197, -93640/58171, 71520/407197, 862304/407197], [280045/6515152, -174525/407197, -1024/1221591, 34872/58171, 1634464/1221591, -606112/407197], [3834477/6515152, -204933/407197, -865280/407197, -19400/58171, -90144/407197, 1081952/407197], [-541683/6515152, 204025/407197, -39576/407197, 62888/174513, 155744/407197, -1222880/1221591], [-228789/930736, 2103/58171, 179456/174513, 220232/174513, -238496/174513, -113696/174513], [-764979/6515152, -61883/407197, 1549056/407197, 83608/58171, -35232/407197, -1963936/407197]] \$ \times \$ [[9/2, 3/2, 3/2, 9/2, 3, 1], [39/8, 9/4, 11/8, 3, 27/8, 9/8], [137/32, 81/32, 3/2, 105/32, 117/32, 3/4], [303/64, 351/128, 161/128, 207/64, 411/128, 105/128], [307/64, 1233/512, 711/512, 195/64, 909/256, 207/256], [4545/1024, 2727/1024, 1435/1024, 3213/1024, 921/256, 195/256]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 4

$$[y_2, 0, y_1, y_4, 0, y_3]$$

$$\begin{aligned} R = & \$ [[0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1]] \$ = \\ & \$ [[0, 0, 9/32, -7/32], [1/3, -4/9, -5/96, 65/288], [0, 0, -7/32, 9/32], [0, 1/3, -7/32, -5/96], [1/3, -4/9, -5/96, 65/288], [0, 0, 9/32, -7/32]] \$ \times \$ [[3, 0, 6, 3, 0, 4], [6, 0, 7, 0, 0, 3], [7, 0, 9, 0, 0, 0], [9, 0, 7, 0, 0, 0]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 2, 5}}, net cycles: 0 . order: 3

$$[y_2, y_3, 0, y_4, y_1, 0]$$

$$B = \$ [[0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[0, -3/112, 29/112, -19/112], [0, 29/112, -19/112, -3/112], [1/5, -19/112, -3/112, 33/560], [0, 29/112, -19/112, -3/112], [0, -19/112, -3/112, 29/112], [1/5, -19/112, -3/112, 33/560]] \$ \times \$ [[5, 2, 0, 5, 4, 0], [7, 4, 0, 0, 5, 0], [4, 5, 0, 0, 7, 0], [5, 7, 0, 0, 4, 0]] \$$$

Â» SYNC'D 53/256 , 0.2070312500

21 . Coloring, {2, 4, 6}

R: [3, 4, 1, 6, 2, 4] **B:** [5, 1, 4, 1, 4, 3]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$[' -32 , 8 ' (' 1 + \tau ') ' (' - 1 + \tau ') ' , 8 ' (' 1 + \tau ') ' (' - 3 + \tau ') ' , -32 , 16 ' (' - 1 + \tau ') ' , -16 ' (' 1 + \tau ') '] '$$

For τ=1/2, [-16, -3, -15, -16, -4, -12] . FixedPtCheck, [16, 3, 15, 16, 4, 12]

$$\det(A + \tau \Delta) = 1 ' (' \tau ') ' (' 1 + \tau ') ' ^ 2 ' (' - 1 + \tau ') ' ^ 2$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	3 vs 5	4 vs 4

bi =

$$\$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 3/4, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 1/4, 0, 3/4, 0, 0], [0, 0, 3/4, 1/4, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ =$$

$$\$ [[-159/272, 0, 53/51, 940/51, 1696/51, -2656/51], [-31/272, 0, -1193/153, 1412/153, -5536/153, 5344/153], [225/272, 0, -211/51, -1556/51, -224/51, 1952/51], [-191/272, 2, 191/153, 4804/153, -6944/153, 1760/153], [257/272, -2, -121/153, -7196/153, 2656/153, 4832/153], [257/272, -2, 967/153, -1756/153, 4832/153, -3872/153]] \$ \times \$ [[9/2, 1/2, 5/2, 9/2, 3, 1], [35/8, 3/4, 15/8, 9/2, 27/8, 9/8], [141/32, 27/32, 31/16, 141/32, 105/32, 9/8], [283/64, 105/128, 249/128, 141/32, 423/128, 141/128], [141/32, 423/512, 989/512, 1131/256, 849/256, 141/128], [2261/512, 849/1024, 987/512, 1131/256, 423/128, 1131/1024]] \$$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3}, {4, 6}}, net cycles: 1 . order: 2

$$\$ [[3, 2, 4, 3, 0, 4], [4, 0, 3, 6, 0, 3], [3, 0, 4, 3, 0, 6], [4, 0, 3, 6, 0, 3], [3, 0, 4, 3, 0, 6]] \$$$

$$[-7 y_1 + 15 y_2 - 7 y_3, 6 y_1, 6 y_2, -15 y_1 + 27 y_2 - 15 y_3, 0, 6 y_3]$$

$$p = s^2 - s^4 \quad p' = -s^2 + s^4$$

Omega Rank for B : cycles: {{1, 4, 5}}, net cycles: 0 . order: 3

$$[y_4, 0, y_1, y_2, y_3, 0]$$

$$\begin{aligned} B = \$ [[0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0], \\ 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0]] \$ \\ = \$ [[0, -5/16, -5/16, 11/16], [0, -5/16, 11/16, -5/16], [0, 11/16, -5/16, -5/16], [0, -5/16, 11/16, -5/16], [0, 11/16, -5/16, -5/16], [1/2, -5/16, -5/16, 3/16]] \$ \times \$ [[5, 0, 2, 5, 4, 0], [5, 0, 0, 6, 5, 0], [6, 0, 0, 5, 5, 0], [5, 0, 0, 5, 6, 0]] \$ \end{aligned}$$

Â» SYNC'D 35/2048 , 0.01708984375

22 . Coloring, {2, 5, 6}

R: [3, 4, 1, 1, 4, 4] **B:** [5, 1, 4, 6, 2, 3]

‘ See graph

‘ ‘ See pair graph

‘

Ω for A+τΔ :

$$\begin{aligned} ['32' ('5 - 2\tau + \tau^2')', 8' ('-1 + \tau')'^2 ('5 - 2\tau + \tau^2')', 8' ('15 - 10\tau + 16\tau^2 - 6\tau^3 + \tau^4')', \\ '32' ('-1 + \tau')' ('-5 + \tau')', -16' ('-1 + \tau')' ('5 - 2\tau + \tau^2')', -16' ('-1 + \tau')'^2 ('-5 + \tau')']' \end{aligned}$$

For τ=1/2, [272, 17, 213, 144, 68, 36] . FixedPtCheck, [272, 17, 213, 144, 68, 36]

$$\det(A + \tau \Delta) = 1' ('\tau')' ('-1 + \tau')'^4$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	3 vs 3	3 vs 6

bi =

$\$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 3/4, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 3/4, 0, 1/4, 0, 0], [0, 0, 3/4, 1/4, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ =$

$\$ [[422523/947696, 81393/236924, 90522/59231, 44024/59231, -28512/59231, -149088/59231], [167355/947696, 472003/710772, 398582/177693, -68104/533079, -176800/177693, -1012064/533079], [132603/947696, -283471/236924, -2446/59231, -147416/177693, 121376/59231, -10528/177693], [-128997/947696, 18057/236924, -124774/59231, -66968/177693, -57184/59231, 634592/177693], [458971/947696, 831787/710772, 69862/177693, -443560/533079, -31456/59231, -331616/533079], [-854757/947696, -129623/236924, -52634/177693, 250120/177693, 150752/177693, -79648/177693]] \$$
 $\times \$ [[5/2, 3/2, 5/2, 7/2, 3, 3], [21/8, 9/4, 23/8, 15/4, 15/8, 21/8], [107/32, 45/32, 21/8, 123/32, 63/32, 45/16], [171/64, 189/128, 377/128, 225/64, 321/128, 369/128], [697/256, 963/512, 1449/512, 1005/256, 513/256, 675/256], [1587/512, 1539/1024, 1361/512, 3843/1024, 2091/1024, 3015/1024]] \$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3}}, net cycles: 0 . order: 2

$$[y_1, 0, y_2, y_3, 0, 0]$$

$R = \$ [[0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[0, -7/32, 9/32], [1/5, -7/32, 13/160], [0, 9/32, -7/32], [0, 9/32, -7/32], [1/5, -7/32, 13/160], [1/5, -7/32, 13/160]] \$ \times \$ [[7, 0, 4, 5, 0, 0], [9, 0, 7, 0, 0, 0], [7, 0, 9, 0, 0, 0]] \$$

Omega Rank for B : cycles: {{1, 2, 5}, {3, 4, 6}}, net cycles: 2 . order: 3

$\$ [[1, 2, 2, 3, 4, 4], [2, 4, 4, 2, 1, 3], [4, 1, 3, 4, 2, 2], [1, 2, 2, 3, 4, 4], [2, 4, 4, 2, 1, 3], [4, 1, 3, 4, 2, 2]] \$$

$$[-20 y_1 + 28 y_2, 16 y_1, 16 y_2, 16 y_3, 25 y_1 - 35 y_2 + 28 y_3, 27 y_1 - 25 y_2 + 20 y_3]$$

$$p' = -1 + s^3 \quad p' = -s + s^4 \quad p' = -s^2 + s^5$$

Â» SYNC'D 1715/16384 , 0.1046752930

23 . Coloring, {3, 4, 5}

R: [3, 1, 4, 6, 4, 3] **B:** [5, 4, 1, 1, 2, 4]

‘ See graph

‘ ‘ See pair graph

‘

Ω for $A+\tau\Delta$:

$$\left[\begin{aligned} & 32^{\tau} (-1 + \tau)^{\tau} (5 + 2\tau + \tau^2)^{\tau}, 8^{\tau} (-1 + \tau)^{3\tau} (5 + 2\tau + \tau^2)^{\tau}, 8^{\tau} (1 + \tau)^{\tau} (-5 \\ & + \tau^2)^{\tau} (3 + \tau^2)^{\tau}, 32^{\tau} (-5 - \tau - 3\tau^2 + \tau^3)^{\tau}, -16^{\tau} (-1 + \tau)^{2\tau} (5 + 2\tau + \tau^2)^{\tau}, 16^{\tau} (1 \\ & + \tau)^{\tau} (-5 - \tau - 3\tau^2 + \tau^3)^{\tau} \end{aligned} \right]$$

For $\tau=1/2$, [-400, -25, -741, -784, -100, -588] . FixedPtCheck, [400, 25, 741, 784, 100, 588]

$$\det(A + \tau \Delta) = 1^{\tau} (1 + \tau)^{2\tau} (\tau)^{\tau} (-1 + \tau)^{2\tau}$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	6 vs 6	6 vs 6	4 vs 4	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 0, 1/4, 0, 3/4, 0], [1/4, 0, 0, 3/4, 0, 0], [3/4, 0, 0, 1/4, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 3/4, 0, 1/4, 0, \\ & 0], [0, 0, 1/4, 3/4, 0, 0]] \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, \\ & 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ = \end{aligned}$$

$$\begin{aligned} & \$ [[-2185107/18804688, 812248/1175293, -774195/1175293, 422276/1175293, -1492320/1175293, \\ & 1242016/1175293], [-838931/18804688, -2808/1175293, -4140449/3525879, 1284740/1175293, \\ & -2760416/1175293, 8953568/3525879], [6927597/18804688, 939672/1175293, 1601925/1175293, \\ & 1375684/1175293, -92128/1175293, -4184672/1175293], [1847565/18804688, -410610/1175293, \\ & 1373517/1175293, -613052/1175293, 3702688/1175293, -4094560/1175293], [-263349/2686384, \\ & -74918/167899, 434905/503697, -131652/167899, 173536/167899, -254944/503697], \\ & [1948941/18804688, -1686954/1175293, -3926235/1175293, -1402780/1175293, -4117088/1175293, \\ & 11084704/1175293]] \$ \times \$ [[11/2, 3/2, 3/2, 7/2, 3, 1], [33/8, 9/4, 13/8, 3, 33/8, 7/8], [129/32, 99/32, \\ & 5/4, 121/32, 99/32, 3/4], [291/64, 297/128, 153/128, 127/32, 387/128, 121/128], [285/64, 1161/512, \\ & 703/512, 897/256, 873/256, 127/128], [2163/512, 2619/1024, 697/512, 233/64, 855/256, 897/1024]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 4, 6}}, net cycles: 0 . order: 3

$$[y_1, 0, y_2, y_3, 0, y_4]$$

$$\begin{aligned} R = & \$ [[0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, \\ & 0]] \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, \\ & 0, 0, 1]] \$ = \$ [[0, -5/16, -5/16, 11/16], [1, -5/16, 11/16, -21/16], [0, 11/16, -5/16, -5/16], [0, -5/16, \\ & 11/16, -5/16], [0, 11/16, -5/16, -5/16], [0, -5/16, -5/16, 11/16]] \times \$ [[1, 0, 6, 5, 0, 4], [0, 0, 5, 6, 0, 5], \\ & [0, 0, 5, 5, 0, 6], [0, 0, 6, 5, 0, 5]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 2, 4, 5}}, net cycles: 1 . order: 4

$$[y_3, y_4, 0, y_1, y_2, 0]$$

$$\begin{aligned} B = & \$ [[0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, \\ & 0]] \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, \\ & 0, 0, 0]] \$ = \$ [[-75/832, 197/832, -107/832, 37/832], [-107/832, 37/832, -75/832, 197/832], [197/832, \end{aligned}$$

-107/832, 37/832, -75/832] , [197/832, -107/832, 37/832, -75/832] , [37/832, -75/832, 197/832, -107/832] ,
 [-107/832, 37/832, -75/832, 197/832]] \$ x \$ [[7, 2, 0, 3, 4, 0] , [3, 4, 0, 2, 7, 0] , [2, 7, 0, 4, 3, 0] , [4, 3, 0,
 7, 2, 0]] \$

Â» SYNC'D 27/128 , 0.2109375000

24 . Coloring, {3, 4, 6}

R: [3, 1, 4, 6, 2, 4] **B:** [5, 4, 1, 1, 4, 3]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

' [-32' (' - 1 + τ ')'' (' - 5 + τ ² ')' , 8' (' 1 + τ ')'' (' - 1 + τ ')' ² ' (' - 5 + τ ² ')' , 8' (' 1 + τ ')''
 (' - 1 + τ ')'' (' 3 + τ ')'' (' 5 - 2τ + τ ² ')' , -32' (' 5 - τ + 3τ ² + τ ³ ')' , 16' (' - 1 + τ ')' ² ' (' - 5 + τ
² ')' , -16' (' 5 - τ + 3τ ² + τ ³ ')'' (' 1 + τ ')'']'

For τ=1/2, [-304, -57, -357, -688, -76, -516] . FixedPtCheck, [304, 57, 357, 688, 76, 516]

det(A + τ Δ) = 1' (' - 1 + τ ')' ² ' (' 1 + τ ')' ² ' (' τ ')'

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	5 vs 5	4 vs 4

bi =

\$ [[0, 0, 1/4, 0, 3/4, 0] , [1/4, 0, 0, 3/4, 0, 0] , [3/4, 0, 0, 1/4, 0, 0] , [3/4, 0, 0, 0, 0, 1/4] , [0, 1/4, 0, 3/4, 0,
 0] , [0, 0, 3/4, 1/4, 0, 0]] \$ x \$ [[1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0,
 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1]] \$ =

\$ [[1780767/8634608, 144453/539663, 1284562/539663, 308776/539663, 839392/539663,
 -2654752/539663] , [21018591/8634608, -432383/539663, 8519770/539663, 3253160/539663,
 3028832/539663, -15649312/539663] , [3402591/8634608, 316521/539663, 671418/539663,
 588712/539663, -760224/539663, -995360/539663] , [-6462081/8634608, 114015/539663,
 -2960862/539663, -480216/539663, -1158688/539663, 4923360/539663] , [-2129729/8634608,
 -226733/539663, -363046/539663, -474552/539663, -219808/539663, 1450976/539663] ,
 [196479/8634608, -548793/539663, -1551366/539663, -1692216/539663, 484320/539663,
 3329504/539663]] \$ x \$ [[11/2, 1/2, 5/2, 7/2, 3, 1] , [37/8, 3/4, 17/8, 7/2, 33/8, 7/8] , [141/32, 33/32,
 29/16, 141/32, 111/32, 7/8] , [315/64, 111/128, 225/128, 259/64, 423/128, 141/128] , [585/128, 423/512,
 1053/512, 123/32, 945/256, 259/256] , [4743/1024, 945/1024, 1947/1024, 4255/1024, 1755/512, 123/128]
] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$[y_1, y_2, y_3, y_4, 0, y_5]$$

$$R = \$ [[0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 1], [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1]] \$ = \$ [[0, 0, 1/2, -3/32, -11/32], [0, 1/2, -1/4, -11/32, 5/32], [0, 0, 0, 5/32, -3/32], [0, 0, 0, -3/32, 5/32], [1/2, -1/4, -7/8, 5/32, 17/32], [0, 0, 0, 5/32, -3/32]] \$ \times \$ [[1, 2, 4, 5, 0, 4], [2, 0, 1, 8, 0, 5], [0, 0, 2, 6, 0, 8], [0, 0, 0, 10, 0, 6], [0, 0, 0, 6, 0, 10]] \$$$

Omega Rank for B : cycles: {{1, 4, 5}}, net cycles: 0 . order: 3

$$[y_1, 0, y_2, y_3, y_4, 0]$$

$$B = \$ [[0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[0, 29/112, -3/112, -19/112], [0, -19/112, 29/112, -3/112], [0, -3/112, -19/112, 29/112], [0, -3/112, -19/112, 29/112], [0, -19/112, 29/112, -3/112], [1/2, -19/112, 29/112, -59/112]] \$ \times \$ [[7, 0, 2, 3, 4, 0], [5, 0, 0, 4, 7, 0], [4, 0, 0, 7, 5, 0], [7, 0, 0, 5, 4, 0]] \$$$

Â» SYNC'D 63/512 , 0.1230468750

25 . Coloring, {3, 5, 6}

R: [3, 1, 4, 1, 4, 4] **B:** [5, 4, 1, 6, 2, 3]

‘ See graph

‘ ‘ See pair graph

‘

Ω for A+τΔ :

$$['32, 8(' - 1 + \tau ')^2, 8('3 + \tau^2'), 32, -16(' - 1 + \tau '), -16(' - 1 + \tau ')]'$$

For τ=1/2, [16, 1, 13, 16, 4, 4] . FixedPtCheck, [16, 1, 13, 16, 4, 4]

$$\det(A + \tau \Delta) = 1(' \tau ')(' - 1 + \tau ')^4$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	3 vs 3	6 vs 6

bi =

$\$ [[0, 0, 1/4, 0, 3/4, 0], [1/4, 0, 0, 3/4, 0, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 3/4, 0, 1/4, 0, 0], [0, 0, 3/4, 1/4, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ =$

$\$ [[1160307/3933296, 808857/983324, 170737/737493, 45084/245831, -343968/245831, -52192/737493], [-2002701/3933296, -64949/2949972, 632683/2212479, 193556/737493, -446560/737493, 1439840/2212479], [1231539/3933296, 58601/983324, -154679/737493, -322404/245831, -61216/245831, 1076768/737493], [153171/3933296, -892191/983324, -11189/245831, 152028/245831, 280672/245831, -192672/245831], [-469229/3933296, 1087027/2949972, 2566571/2212479, 526132/737493, 215456/737493, -5204384/2212479], [-1037037/3933296, -272751/983324, -3009877/2212479, -354764/737493, 221024/245831, 3420256/2212479]] \$ \times \$ [[7/2, 3/2, 5/2, 5/2, 3, 3], [23/8, 9/4, 25/8, 13/4, 21/8, 15/8], [119/32, 63/32, 17/8, 115/32, 69/32, 39/16], [191/64, 207/128, 353/128, 101/32, 357/128, 345/128], [835/256, 1071/512, 1417/512, 419/128, 573/256, 303/128], [3499/1024, 1719/1024, 2653/1024, 1747/512, 2505/1024, 1257/512]] \$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4}}, net cycles: 1 . order: 3

$[y_3, 0, y_2, y_1, 0, 0]$

$R = \$ [[0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[-19/112, -3/112, 29/112], [-3/112, 29/112, -19/112], [29/112, -19/112, -3/112], [-3/112, 29/112, -19/112], [29/112, -19/112, -3/112], [29/112, -19/112, -3/112]] \$ \times \$ [[5, 0, 4, 7, 0, 0], [7, 0, 5, 4, 0, 0], [4, 0, 7, 5, 0, 0]] \$$

Omega Rank for B : cycles: {{1, 2, 3, 4, 5, 6}}, net cycles: 1 . order: 6

$[y_1, y_2, y_3, y_4, y_5, y_6]$

$B = \$ [[0, 0, 0, 0, 1, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ = \$ [[3/32, 7/32, -1/32, 3/32, -9/32, -1/32], [-9/32, -1/32, 3/32, 7/32, -1/32, 3/32], [7/32, -1/32, 3/32, -9/32, -1/32, 3/32], [3/32, -9/32, -1/32, 3/32, 7/32, -1/32], [-1/32, 3/32, 7/32, -1/32, 3/32, -9/32], [-1/32, 3/32, -9/32, -1/32, 3/32, 7/32]] \$ \times \$ [[3, 2, 2, 1, 4, 4], [2, 4, 4, 2, 3, 1], [4, 3, 1, 4, 2, 2], [1, 2, 2, 3, 4, 4], [2, 4, 4, 2, 1, 3], [4, 1, 3, 4, 2, 2]] \$$

Â» SYNC'D 4007/16384 , 0.2445678711

26 . Coloring, {4, 5, 6}

R: [3, 1, 1, 6, 4, 4] **B:** [5, 4, 4, 1, 2, 3]

‘ See graph

‘ ‘ See pair graph

Ω for A+τΔ :

$$[4, 1, (-1 + \tau)^2, -1(1 + \tau)(-3 + \tau), 4, -2(-1 + \tau), 2(1 + \tau)]$$

For τ=1/2, [16, 1, 15, 16, 4, 12] . FixedPtCheck, [16, 1, 15, 16, 4, 12]

$$\det(A + \tau \Delta) = 0$$

Delta Range : [-y₅ - y₃ - y₄ - y₂ - y₁, y₅, y₃, y₄, y₂, y₁]

$$[4, 1, 3, 4, 2, 2]$$

$$+ \quad \setminus ; \quad - \quad \setminus ; \quad \Delta$$

$$[[4, 0, 4, 4, 0, 4], [4, 2, 2, 4, 2, 2], [4, 1, 3, 4, 2, 2], [4, 1, 3, 4, 2, 2], [4, 1, 3, 4, 2, 2]] [[4, 2, 2, 4, 4, 0], [4, 0, 4, 4, 2, 2], [4, 1, 3, 4, 2, 2], [4, 1, 3, 4, 2, 2], [4, 1, 3, 4, 2, 2]] [[0, -1, 1, 0, -2, 2], [0, 1, -1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0]]$$

$$[0, -y_1, y_1, 0, -y_2, y_2]$$

$$p' = s^4 \quad p = s^3$$

$$S+ \quad \setminus ; \quad S- \quad \setminus ; \quad NM$$

$$[[3, 0, 1, 0, 0, 0], [1, 1, 2, 0, 0, 0], [1, 1, 2, 0, 0, 0], [0, 0, 0, 3, 0, 1], [0, 0, 0, 1, 2, 1], [0, 0, 0, 1, 2, 1]] [[0, 0, 0, 1, 1, 2], [0, 0, 0, 3, 1, 0], [0, 0, 0, 3, 1, 0], [3, 1, 0, 0, 0, 0], [1, 0, 3, 0, 0, 0], [1, 0, 3, 0, 0, 0]] [[104, 18, 54, 68, 34, 34], [72, 26, 78, 68, 34, 34], [72, 26, 78, 68, 34, 34], [68, 17, 51, 104, 36, 36], [68, 17, 51, 72, 52, 52], [68, 17, 51, 72, 52, 52]]$$

CmmCk true, true, true

$$p' = s^3$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
2 vs 5	2 vs 5	2 vs 5	1 vs 4	2 vs 5

Omega Rank for R : cycles: {{1, 3}, {4, 6}}, net cycles: 2 . order: 2

$$[[4, 0, 4, 4, 0, 4], [4, 0, 4, 4, 0, 4], [4, 0, 4, 4, 0, 4], [4, 0, 4, 4, 0, 4]]$$

$$[y_1, 0, y_1, y_1, 0, y_1]$$

$$p = -s + s^2 \quad p = -s + s^3 \quad p = -s + s^4$$

Omega Rank for B : cycles: {{1, 2, 4, 5}}, net cycles: 0 . order: 4

$$[[4, 2, 2, 4, 4, 0], [4, 4, 0, 4, 4, 0], [4, 4, 0, 4, 4, 0], [4, 4, 0, 4, 4, 0], [4, 4, 0, 4, 4, 0]]$$

$$[y_2, -y_1 + y_2, y_1, y_2, y_2, 0]$$

$$p = s^2 - s^3 \quad p' = -s^2 + s^4 \quad p'' = -s^2 + s^3$$

Â« NOT SYNC'D Â»

Nullspace of $\{\Omega\Delta^i\}$:

$$[0, 0, x_1, x_2, x_3]$$

$$\text{For } A+2\Delta: [y_1, y_2, y_2, y_3, -4y_1 - 4y_2 - 4y_3 - 3y_4, y_4]$$

$$\text{For } A-2\Delta: [y_1, y_2, y_2, y_3, y_4, -4y_1 - 4y_2 - 4y_3 - 3y_4]$$

$$\text{Range of } \{\Omega\Delta^i\}: [0, -\mu_1, \mu_1, 0, -\mu_2, \mu_2]$$

rank of M is 6 , rank of N is 4

M N

$$\begin{aligned} & \$ [[0, 8, 24, 36, 18, 18] , [8, 0, 0, 9, 9, 0] , [24, 0, 0, 27, 9, 18] , [36, 9, 27, 0, 16, 16] , [18, 9, 9, 16, 0, 0] , \\ & [18, 0, 18, 16, 0, 0]] \$ \quad \$ [[0, 1, 1, 1, 1, 1] , [1, 0, 0, 1, 1, 1] , [1, 0, 0, 1, 1, 1] , [1, 1, 1, 0, 1, 1] , [1, 1, 1, \\ & 1, 0, 0] , [1, 1, 1, 1, 0, 0]] \$ \end{aligned}$$

Check is $\Omega\Delta N$ zero? *true*, $\pi\Delta = [0, -1, 1, 0, -2, 2]$

ker M, $[0, 0, 0, 0, 0, 0]$

Range M, $[x_1, x_2, x_3, x_4, x_5, x_6]$

$$\tau = 10, r' = 3/4$$

Ranges

Action of R on ranges, $[[3], [3], [3]]$

Action of B on ranges, $[[1], [1], [2]]$

$$\beta(\{1, 2, 4, 5\}) = 1/4$$

$$\beta(\{1, 3, 4, 5\}) = 1/4$$

$$\beta(\{1, 3, 4, 6\}) = 1/2$$

ker N, $[0, \mu_1, -\mu_1, 0, -\mu_2, \mu_2]$

Range of N

$$[y_4, y_3, y_3, y_1, y_2, y_2]$$

Partitions

$$\alpha(\{\{1\}, \{5, 6\}, \{2, 3\}, \{4\}\}) = 1/1$$

$$b1 = \{1\} \text{ ' , ' } b2 = \{5, 6\} \text{ ' , ' } b3 = \{2, 3\} \text{ ' , ' } b4 = \{4\}$$

Action of R and B on the blocks of the partitions: $\$ [[0, 0, 1, 1], [1, 0, 0, 1], [1, 1, 0, 0], [0, 1, 1, 0]] \$ =$
 $\$ [[0, 0, 1, 0], [0, 0, 0, 1], [1, 0, 0, 0], [0, 1, 0, 0]] \$ + \$ [[0, 0, 0, 1], [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1,$
 $0]] \$$
 $['3', '4', '1', '2', ['4', '1', '2', '3']$ with invariant measure $[1, 1, 1, 1]$

N by blocks, check: true . ' See partition graph.

' ' See level-4 partition graph.

'

Right Group	
Coloring	{4, 5, 6}
Rank	4
R,B	[3, 1, 1, 6, 4, 4], [5, 4, 4, 1, 2, 3]
π_2	[8, 24, 36, 18, 18, 0, 9, 9, 0, 27, 9, 18, 16, 16, 0]
u_2	[1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0] (dim 2)
wpp	[1, 2, 2, 1, 2, 2]
π_4	[0, 0, 0, 1, 0, 0, 1, 2, 0, 0, 0, 0, 0, 0, 0]
u_4	[0, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0]

27 . Coloring, {2, 3, 4, 5}

R: [3, 4, 4, 6, 4, 3] **B:** [5, 1, 1, 1, 2, 4]

' See graph

' ' See pair graph

'

Ω for $A+\tau\Delta$:

$['-16' (' - 1 + \tau ')'' (' 5 + 2\tau + \tau^2 ')', -4' (' - 1 + \tau ')'^3 (' 5 + 2\tau + \tau^2 ')', 4' (' 15 + 2\tau - 2\tau^3$
 $+ \tau^4 ')'' (' 1 + \tau ')', 16' (' 1 + \tau ')'' (' 5 - 2\tau + \tau^2 ')', 8' (' - 1 + \tau ')'^2 (' 5 + 2\tau + \tau^2 ')', 8' (' 1$
 $+ \tau ')'^2 (' 5 - 2\tau + \tau^2 ')'']'$

For $\tau=1/2$, [400, 25, 759, 816, 100, 612] . FixedPtCheck, [400, 25, 759, 816, 100, 612]

$$\det(A + \tau \Delta) = 0$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	5 vs 5	5 vs 5	3 vs 3	4 vs 4

bi =

$$\begin{aligned} & \$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [3/4, 0, 0, 1/4, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 3/4, 0, 1/4, 0, 0], \\ & [0, 0, 1/4, 3/4, 0, 0]] \$ \times \$ [[109/110, -1/110, -9/110, 3/110, 3/110, 3/110], [-1/110, 109/110, -9/110, 3/110, 3/110, 3/110], \\ & [-9/110, -9/110, 29/110, 27/110, 27/110, 27/110], [3/110, 3/110, 27/110, 101/110, -9/110, -9/110], \\ & [-9/110, -9/110], [3/110, 3/110, 27/110, -9/110, 101/110, -9/110], [3/110, 3/110, 27/110, -9/110, -9/110, 101/110]] \$ = \end{aligned}$$

$$\begin{aligned} & \$ [[155/12208, -541/2289, -299/763, -344/763, 2584/2289], [2907/12208, -783/763, -311/763, -72/763, 1032/763], \\ & [-5109/12208, 1558/763, 345/763, 1336/763, -2872/763], [-323/1744, 152/327, 83/109, 8/109, -344/327], [12459/12208, -1544/763, -51/763, -1896/763, 2760/763]] \$ \times \$ [[6, 3/2, 3/2, 3, 3, 1], [9/2, 9/4, 7/4, 9/4, 9/2, 3/4], \\ & [75/16, 27/8, 21/16, 43/16, 27/8, 9/16], [177/32, 81/32, 21/16, 39/16, 225/64, 43/64], [603/128, 675/256, 397/256, 75/32, 531/128, 39/64]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 4, 6}}, net cycles: 1 . order: 3

$$[0, 0, y_1, y_2, 0, y_3]$$

$$\begin{aligned} R = & \$ [[0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0], \\ & [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1]] \$ = \\ & \$ [[3/16, -5/16, 3/16], [3/16, 3/16, -5/16], [3/16, 3/16, -5/16], [-5/16, 3/16, 3/16], [3/16, 3/16, -5/16], [3/16, -5/16, 3/16]] \$ \times \$ [[0, 0, 6, 6, 0, 4], [0, 0, 4, 6, 0, 6], [0, 0, 6, 4, 0, 6]] \$ \end{aligned}$$

Omega Rank for B : cycles: {{1, 2, 5}}, net cycles: 0 . order: 3

$$[y_1, y_2, 0, y_3, y_4, 0]$$

$$\begin{aligned} B = & \$ [[0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], \\ & [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0]] \$ = \\ & \$ [[0, 3/16, -1/16, -1/16], [0, -1/16, -1/16, 3/16], [0, -1/16, -1/16, 3/16], [0, -1/16, -1/16, 3/16], [0, -1/16, 3/16, -1/16], \\ & [0, -1/16, 3/16, -1/16], [1/2, -1/16, 3/16, -9/16]] \$ \times \$ [[8, 2, 0, 2, 4, 0], [4, 4, 0, 0, 8, 0], [4, 8, 0, 0, 4, 0], [8, 4, 0, 0, 4, 0]] \$ \end{aligned}$$

Â» SYNC'D 3/8 , 0.3750000000

28 . Coloring, {2, 3, 4, 6}

R: [3, 4, 4, 6, 2, 4] **B:** [5, 1, 1, 1, 4, 3]

‘ See graph

‘ ‘ See pair graph

‘

Ω for $A+\tau\Delta$:

$$\begin{aligned} & ['-16' ('-5 + \tau^2')'' ('-1 + \tau')', 4' ('1 + \tau')'' ('-5 + \tau^2')'' ('-1 + \tau')'^2, -4' ('1 + \tau')'' \\ & ('5 + 2\tau + \tau^2')'' ('-1 + \tau')'' ('-3 + \tau')', 16' ('-5 - \tau - 3\tau^2 + \tau^3')', 8' ('-5 + \tau^2')'' ('-1 + \\ & \tau')'^2, 8' ('1 + \tau')'' ('-5 - \tau - 3\tau^2 + \tau^3')'']' \end{aligned}$$

For $\tau=1/2$, [-304, -57, -375, -784, -76, -588] . FixedPtCheck, [304, 57, 375, 784, 76, 588]

$$\det(A + \tau \Delta) = 0$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	5 vs 5	5 vs 5	3 vs 4	4 vs 4

bi =

$$\begin{aligned} \$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [3/4, 0, 0, 1/4, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 1/4, 0, 3/4, 0, \\ 0], [0, 0, 3/4, 1/4, 0, 0]] \$ \times \$ [[901/910, -81/910, -9/910, 27/910, 3/910, 27/910], [-81/910, 181/910, \\ -81/910, 243/910, 27/910, 243/910], [-9/910, -81/910, 901/910, 27/910, 3/910, 27/910], [27/910, \\ 243/910, 27/910, 829/910, -9/910, -81/910], [3/910, 27/910, 3/910, -9/910, 909/910, -9/910], [27/910, \\ 243/910, 27/910, -81/910, -9/910, 829/910]] \$ = \end{aligned}$$

$$\begin{aligned} \$ [[-1055/7376, 1993/5532, -358/1383, -40/1383, 184/1383], [1665/7376, 1451/1844, 766/1383, \\ 56/461, -2248/1383], [1665/7376, 1451/1844, 766/1383, 56/461, -2248/1383], [-207/7376, -609/1844, \\ -158/461, 312/461, 40/461], [-463/7376, -2483/5532, 190/461, -472/1383, 232/461], [3345/7376, \\ -2185/1844, -146/461, -552/461, 1064/461]] \$ \times \$ [[6, 1/2, 5/2, 3, 3, 1], [9/2, 3/4, 9/4, 13/4, 9/2, 3/4], \\ [75/16, 9/8, 27/16, 69/16, 27/8, 13/16], [171/32, 27/32, 57/32, 55/16, 225/64, 69/64], [291/64, 225/256, \\ 549/256, 57/16, 513/128, 55/64]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{4, 6}}, net cycles: -1 . order: 2

$$\$ [[0, 2, 4, 6, 0, 4], [0, 0, 0, 10, 0, 6], [0, 0, 0, 6, 0, 10], [0, 0, 0, 10, 0, 6]] \$$$

$$[0, y_1, 2y_1, y_3, 0, y_2]$$

$$p = s^2 - s^4$$

Omega Rank for B : cycles: {{1, 4, 5}}, net cycles: 0 . order: 3

$$[y_1, 0, y_2, y_3, y_4, 0]$$

$$B = \$ [[0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[0, 3/16, -1/16, -1/16], [0, -1/16, -1/16, 3/16], [0, -1/16, -1/16, 3/16], [0, -1/16, -1/16, 3/16], [0, -1/16, 3/16, -1/16], [1/2, -1/16, 3/16, -9/16]] \$ \times \$ [[8, 0, 2, 2, 4, 0], [4, 0, 0, 4, 8, 0], [4, 0, 0, 8, 4, 0], [8, 0, 0, 4, 4, 0]] \$$$

Â» SYNC'D 9/32 , 0.2812500000

29 . Coloring, {2, 3, 5, 6}

R: [3, 4, 4, 1, 4, 4] **B:** [5, 1, 1, 6, 2, 3]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$['-4' (' - 5 - \tau - 3\tau^2 + \tau^3 ')', -1' (' - 5 - \tau - 3\tau^2 + \tau^3 ')'' (' - 1 + \tau ')'^2 , 1' (' 1 + \tau ')'' (' 15 - 10\tau + 16\tau^2 - 6\tau^3 + \tau^4 ')', 4' (' 1 + \tau ')'' (' 5 - 2\tau + \tau^2 ')', 2' (' - 5 - \tau - 3\tau^2 + \tau^3 ')'' (' - 1 + \tau ')', -2' (' 1 + \tau ')'' (' - 1 + \tau ')'' (' 5 - 2\tau + \tau^2 ')'']'$$

For τ=1/2, [784, 49, 639, 816, 196, 204] . FixedPtCheck, [784, 49, 639, 816, 196, 204]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	5 vs 5	5 vs 5	3 vs 3	5 vs 5

bi =

$$\$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 3/4, 0, 1/4, 0, 0], [0, 0, 3/4, 1/4, 0, 0]] \$ \times \$ [[101/110, -9/110, -9/110, 27/110, 3/110, 3/110], [-9/110, 101/110, -9/110, 27/110, 3/110, 3/110], [-9/110, -9/110, 101/110, 27/110, 3/110, 3/110], [27/110, 27/110, 27/110, 29/110, -9/110, -9/110], [3/110, 3/110, 3/110, -9/110, 109/110, -1/110], [3/110, 3/110, 3/110, -9/110, -1/110, 109/110]] \$ = \$ [[-3/16, -1/6, -1/4, 4/5, -2/15], [-3/16, 1/6, 3/4, 0, -2/3], [-3/16, 1/6, 3/4, 0, -2/3], [9/16, -1/4, -1/4, -2/5, 2/5], [1/16, -11/36, -7/12, -2/5, 58/45], [1/16, 29/36, 1/12, -2/5, -22/45]] \$ \times \$ [[4, 3/2, 5/2, 2, 3, 3], [7/2, 9/4, 13/4, 5/2, 3, 3/2], [19/4, 9/4, 2, 5/2, 21/8, 15/8], [61/16, 63/32, 83/32, 35/16, 57/16, 15/8], [127/32, 171/64, 151/64, 5/2, 183/64, 105/64]] \$$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3, 4}}, net cycles: 1 . order: 3

$$[y_1, 0, y_2, y_3, 0, 0]$$

$$R = \$ [[0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[-1/16, -1/16, 3/16], [3/16, -1/16, -1/16], [3/16, -1/16, -1/16], [-1/16, 3/16, -1/16], [3/16, -1/16, -1/16], [3/16, -1/16, -1/16]] \$ \times \$ [[4, 0, 4, 8, 0, 0], [8, 0, 4, 4, 0, 0], [4, 0, 8, 4, 0, 0]] \$$$

Omega Rank for B : cycles: {{1, 2, 5}}, net cycles: 0 . order: 3

$$[y_1, y_2, y_3, 0, y_4, y_5]$$

$$B = \$ [[0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ = \$ [[0, 0, -1/16, 3/16, -1/16], [0, 0, 3/16, -1/16, -1/16], [0, 0, 3/16, -1/16, -1/16], [1/4, -1/8, -1/16, -1/16, 1/16], [0, 0, -1/16, -1/16, 3/16], [0, 1/4, -1/16, -1/16, -1/16]] \$ \times \$ [[4, 2, 2, 0, 4, 4], [4, 4, 4, 0, 4, 0], [8, 4, 0, 0, 4, 0], [4, 4, 0, 0, 8, 0], [4, 8, 0, 0, 4, 0]] \$$$

Â» SYNC'D 3/8 , 0.3750000000

30 . Coloring, {2, 4, 5, 6}

R: [3, 4, 1, 6, 4, 4] **B:** [5, 1, 4, 1, 2, 3]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$['32' ('5 + 2\tau + \tau^2'), '8' ('-1 + \tau')^2 ('5 + 2\tau + \tau^2'), '8' ('1 + \tau') ('15 + 3\tau - 3\tau^2 + \tau^3'), '-32' ('1 + \tau') ('-5 + \tau'), '-16' ('-1 + \tau') ('5 + 2\tau + \tau^2'), '-16' ('1 + \tau')^2 ('-5 + \tau')]$$

For τ=1/2, [400, 25, 381, 432, 100, 324] . FixedPtCheck, [400, 25, 381, 432, 100, 324]

$$\det(A + \tau \Delta) = 1' ('-1 + \tau')^3 ('\tau') ('1 + \tau')$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	6 vs 6	6 vs 6	2 vs 4	5 vs 5

bi =

$\$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [1/4, 0, 0, 3/4, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 3/4, 0, 1/4, 0, 0], [0, 0, 3/4, 1/4, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ =$

$\$ [[-5823/25040, -9/1252, -601/4695, 14116/4695, 16096/4695, -28192/4695], [5121/25040, 821/3756, -10753/4695, 1828/4695, -24992/4695, 32224/4695], [11841/25040, 2263/1252, -25553/4695, -21532/4695, -18592/4695, 55264/4695], [-18783/25040, 447/1252, 6333/1565, 23396/4695, -8288/1565, -15392/4695], [1325/5008, -2899/3756, 931/939, -5068/939, 4064/939, 608/939], [6957/5008, -3441/1252, 451/939, -1220/313, 7520/939, -992/313]] \$ \times \$ [[9/2, 3/2, 5/2, 7/2, 3, 1], [35/8, 9/4, 15/8, 13/4, 27/8, 7/8], [147/32, 81/32, 7/4, 97/32, 105/32, 13/16], [295/64, 315/128, 225/128, 95/32, 441/128, 97/128], [1155/256, 1323/512, 881/512, 191/64, 885/256, 95/128], [4717/1024, 2655/1024, 1725/1024, 1529/512, 3465/1024, 191/256]] \$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 3}, {4, 6}}, net cycles: 2 . order: 2

$\$ [[3, 0, 4, 5, 0, 4], [4, 0, 3, 4, 0, 5], [3, 0, 4, 5, 0, 4], [4, 0, 3, 4, 0, 5]] \$$

$[-8 y_1 + 7 y_2, 0, y_1, y_2, 0, -9 y_1 + 8 y_2]$

$$p = s - s^3 \quad p' = -s + s^3$$

Omega Rank for B : cycles: {{1, 2, 5}}, net cycles: 0 . order: 3

$[y_1, y_4, y_5, y_2, y_3, 0]$

$B = \$ [[0, 0, 0, 0, 1, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0]] \$ = \$ [[0, 0, -5/16, 11/16, -5/16], [0, 0, 11/16, -5/16, -5/16], [0, 1/2, -5/16, -5/16, 3/16], [0, 0, 11/16, -5/16, -5/16], [0, 0, -5/16, -5/16, 11/16], [1/2, -3/4, -5/16, 3/16, 7/16]] \$ \times \$ [[5, 2, 2, 3, 4, 0], [5, 4, 0, 2, 5, 0], [6, 5, 0, 0, 5, 0], [5, 5, 0, 0, 6, 0], [5, 6, 0, 0, 5, 0]] \$$

Â» SYNC'D 1/64 , 0.01562500000

31 . Coloring, {3, 4, 5, 6}

R: [3, 1, 4, 6, 4, 4] **B:** [5, 4, 1, 1, 2, 3]

‘ See graph

‘ ‘ See pair graph

‘

Ω for $A+\tau\Delta$:

$$\begin{aligned} & ['32' ('-1+\tau')^{''} ('-5+\tau^2')^{''} , 8' ('-1+\tau')^{''3} ('-5+\tau^2')^{''} , 8' ('-1+\tau')^{''} ('1+\tau' \\ &)^{''} ('5+2\tau+\tau^2')^{''} ('-3+\tau')^{''} , -32' ('-5-\tau-3\tau^2+\tau^3')^{''} , -16' ('-1+\tau')^{''2} ('-5+\tau^2')^{''} \\ & , -16' ('1+\tau')^{''} ('-5-\tau-3\tau^2+\tau^3')^{''}]^{''} \end{aligned}$$

For $\tau=1/2$, [304, 19, 375, 784, 76, 588] . FixedPtCheck, [304, 19, 375, 784, 76, 588]

$$\det(A + \tau \Delta) = 1' ('-1+\tau')^{''3} ('\tau')^{''} ('1+\tau')^{''}$$

Δ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	R	B
5 vs 5	6 vs 6	6 vs 6	3 vs 4	5 vs 5

bi =

$$\begin{aligned} & \$ [[0, 0, 1/4, 0, 3/4, 0] , [1/4, 0, 0, 3/4, 0, 0] , [3/4, 0, 0, 1/4, 0, 0] , [3/4, 0, 0, 0, 0, 1/4] , [0, 3/4, 0, 1/4, 0, \\ & 0] , [0, 0, 3/4, 1/4, 0, 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, \\ & 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1]] \$ = \end{aligned}$$

$$\begin{aligned} & \$ [[-391747/1864080, 121169/279612, -16252/38835, -27896/209709, 93856/349515, \\ & 128224/1048545] , [261047/5592240, -78733/838836, -22348/116505, 648088/629127, \\ & -1348256/1048545, 1752736/3145635] , [654589/1864080, 139585/279612, 32764/38835, \\ & 248840/209709, -121312/349515, -2591008/1048545] , [67871/621360, -6445/93204, 3116/12945, \\ & 4696/69903, 171232/116505, -613472/349515] , [-121301/1118448, -372229/838836, 8068/23301, \\ & -561416/629127, -4384/209709, 743456/629127] , [32339/124272, -91949/93204, -3004/2589, \\ & -88840/69903, -53408/23301, 385312/69903]] \$ \times \$ [[11/2, 3/2, 5/2, 5/2, 3, 1] , [33/8, 9/4, 17/8, 11/4, \\ & 33/8, 5/8] , [135/32, 99/32, 3/2, 109/32, 99/32, 11/16] , [285/64, 297/128, 201/128, 233/64, 405/128, \\ & 109/128] , [1149/256, 1215/512, 897/512, 803/256, 855/256, 233/256] , [2181/512, 2565/1024, 231/128, \\ & 3359/1024, 3447/1024, 803/1024]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{4, 6}}, net cycles: 0 . order: 4

$$\$ [[1, 0, 4, 7, 0, 4] , [0, 0, 1, 8, 0, 7] , [0, 0, 0, 8, 0, 8] , [0, 0, 0, 8, 0, 8]] \$$$

$$[y_1, 0, y_2, y_3, 0, y_1 - y_2 + y_3]$$

$$p = -s^3 + s^4$$

Omega Rank for B : cycles: {{1, 2, 4, 5}}, net cycles: 0 . order: 4

$$[y_2, y_1, y_3, y_5, y_4, 0]$$

$$\begin{aligned} & B = \$ [[0, 0, 0, 0, 1, 0] , [0, 0, 0, 1, 0, 0] , [1, 0, 0, 0, 0, 0] , [1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, \\ & 0]] \$ \times \$ [[1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, \\ & 0, 0, 0]] \$ = \$ [[0, 197/832, -107/832, 37/832, -75/832] , [0, 37/832, -75/832, 197/832, -107/832] , [0, \\ & -107/832, 37/832, -75/832, 197/832] , [0, -107/832, 37/832, -75/832, 197/832] , [0, -75/832, 197/832, \\ & -107/832, 37/832] , [1/2, 37/832, -75/832, 197/832, -523/832]] \$ \times \$ [[7, 2, 2, 1, 4, 0] , [3, 4, 0, 2, 7, 0] , \end{aligned}$$

$$[2, 7, 0, 4, 3, 0], [4, 3, 0, 7, 2, 0], [7, 2, 0, 3, 4, 0]] \$$$

Â» SYNC'D 39/256 , 0.1523437500

32 . Coloring, {2, 3, 4, 5, 6}

R: [3, 4, 4, 6, 4, 4] **B:** [5, 1, 1, 1, 2, 3]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$\begin{aligned} & \text{' ['8' (' - 5 + \tau^2 \text{'})'' (' - 1 + \tau \text{'})', 2' (' - 5 + \tau^2 \text{'})'' (' - 1 + \tau \text{'})^3 , -2' (' 1 + \tau \text{'})'' (' 15 + 3\tau - } \\ & 3\tau^2 + \tau^3 \text{'})'' (' - 1 + \tau \text{'})', 8' (' 1 + \tau \text{'})'' (' 5 - 2\tau + \tau^2 \text{'})', -4' (' - 5 + \tau^2 \text{'})'' (' - 1 + \tau \text{'})^2 , 4' (' } \\ & 1 + \tau \text{'})^2 \text{' (' 5 - 2\tau + \tau^2 \text{'})'']' } \end{aligned}$$

For τ=1/2, [304, 19, 381, 816, 76, 612] . FixedPtCheck, [304, 19, 381, 816, 76, 612]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
5 vs 5	4 vs 5	5 vs 5	2 vs 3	4 vs 4

bi =

$$\begin{aligned} \$ [[0, 0, 1/4, 0, 3/4, 0], [3/4, 0, 0, 1/4, 0, 0], [3/4, 0, 0, 1/4, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 3/4, 0, 1/4, 0, \\ 0], [0, 0, 3/4, 1/4, 0, 0]] \$ x \$ [[181/190, -9/190, -9/190, 27/190, 3/190, 27/190], [-9/190, 181/190, \\ -9/190, 27/190, 3/190, 27/190], [-9/190, -9/190, 181/190, 27/190, 3/190, 27/190], [27/190, 27/190, \\ 27/190, 109/190, -9/190, -81/190], [3/190, 3/190, 3/190, -9/190, 189/190, -9/190], [27/190, 27/190, \\ 27/190, -81/190, -9/190, 109/190]] \$ = \end{aligned}$$

$$\begin{aligned} \$ [[-3/32, 17/160, -5/12, -2/15, 3/5], [-3/32, 81/160, 1/4, 14/15, -23/15], [-3/32, 81/160, 1/4, 14/15, \\ -23/15], [9/32, -203/160, -3/4, -2/15, 29/15], [1/32, -7/32, 7/12, -2/3, 1/3], [9/32, 49/32, 5/4, -2/3, -7/3]] \\ \$ x \$ [[6, 3/2, 5/2, 2, 3, 1], [9/2, 9/4, 9/4, 2, 9/2, 1/2], [39/8, 27/8, 3/2, 19/8, 27/8, 1/2], [87/16, 81/32, \\ 51/32, 35/16, 117/32, 19/32], [303/64, 351/128, 231/128, 67/32, 261/64, 35/64]] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{4, 6}}, net cycles: 0 . order: 2

$$\$ [[0, 0, 4, 8, 0, 4], [0, 0, 0, 8, 0, 8], [0, 0, 0, 8, 0, 8]] \$$$

$$[0, 0, y_1 - y_2, y_1, 0, y_2]$$

$$p = s^2 - s^3$$

Omega Rank for B : cycles: {{1, 2, 5}}, net cycles: 0 . order: 3

$$[y_1, y_2, y_3, 0, y_4, 0]$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3/16 & -1/16 & -1/16 \\ 0 & -1/16 & -1/16 & 3/16 \\ 0 & -1/16 & -1/16 & 3/16 \\ 0 & -1/16 & -1/16 & 3/16 \\ 0 & -1/16 & 3/16 & -1/16 \\ 1/2 & -1/16 & 3/16 & -9/16 \end{bmatrix} \times \begin{bmatrix} 8 & 2 & 2 & 0 & 4 & 0 \\ 4 & 4 & 0 & 0 & 8 & 0 \\ 4 & 8 & 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 & 4 & 0 \\ 8 & 4 & 0 & 0 & 4 & 0 \end{bmatrix}$$

Â» SYNC'D 1/4 , 0.2500000000

SUMMARY	
Graph Type	CC
v(A)	1
v(Δ)	1
π	[4, 1, 3, 4, 2, 2]
Dbly Stoch	false

SANDWICH		Total 1
No .	Coloring	Rank
1	{}	2

RT GROUPS		Total 1	
No .	Coloring	Rank	Solv
1	{4, 5, 6}	4	Not Solvable

CC Colorings		Total 1
No .	Coloring	Sandwich,Rank
1	{}	true, 2

Δ-RANK'D	SC'D !RK'D	τ-RANK'D	R/B RANK'D	NOT SYNC'D	Total Runs	2^{n-1}
30	0	29 , 29	19 , 24	2	32	32
