

## New Graph

[2, 4, 4, 2, 6, 5], [3, 6, 5, 3, 1, 4]

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$$\pi = [1, 2, 2, 3, 2, 2]$$

POSSIBLE RANKS

1 x 12

2 x 6

3 x 4

BASE DETERMINANT 231/2048, .1127929688

*NullSpace* of  $\Delta$

{2, 3}, {1, 4, 5, 6}

*Range* of  $\Delta$ :  $[\lambda_4, -\lambda_1, \lambda_1, -\lambda_4 - \lambda_2 - \lambda_3, \lambda_2, \lambda_3]$

1 . Coloring, {}

**R**: [2, 4, 4, 2, 6, 5]   **B**: [3, 6, 5, 3, 1, 4]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A + \tau \Delta$  :

[ ‘ -1‘ (‘ - 1 +  $\tau$  ‘)‘ (‘ 3 +  $\tau^2$  ‘)‘ , 2‘ (‘ 3 +  $\tau$  ‘)‘ (‘ 1 +  $\tau$  ‘)‘ , -2‘ (‘ - 1 +  $\tau$  ‘)‘ (‘ 3 +  $\tau$  ‘)‘ , 1‘ (‘ 9 - 2 $\tau$  +  $\tau^2$  ‘)‘ (‘ 1 +  $\tau$  ‘)‘ , 2‘ (‘ 3 +  $\tau^2$  ‘)‘ , -2‘ (‘ 1 +  $\tau$  ‘)‘ (‘ - 3 +  $\tau$  ‘)‘ ]‘

For  $\tau=1/2$ , [13, 84, 28, 99, 52, 60] . FixedPtCheck, [13, 84, 28, 99, 52, 60]

$\det(A + \tau \Delta) = 0$

Delta Range :  $[-y_2 - y_3 - y_4, -y_1, y_1, y_2, y_3, y_4]$

[1, 2, 2, 3, 2, 2]

$$+ \quad \backslash; \quad - \quad \backslash; \quad \Delta$$

$$\begin{aligned} & \$ [ [0, 4, 0, 4, 2, 2], [1, 2, 2, 3, 3, 1], [1, 4, 4, 7, 3, 5], [5, 8, 8, 11, 9, 7] ] \$ \quad \$ [ [2, 0, 4, 2, 2, 2], [1, 2, 2, \\ & 3, 1, 3], [3, 4, 4, 5, 5, 3], [3, 8, 8, 13, 7, 9] ] \$ \quad \$ [ [-1, 2, -2, 1, 0, 0], [0, 0, 0, 0, 1, -1], [-1, 0, 0, 1, -1, 1], \\ & [1, 0, 0, -1, 1, -1] ] \$ \end{aligned}$$

$$[-y_2, -y_1, y_1, y_2, -y_3, y_3]$$

$$p = s^3 + 2s^4$$

$$S+ \quad \backslash; \quad S- \quad \backslash; \quad NM$$

$$\begin{aligned} & \$ [ [1, 2, 2, 3, 2, 2], [1, 3, 1, 4, 2, 1], [1, 3, 1, 4, 2, 1], [1, 2, 2, 3, 2, 2], [1, 1, 3, 2, 2, 3], [1, 1, 3, 2, 2, 3] ] \\ & \$ \quad \$ [ [1, 2, 2, 3, 2, 2], [1, 1, 3, 2, 2, 3], [1, 1, 3, 2, 2, 3], [1, 2, 2, 3, 2, 2], [1, 3, 1, 4, 2, 1], [1, 3, 1, 4, 2, \\ & 1] ] \$ \quad \$ [ [4, 4, 4, 12, 4, 4], [2, 8, 8, 6, 4, 4], [2, 8, 8, 6, 4, 4], [4, 4, 4, 12, 4, 4], [2, 4, 4, 6, 8, 8], [2, 4, \\ & 4, 6, 8, 8] ] \$ \end{aligned}$$

CmmCk true, true, true

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
3 vs 4	3 vs 5	3 vs 5	1 vs 4	3 vs 5

Omega Rank for R : cycles:  $\{\{5, 6\}, \{2, 4\}\}$ , net cycles: 2 . order: 2

$$\begin{aligned} & \$ [ [0, 4, 0, 4, 2, 2], [0, 4, 0, 4, 2, 2], [0, 4, 0, 4, 2, 2], [0, 4, 0, 4, 2, 2] ] \$ \\ & [0, 2y_1, 0, 2y_1, y_1, y_1] \end{aligned}$$

$$p = -s + s^2 \quad p = -s + s^4 \quad p = -s + s^3$$

Omega Rank for B : cycles:  $\{\{1, 3, 5\}\}$ , net cycles: 0 . order: 3

$$\begin{aligned} & \$ [ [2, 0, 4, 2, 2, 2], [2, 0, 4, 2, 4, 0], [4, 0, 4, 0, 4, 0], [4, 0, 4, 0, 4, 0], [4, 0, 4, 0, 4, 0] ] \$ \\ & [y_2 + y_3 - y_1, 0, y_2 + y_3, y_1, y_2, y_3] \end{aligned}$$

$$p = -s^3 + s^4 \quad p = -s^3 + s^5$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[0, 0, x_1, 2x_1]$$

$$\text{For } A+2\Delta: [y_2, y_1, -3y_1 - 4y_2 - 4y_3, y_2, y_3, y_3]$$

$$\text{For } A-2\Delta: [y_3, -3y_1 - 4y_2 - 4y_3, y_1, y_3, y_2, y_2]$$

Range of  $\{\Omega\Delta^i\}$ :  $[-\mu_1, \mu_3, -\mu_3, \mu_1, -\mu_2, \mu_2]$

rank of M is 5 , rank of N is 3

M            N

\$ [ [0, 0, 2, 0, 1, 1] , [0, 0, 0, 4, 2, 2] , [2, 0, 0, 2, 2, 2] , [0, 4, 2, 0, 3, 3] , [1, 2, 2, 3, 0, 0] , [1, 2, 2, 3, 0, 0] ]  
 \$ \$ [ [0, 1, 1, 0, 1, 1] , [1, 0, 0, 1, 1, 1] , [1, 0, 0, 1, 1, 1] , [0, 1, 1, 0, 1, 1] , [1, 1, 1, 1, 0, 0] , [1, 1, 1, 1, 0, 0] ] \$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [-1, 2, -2, 1, 0, 0]$

ker M,  $[0, 0, 0, 0, -\lambda_1, \lambda_1]$

Range M,  $[x_1, x_2, x_3, x_4, x_5, x_5]$

$\tau = 12$  ,  $r' = 2/3$

Ranges

Action of R on ranges,  $[[4], [3], [4], [3], [4], [3]]$

Action of B on ranges,  $[[1], [5], [2], [6], [1], [5]]$

$\beta(\{1, 3, 5\}) = 1/8$

$\beta(\{1, 3, 6\}) = 1/8$

$\beta(\{2, 4, 5\}) = 1/4$

$\beta(\{2, 4, 6\}) = 1/4$

$\beta(\{3, 4, 5\}) = 1/8$

$\beta(\{3, 4, 6\}) = 1/8$

ker N,  $[-\mu_2, -\mu_1, \mu_1, \mu_2, -\mu_3, \mu_3]$

Range of N

$[y_3, y_1, y_1, y_3, y_2, y_2]$

Partitions

$\alpha(\{\{5, 6\}, \{1, 4\}, \{2, 3\}\}) = 1/1$

$b_1 = \{5, 6\}$  , ,  $b_2 = \{1, 4\}$  , ,  $b_3 = \{2, 3\}$

Action of R and B on the blocks of the partitions: \$ [ [1, 0, 1] , [1, 0, 1] , [0, 2, 0] ] \$ = \$ [ [1, 0, 0] , [0, 0, 1] , [0, 1, 0] ] \$ + \$ [ [0, 0, 1] , [1, 0, 0] , [0, 1, 0] ] \$  
 $['1'$  ,  $'3'$  ,  $'2'$  ,  $['3'$  ,  $'1'$  ,  $'2'$  with invariant measure  $[1, 1, 1]$

N by blocks, check: true . ' See partition graph.

' ' See level-3 partition graph.

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Right Group	
<b>Coloring</b>	{ }
<b>Rank</b>	3
<b>R,B</b>	[2, 4, 4, 2, 6, 5], [3, 6, 5, 3, 1, 4]
$\pi_2$	[0, 2, 0, 1, 1, 0, 4, 2, 2, 2, 2, 2, 3, 3, 0]
$u_2$	[1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0] (dim 1)
<b>wpp</b>	[2, 2, 2, 2, 2, 2]
$\pi_3$	[0, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 2, 2, 0, 1, 1, 0, 0]
$u_3$	[0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0]

2. Coloring, {2}

**R:** [2, 6, 4, 2, 6, 5] **B:** [3, 4, 5, 3, 1, 4]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

‘ [ ‘ -3‘ (‘ 3 -  $\tau$  + 5 $\tau^2$  +  $\tau^3$  ‘)‘ (‘ - 1 +  $\tau$  ‘)‘ , -6‘ (‘ - 1 +  $\tau$  ‘)‘ (‘ 3 +  $\tau$  ‘)‘ (‘ 1 +  $\tau$  ‘)‘ , 6‘ (‘ - 1 +  $\tau$  ‘)‘  
 )‘<sup>2</sup> (‘ 3 +  $\tau$  ‘)‘ , 3‘ (‘ - 1 +  $\tau$  ‘)‘ (‘ 1 +  $\tau$  ‘)‘ (‘ - 9 + 4 $\tau$  +  $\tau^2$  ‘)‘ , 6‘ (‘ 3 -  $\tau$  + 5 $\tau^2$  +  $\tau^3$  ‘)‘ , 6‘ (‘ 3  
 +  $\tau^2$  ‘)‘ (‘ 1 +  $\tau$  ‘)‘ ‘ ]‘

For  $\tau=1/2$ , [31, 84, 28, 81, 124, 156] . FixedPtCheck, [31, 84, 28, 81, 124, 156]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	4 vs 5	5 vs 5	3 vs 4	4 vs 4

bi =

\$ [ [0, 1/4, 3/4, 0, 0, 0] , [0, 0, 0, 3/4, 0, 1/4] , [0, 0, 0, 1/4, 3/4, 0] , [0, 1/4, 3/4, 0, 0, 0] , [3/4, 0, 0, 0, 0, 1/4] , [0, 0, 0, 3/4, 1/4, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0] , [0, 1/10, 3/10, 0, 0, 0] , [0, 3/10, 9/10, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1] ] \$ =  
 \$ [ [0, 11/8, 3/8, 1/3, -2] , [3/4, 27/8, 29/8, 1, -26/3] , [-1/4, -11/8, -5/8, -1, 10/3] , [0, 11/8, 3/8, 1/3,

$-2]$ ,  $[0, -3/8, 1/8, 1, -2/3]$ ,  $[0, -35/8, -31/8, -5/3, 10]$  ] \$ x \$ [  $[3/2, 1, 3, 7/2, 2, 1]$ ,  $[3/2, 5/4, 15/4, 9/4, 5/2, 3/4]$ ,  $[15/8, 15/16, 45/16, 39/16, 3, 15/16]$ ,  $[9/4, 69/64, 207/64, 135/64, 75/32, 63/64]$ ,  $[225/128, 279/256, 837/256, 603/256, 171/64, 219/256]$  ] \$

Check x AllOnes:  $[1, 1, 1, 1, 1, 1]$

Omega Rank for R : cycles:  $\{\{5, 6\}\}$ , net cycles: 0 . order: 4

\$ [  $[0, 4, 0, 2, 2, 4]$ ,  $[0, 2, 0, 0, 4, 6]$ ,  $[0, 0, 0, 0, 6, 6]$ ,  $[0, 0, 0, 0, 6, 6]$  ] \$

$[0, y_2, 0, y_3, y_1, y_2 - y_3 + y_1]$

$$p = -s^3 + s^4$$

Omega Rank for B : cycles:  $\{\{1, 3, 5\}\}$ , net cycles: 0 . order: 3

$[y_1, 0, y_4, y_2, y_3, 0]$

$B = \$ [ [0, 0, 1, 0, 0, 0]$ ,  $[0, 0, 0, 1, 0, 0]$ ,  $[0, 0, 0, 0, 1, 0]$ ,  $[0, 0, 1, 0, 0, 0]$ ,  $[1, 0, 0, 0, 0, 0]$ ,  $[0, 0, 0, 1, 0, 0]$  ] \$ x \$ [  $[1, 0, 0, 0, 0, 0]$ ,  $[0, 0, 0, 0, 0, 0]$ ,  $[0, 0, 1, 0, 0, 0]$ ,  $[0, 0, 0, 1, 0, 0]$ ,  $[0, 0, 0, 0, 1, 0]$ ,  $[0, 0, 0, 0, 0, 0]$  ] \$ = \$ [  $[0, 7/36, -5/36, 1/36]$ ,  $[1/4, -5/36, 1/36, -1/18]$ ,  $[0, 1/36, 7/36, -5/36]$ ,  $[0, 7/36, -5/36, 1/36]$ ,  $[0, -5/36, 1/36, 7/36]$ ,  $[1/4, -5/36, 1/36, -1/18]$  ] \$ x \$ [  $[2, 0, 4, 4, 2, 0]$ ,  $[2, 0, 6, 0, 4, 0]$ ,  $[4, 0, 2, 0, 6, 0]$ ,  $[6, 0, 4, 0, 2, 0]$  ] \$

Â» SYNC'D 1/4 , 0.2500000000

3 . Coloring, {3}

**R:**  $[2, 4, 5, 2, 6, 5]$  **B:**  $[3, 6, 4, 3, 1, 4]$

' See graph

' ' See pair graph

'

$\Omega$  for  $A + \tau \Delta$  :

' [ '-3' (' 1 + \tau ')'' (' - 1 + \tau ')', 6' (' 1 + \tau ')', -6' (' - 1 + \tau ')', 3' (' 3 + \tau ^ 2 ')', 6' (' 1 + \tau ')', 6' (' 1 + \tau ')'' ]'

For  $\tau=1/2$ ,  $[3, 12, 4, 13, 12, 12]$  . FixedPtCheck,  $[3, 12, 4, 13, 12, 12]$

$\det(A + \tau \Delta) = 0$

Delta Range :  $[-y_2 - y_3 - y_4, -y_1, y_1, y_2, y_3, y_4]$

$[1, 2, 2, 3, 2, 2]$

$$+ \quad \backslash; \quad - \quad \backslash; \quad \Delta$$

\$ [ [0, 4, 0, 2, 4, 2], [0, 1, 3, 5, 1, 2], [3, 5, 3, 4, 5, 4], [3, 7, 9, 14, 7, 8] ] \$ \$ [ [2, 0, 4, 4, 0, 2], [2, 3, 1, 1, 3, 2], [1, 3, 5, 8, 3, 4], [5, 9, 7, 10, 9, 8] ] \$ \$ [ [-1, 2, -2, -1, 2, 0], [-1, -1, 1, 2, -1, 0], [1, 1, -1, -2, 1, 0], [-1, -1, 1, 2, -1, 0] ] \$

$$[-y_2 - y_1, y_2, -y_2, y_1, y_2, 0]$$

$$p' = s^2 + 2s^3 \quad p = s^2 - 4s^4$$

$$S+ \quad \backslash; \quad S- \quad \backslash; \quad NM$$

\$ [ [0, 0, 2, 2, 3, 3], [0, 1, 3, 1, 3, 2], [2, 3, 1, 3, 0, 1], [0, 0, 2, 2, 3, 3], [1, 3, 1, 4, 1, 0], [2, 3, 1, 3, 0, 1] ] \$ \$ [ [1, 2, 0, 5, 1, 1], [1, 3, 1, 4, 1, 0], [1, 1, 3, 0, 2, 3], [1, 2, 0, 5, 1, 1], [0, 1, 3, 1, 3, 2], [1, 1, 3, 0, 2, 3] ] \$ \$ [ [1, 2, 0, 3, 0, 0], [1, 2, 0, 3, 0, 0], [0, 0, 2, 0, 2, 2], [1, 2, 0, 3, 0, 0], [0, 0, 2, 0, 2, 2], [0, 0, 2, 0, 2, 2] ] \$

CmmCk true, true, true

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
2 vs 4	2 vs 4	2 vs 4	2 vs 4	2 vs 4

Omega Rank for R : cycles: {{5, 6}, {2, 4}}, net cycles: 2 . order: 2

\$ [ [0, 4, 0, 2, 4, 2], [0, 2, 0, 4, 2, 4], [0, 4, 0, 2, 4, 2], [0, 2, 0, 4, 2, 4] ] \$

$$[0, y_1, 0, y_2, y_1, y_2]$$

$$p = -s + s^3 \quad p' = -s + s^3$$

Omega Rank for B : cycles: {{3, 4}}, net cycles: -1 . order: 2

\$ [ [2, 0, 4, 4, 0, 2], [0, 0, 6, 6, 0, 0], [0, 0, 6, 6, 0, 0], [0, 0, 6, 6, 0, 0] ] \$

$$[y_2, 0, y_1, y_1, 0, y_2]$$

$$p' = s^2 - s^3 \quad p = s^2 - s^4$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[0, x_1, x_2, -4x_1 + 2x_2]$$

$$\text{For } A+2\Delta: [-3y_2 - 4y_4, y_1, -3y_1 - y_2 - 3y_3, y_2, y_3, y_4]$$

$$\text{For } A-2\Delta: [y_1, 9y_1 - 3y_4 + 12y_2 - y_3, y_4, -3y_1 - 4y_2, y_3, y_2]$$

Range of  $\{\Omega\Delta^i\}$ :  $[\mu_2, \mu_1, -\mu_1, -\mu_2 - \mu_1, \mu_1, 0]$

rank of M is 6 , rank of N is 2

M            N

\$ [ [0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 2, 0] , [0, 0, 0, 2, 0, 0] , [0, 0, 2, 0, 0, 1] , [0, 2, 0, 0, 0, 0] , [1, 0, 0, 1, 0, 0] ]  
 \$ \$ [ [0, 0, 1, 0, 1, 1] , [0, 0, 1, 0, 1, 1] , [1, 1, 0, 1, 0, 0] , [0, 0, 1, 0, 1, 1] , [1, 1, 0, 1, 0, 0] , [1, 1, 0, 1, 0, 0] ] \$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [-1, 2, -2, -1, 2, 0]$

ker M, [0, 0, 0, 0, 0, 0]

Range M, [x<sub>6</sub>, x<sub>5</sub>, x<sub>4</sub>, x<sub>3</sub>, x<sub>2</sub>, x<sub>1</sub>]

$\tau = 18$  ,  $r' = 1/2$

Ranges

Action of R on ranges, [[2], [4], [2], [2]]

Action of B on ranges, [[3], [1], [3], [3]]

$\beta(\{1, 6\}) = 1/6$

$\beta(\{2, 5\}) = 1/3$

$\beta(\{3, 4\}) = 1/3$

$\beta(\{4, 6\}) = 1/6$

ker N, [ $\mu_4, \mu_3, \mu_2, -\mu_4 - \mu_3, \mu_1, -\mu_2 - \mu_1$ ]

Range of N

[y<sub>1</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>2</sub>]

Partitions

$\alpha(\{\{1, 2, 4\}, \{3, 5, 6\}\}) = 1/1$

b1 = {1, 2, 4} ‘ , ‘ b2 = {3, 5, 6}

Action of R and B on the blocks of the partitions: \$ [ [1, 1] , [1, 1] ] \$ = \$ [ [1, 0] , [0, 1] ] \$ + \$ [ [0, 1] , [1, 0] ] \$

[‘1‘, ‘2‘], [‘2‘, ‘1‘] with invariant measure [1, 1]

N by blocks, check: true . ‘ See partition graph.

‘ ‘ See level-2 partition graph.

‘

Right Group	
Coloring	{3}
Rank	2
R,B	[2, 4, 5, 2, 6, 5], [3, 6, 4, 3, 1, 4]
$\pi_2$	[0, 0, 0, 0, 1, 0, 0, 2, 0, 2, 0, 0, 0, 1, 0]
$u_2$	[0, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, 0, 1, 1, 0] (dim 1)
wpp	[3, 3, 3, 3, 3, 3]

4. Coloring, {4}

**R:** [2, 4, 4, 3, 6, 5]    **B:** [3, 6, 5, 2, 1, 4]

‘ See graph

‘ ‘ See pair graph

$\Omega$  for  $A+\tau\Delta$  :

‘ [ ‘2‘ (‘ - 3 +  $\tau$  ‘)‘ (‘ - 1 +  $\tau$  ‘)‘ , -12‘ (‘ - 1 +  $\tau$  ‘)‘ , 4‘ (‘ 3 +  $\tau^2$  ‘)‘ , 2‘ (‘ 9 - 2 $\tau$  +  $\tau^2$  ‘)‘ , -4‘ (‘ - 3 +  $\tau$  ‘)‘ , 4‘ (‘ 3 - 2 $\tau$  +  $\tau^2$  ‘)‘ ]‘

For  $\tau=1/2$ , [5, 12, 26, 33, 20, 18] . FixedPtCheck, [5, 12, 26, 33, 20, 18]

$\det(A + \tau\Delta) = 1‘ (‘ \tau ‘)^2 ‘ (‘ - 1 + \tau ‘)^2$

Delta Range : [- $y_2 - y_3 - y_4$ , - $y_1$ ,  $y_1$ ,  $y_2$ ,  $y_3$ ,  $y_4$ ]

[1, 2, 2, 3, 2, 2]

+ \; - \; \Delta

\$ [ [0, 1, 3, 4, 2, 2] , [2, 2, 6, 6, 3, 5] , [5, 8, 8, 11, 7, 9] , [9, 18, 14, 23, 17, 15] ] \$ \$ [ [2, 3, 1, 2, 2, 2] , [2, 6, 2, 6, 5, 3] , [3, 8, 8, 13, 9, 7] , [7, 14, 18, 25, 15, 17] ] \$ \$ [ [-1, -1, 1, 1, 0, 0] , [0, -2, 2, 0, -1, 1] , [1, 0, 0, -1, -1, 1] , [1, 2, -2, -1, 1, -1] ] \$

[- $y_2$ , - $y_1$ ,  $y_1$ ,  $y_2$ , - $y_3$ ,  $y_3$ ]

$p = s + 2s^3 + 4s^4$

S+ \; S- \; NM

\$ [ [1, 3, 3, 3, 2, 2] , [2, 2, 2, 5, 1, 2] , [2, 2, 2, 5, 2, 1] , [1, 3, 3, 3, 2, 2] , [1, 2, 2, 2, 3, 4] , [0, 2, 2, 3, 4, 3] ]



\$ \$ [ [1, 3, 3, 3, 2, 2], [0, 2, 2, 3, 3, 4], [0, 2, 2, 3, 4, 3], [1, 3, 3, 3, 2, 2], [3, 2, 2, 4, 1, 2], [2, 2, 2, 5, 2, 1] ] \$ \$ [ [10, 10, 10, 30, 10, 10], [5, 20, 20, 15, 10, 10], [5, 20, 20, 15, 10, 10], [10, 10, 10, 30, 10, 10], [5, 10, 10, 15, 20, 20], [5, 10, 10, 15, 20, 20] ] \$

CmmCk true, true, true

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
3 vs 4	4 vs 6	4 vs 6	2 vs 5	3 vs 6

Omega Rank for R : cycles: {{3, 4}, {5, 6}}, net cycles: 1 . order: 2

\$ [ [0, 1, 3, 4, 2, 2], [0, 0, 4, 4, 2, 2], [0, 0, 4, 4, 2, 2], [0, 0, 4, 4, 2, 2], [0, 0, 4, 4, 2, 2] ] \$

$[0, -y_1 + 2y_2, y_1, 2y_2, y_2, y_2]$

$$p = -s^2 + s^3 \quad p = -s^2 + s^5 \quad p = -s^2 + s^4$$

Omega Rank for B : cycles: {{1, 3, 5}, {2, 4, 6}}, net cycles: 2 . order: 3

\$ [ [2, 3, 1, 2, 2, 2], [2, 2, 2, 2, 1, 3], [1, 2, 2, 3, 2, 2], [2, 3, 1, 2, 2, 2], [2, 2, 2, 2, 1, 3], [1, 2, 2, 3, 2, 2] ] \$

$[y_3, 4y_3 + 3y_2 - 4y_1, y_2, 3y_3 + 4y_2 - 4y_1, 4y_3 + 4y_2 - 5y_1, y_1]$

$$p' = -1 + s^3 \quad p' = -s + s^4 \quad p' = -s^2 + s^5$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$[x_1, 0, 2x_1, 4x_1]$

For  $A+2\Delta$  :  $[-y_2 - y_1, y_2, y_2, -y_2 - y_1, y_1, y_1]$

For  $A-2\Delta$  :  $[y_1, -y_2 - y_1, -y_2 - y_1, y_1, y_2, y_2]$

Range of  $\{\Omega\Delta^i\}$ :  $[-\mu_2, -\mu_1, \mu_1, \mu_2, -\mu_3, \mu_3]$

rank of M is 6 , rank of N is 3

M N

\$ [ [0, 4, 1, 0, 2, 3], [4, 0, 0, 6, 6, 4], [1, 0, 0, 9, 4, 6], [0, 6, 9, 0, 8, 7], [2, 6, 4, 8, 0, 0], [3, 4, 6, 7, 0, 0] ] \$

\$ \$ [ [0, 1, 1, 0, 1, 1], [1, 0, 0, 1, 1, 1], [1, 0, 0, 1, 1, 1], [0, 1, 1, 0, 1, 1], [1, 1, 1, 1, 0, 0], [1, 1, 1, 1, 0, 0] ] \$

Check is  $\Omega\Delta N$  zero? true,  $\pi\Delta = [-1, -1, 1, 1, 0, 0]$

ker M, [0, 0, 0, 0, 0, 0]

Range M, [x<sub>1</sub>, x<sub>2</sub>, x<sub>3</sub>, x<sub>5</sub>, x<sub>6</sub>, x<sub>4</sub>]

$\tau = 12$ ,  $r' = 2/3$

Ranges

Action of R on ranges, [[5], [4], [4], [7], [6], [7], [6]]

Action of B on ranges, [[3], [7], [6], [2], [5], [1], [4]]

$\beta(\{1, 2, 5\}) = 1/10$

$\beta(\{1, 2, 6\}) = 1/10$

$\beta(\{1, 3, 6\}) = 1/20$

$\beta(\{2, 4, 5\}) = 1/5$

$\beta(\{2, 4, 6\}) = 1/10$

$\beta(\{3, 4, 5\}) = 1/5$

$\beta(\{3, 4, 6\}) = 1/4$

ker N, [- $\mu_2$ , - $\mu_1$ ,  $\mu_1$ ,  $\mu_2$ , - $\mu_3$ ,  $\mu_3$ ]

Range of N

[y<sub>3</sub>, y<sub>2</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>1</sub>, y<sub>1</sub>]

Partitions

$\alpha(\{\{5, 6\}, \{2, 3\}, \{1, 4\}\}) = 1/1$

b<sub>1</sub> = {5, 6} ‘, ‘ b<sub>2</sub> = {2, 3} ‘, ‘ b<sub>3</sub> = {1, 4}

Action of R and B on the blocks of the partitions: \$ [ [1, 1, 0], [0, 0, 2], [1, 1, 0] ] \$ = \$ [ [1, 0, 0], [0, 0, 1], [0, 1, 0] ] \$ + \$ [ [0, 1, 0], [0, 0, 1], [1, 0, 0] ] \$

[‘1‘, ‘3‘, ‘2‘], [‘2‘, ‘3‘, ‘1‘] with invariant measure [1, 1, 1]

N by blocks, check: true . ‘ See partition graph.

‘ ‘ See level-3 partition graph.

‘

Right Group	
<b>Coloring</b>	{4}
<b>Rank</b>	3
<b>R,B</b>	[2, 4, 4, 3, 6, 5], [3, 6, 5, 2, 1, 4]
$\pi_2$	[4, 1, 0, 2, 3, 0, 6, 6, 4, 9, 4, 6, 8, 7, 0]
$u_2$	[1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0] (dim 1)
<b>wpp</b>	[2, 2, 2, 2, 2, 2]
$\pi_3$	[0, 0, 2, 2, 0, 0, 1, 0, 0, 0, 0, 0, 0, 4, 2, 0, 4, 5, 0, 0]
$u_3$	[0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0]

5. Coloring, {5}

**R:** [2, 4, 4, 2, 1, 5]    **B:** [3, 6, 5, 3, 6, 4]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

‘ [ ‘3‘ (‘ - 1 +  $\tau$  ‘) ‘ (‘ 1 +  $\tau$  ‘) ‘ , -6‘ (‘ 1 +  $\tau$  ‘) ‘ , 6‘ (‘ - 1 +  $\tau$  ‘) ‘ , -3‘ (‘ 3 +  $\tau^2$  ‘) ‘ , 6‘ (‘ - 1 +  $\tau$  ‘) ‘ , 6‘ (‘ - 1 +  $\tau$  ‘) ‘ ] ‘

For  $\tau=1/2$ , [-3, -12, -4, -13, -4, -4] . FixedPtCheck, [3, 12, 4, 13, 4, 4]

$\det(A + \tau \Delta) = 0$

Delta Range : [- $y_2 - y_3 - y_4$ , - $y_1$ ,  $y_1$ ,  $y_2$ ,  $y_3$ ,  $y_4$ ]

[1, 2, 2, 3, 2, 2]

+ \ ; - \ ;  $\Delta$

\$ [ [2, 4, 0, 4, 2, 0] , [1, 3, 1, 4, 2, 1] , [2, 5, 3, 7, 4, 3] , [4, 9, 7, 13, 8, 7] ] \$ \$ [ [0, 0, 4, 2, 2, 4] , [1, 1, 3, 2, 2, 3] , [2, 3, 5, 5, 4, 5] , [4, 7, 9, 11, 8, 9] ] \$ \$ [ [1, 2, -2, 1, 0, -2] , [0, 1, -1, 1, 0, -1] , [0, 1, -1, 1, 0, -1] , [0, 1, -1, 1, 0, -1] ] \$

[ $y_2$ , - $y_1$ ,  $y_1$ , - $y_2 - y_1$ , 0,  $y_1$ ]

$$p = s^2 - 4s^4$$

S+ \; S- \; NM

\$ [ [1, 0, 2, 3, 3, 1], [0, 3, 1, 3, 1, 2], [0, 3, 1, 1, 2, 3], [1, 0, 2, 3, 3, 1], [0, 3, 1, 3, 1, 2], [3, 1, 3, 2, 0, 1] ]  
 \$ \$ [ [0, 2, 0, 4, 1, 3], [1, 1, 3, 2, 3, 0], [3, 1, 3, 2, 0, 1], [0, 2, 0, 4, 1, 3], [1, 1, 3, 2, 3, 0], [0, 3, 1, 1, 2, 3] ] \$  
 \$ [ [3, 0, 2, 9, 0, 4], [0, 6, 4, 0, 6, 2], [1, 4, 6, 3, 4, 0], [3, 0, 2, 9, 0, 4], [0, 6, 4, 0, 6, 2], [2, 2, 0, 6, 2, 6] ] \$

CmmCk true, true, true

$$p' = s^2 - 2s^3$$

$\Delta$ -Rank	A+(1/2) $\Delta$	A-(1/2) $\Delta$	R	B
2 vs 4	3 vs 5	3 vs 5	3 vs 4	2 vs 4

Omega Rank for R : cycles: {{2, 4}}, net cycles: 0 . order: 4

\$ [ [2, 4, 0, 4, 2, 0], [2, 6, 0, 4, 0, 0], [0, 6, 0, 6, 0, 0], [0, 6, 0, 6, 0, 0] ] \$

[y<sub>3</sub>, y<sub>2</sub>, 0, y<sub>1</sub>, y<sub>3</sub> - y<sub>2</sub> + y<sub>1</sub>, 0]

$$p = s^3 - s^4$$

Omega Rank for B : cycles: {{3, 4, 5, 6}}, net cycles: 1 . order: 4

\$ [ [0, 0, 4, 2, 2, 4], [0, 0, 2, 4, 4, 2], [0, 0, 4, 2, 2, 4], [0, 0, 2, 4, 4, 2] ] \$

[0, 0, y<sub>1</sub>, y<sub>2</sub>, y<sub>2</sub>, y<sub>1</sub>]

$$p' = -s + s^3 \quad p = -s + s^3$$

Â« NOT SYNC'D Â»

Nullspace of { $\Omega\Delta^i$ } :

[0, x<sub>1</sub>, x<sub>2</sub>, -4x<sub>1</sub> - 2x<sub>2</sub>]

For A+2 $\Delta$  : [-y<sub>2</sub>, y<sub>1</sub>, 3y<sub>2</sub> - 3y<sub>1</sub> - y<sub>3</sub>, -y<sub>2</sub>, y<sub>2</sub>, y<sub>3</sub>]

For A-2 $\Delta$  : [-y<sub>2</sub>, -3y<sub>1</sub> + y<sub>2</sub> - 3y<sub>3</sub>, y<sub>1</sub>, -y<sub>2</sub>, y<sub>2</sub>, y<sub>3</sub>]

Range of { $\Omega\Delta^i$ } : [- $\mu_2$  -  $\mu_1$ , - $\mu_2$ ,  $\mu_2$ ,  $\mu_1$ , 0,  $\mu_2$ ]

rank of M is 6 , rank of N is 3

M N

\$ [ [0, 1, 0, 0, 0, 0], [1, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 2], [0, 1, 0, 0, 2, 0], [0, 0, 0, 2, 0, 0], [0, 0, 2, 0, 0, 0] ]  
 \$ \$ [ [0, 3, 2, 0, 3, 1], [3, 0, 1, 3, 0, 2], [2, 1, 0, 2, 1, 3], [0, 3, 2, 0, 3, 1], [3, 0, 1, 3, 0, 2], [1, 2, 3, 1, 2, 0] ] \$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [1, 2, -2, 1, 0, -2]$

ker M, [0, 0, 0, 0, 0, 0]

Range M, [ $x_5, x_1, x_6, x_2, x_3, x_4$ ]

$\tau = 18$ ,  $r' = 1/2$

Ranges

Action of R on ranges, [[2], [2], [4], [1]]

Action of B on ranges, [[3], [3], [4], [3]]

$\beta(\{1, 2\}) = 1/6$

$\beta(\{2, 4\}) = 1/6$

$\beta(\{3, 6\}) = 1/3$

$\beta(\{4, 5\}) = 1/3$

ker N, [ $-\mu_1 - \mu_3, -\mu_3 - \mu_2, \mu_3, \mu_1, \mu_2, \mu_3$ ]

Range of N

[ $y_1, y_2, y_1 + y_2 - y_3, y_1, y_2, y_3$ ]

Partitions

Action of R on partitions, [[2], [2]]

Action of B on partitions, [[2], [1]]

$\alpha(\{\{1, 3, 4\}, \{2, 5, 6\}\}) = 1/3$

$\alpha(\{\{1, 4, 6\}, \{2, 3, 5\}\}) = 2/3$

$b_1 = \{1, 4, 6\}$ ,  $b_2 = \{1, 3, 4\}$ ,  $b_3 = \{2, 3, 5\}$ ,  $b_4 = \{2, 5, 6\}$

Action of R and B on the blocks of the partitions:  $\$ [ [0, 0, 1, 1], [1, 0, 1, 0], [1, 1, 0, 0], [1, 0, 1, 0] ] \$ =$   
 $\$ [ [0, 0, 1, 0], [0, 0, 1, 0], [1, 0, 0, 0], [1, 0, 0, 0] ] \$ + \$ [ [0, 0, 0, 1], [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1,$   
 $0] ] \$$

[ $'3', '3', '1', '1'$ ], [ $'4', '1', '2', '3'$ ] with invariant measure [2, 1, 2, 1]

N by blocks, check: true. ' See partition graph.

' ' See level-2 partition graph.

,

Sandwich	
<b>Coloring</b>	{5}
<b>Rank</b>	2
<b>R,B</b>	[2, 4, 4, 2, 1, 5], [3, 6, 5, 3, 6, 4]
$\pi_2$	[1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 2, 2, 0, 0]
$u_2$	[3, 2, 0, 3, 1, 1, 3, 0, 2, 2, 1, 3, 3, 1, 2] (dim 1)
<b>wpp</b>	[3, 3, 3, 3, 3, 3]

6. Coloring, {6}

**R:** [2, 4, 4, 2, 6, 4]    **B:** [3, 6, 5, 3, 1, 5]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

‘ [ ‘ -3‘ (‘ - 1 +  $\tau$  ‘)‘<sup>3</sup> ‘ (‘ 3 +  $\tau$  ‘)‘ , 6‘ (‘ 1 +  $\tau$  ‘)‘ (‘ 3 +  $\tau$  ‘<sup>2</sup> ‘)‘ , -6‘ (‘ - 1 +  $\tau$  ‘)‘ (‘ 3 +  $\tau$  ‘<sup>2</sup> ‘)‘ , 3‘ (‘ 9 -  $\tau$  -  $\tau$  ‘<sup>2</sup> +  $\tau$  ‘<sup>3</sup> ‘)‘ (‘ 1 +  $\tau$  ‘)‘ , 6‘ (‘ - 1 +  $\tau$  ‘)‘<sup>2</sup> ‘ (‘ 3 +  $\tau$  ‘)‘ , 6‘ (‘ - 1 +  $\tau$  ‘)‘ (‘ 1 +  $\tau$  ‘)‘ (‘ - 3 +  $\tau$  ‘)‘ ] ‘

For  $\tau=1/2$ , [7, 156, 52, 201, 28, 60] . FixedPtCheck, [7, 156, 52, 201, 28, 60]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	3 vs 4	4 vs 4	2 vs 3	4 vs 4

bi =

$\$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 1/4, 0, 3/4], [0, 0, 0, 1/4, 3/4, 0], [0, 1/4, 3/4, 0, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 0, 0, 1/4, 3/4, 0] ] \times \$ [ [99/100, 0, 0, -9/100, 3/100, 3/100], [0, 1/10, 3/10, 0, 0, 0], [0, 3/10, 9/10, 0, 0, 0], [-9/100, 0, 0, 19/100, 27/100, 27/100], [3/100, 0, 0, 27/100, 91/100, -9/100], [3/100, 0, 0, 27/100, -9/100, 91/100] ] \times \$ =$

$\$ [ [-1/16, -11/16, -1/6, 1], [15/16, 5/16, 1/2, -5/3], [-3/16, 7/16, -1/2, 1/3], [-1/16, -11/16, -1/6, 1], [1/16, 3/16, 5/6, -1], [-3/16, 7/16, -1/2, 1/3] ] \times \$ [ [3/2, 1, 3, 3/2, 3, 2], [9/4, 3/4, 9/4, 3/2, 15/4, 3/2], [45/16, 15/16, 45/16, 9/8, 45/16, 3/2], [135/64, 63/64, 189/64, 21/16, 207/64, 45/32] ] \times \$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{2, 4}}, net cycles: 0 . order: 2

$$\$ [ [0, 4, 0, 6, 0, 2], [0, 6, 0, 6, 0, 0], [0, 6, 0, 6, 0, 0] ] \$$$

$$[0, y_1 - y_2, 0, y_1, 0, y_2]$$

$$p = -s^2 + s^3$$

Omega Rank for B : cycles: {{1, 3, 5}}, net cycles: 0 . order: 3

$$[y_3, 0, y_4, 0, y_1, y_2]$$

$$B = \$ [ [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, -5/36, 1/36, 7/36], [1/2, -5/36, 1/36, -11/36], [0, 7/36, -5/36, 1/36], [0, -5/36, 1/36, 7/36], [0, 1/36, 7/36, -5/36], [0, 7/36, -5/36, 1/36] ] \$ \times \$ [ [2, 0, 4, 0, 4, 2], [4, 0, 2, 0, 6, 0], [6, 0, 4, 0, 2, 0], [2, 0, 6, 0, 4, 0] ] \$$$

Â» SYNC'D 1/4 , 0.2500000000

7 . Coloring, {2, 3}

**R:** [2, 6, 5, 2, 6, 5] **B:** [3, 4, 4, 3, 1, 4]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[ '3' ('1 + \tau')'' ('3 + \tau^2')'' ('-1 + \tau')', 6' ('3 + \tau')'' ('1 + \tau')'' ('-1 + \tau')', -6' ('3 + \tau')'' ('-1 + \tau')'^2, -3' ('-9 - \tau + \tau^2 + \tau^3')'' ('-1 + \tau')', -6' ('1 + \tau')'' ('3 + \tau^2')', 6' ('1 + \tau')'^2 '(-3 + \tau')' ]'$$

For τ=1/2, [-39, -84, -28, -73, -156, -180] . FixedPtCheck, [39, 84, 28, 73, 156, 180]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	4 vs 4	4 vs 4	3 vs 3	3 vs 3

bi =

$\$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 3/4, 0, 1/4], [0, 0, 0, 3/4, 1/4, 0], [0, 1/4, 3/4, 0, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 0, 0, 3/4, 1/4, 0] ] \$ \times \$ [ [19/20, 0, 0, -1/20, 3/20, 3/20], [0, 1/10, 3/10, 0, 0, 0], [0, 3/10, 9/10, 0, 0, 0], [-1/20, 0, 0, 19/20, 3/20, 3/20], [3/20, 0, 0, 3/20, 11/20, -9/20], [3/20, 0, 0, 3/20, -9/20, 11/20] ] \$$

$=$   
 $\$ [ [1/4, 3/2, -1/3, -4/3], [9/4, 15/2, -1/3, -28/3], [-3/4, -5/2, 2/3, 8/3], [1/4, 3/2, -1/3, -4/3], [-3/4, -11/2, -1/3, 20/3], [-3/4, -5/2, 2/3, 8/3] ] \$ \times \$ [ [3/2, 1, 3, 9/2, 1, 1], [3/4, 3/2, 9/2, 15/4, 1, 1/2], [3/4, 9/8, 27/8, 39/8, 5/4, 5/8], [15/16, 45/32, 135/32, 123/32, 1, 19/32] ] \$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{5, 6}}, net cycles: 0 . order: 2

$$[0, y_1, 0, 0, y_3, y_2]$$

$R = \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0] ] \$ \times \$ [ [0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [1/4, -1/12, -1/12], [0, 1/6, -1/12], [0, -1/12, 1/6], [1/4, -1/12, -1/12], [0, 1/6, -1/12], [0, -1/12, 1/6] ] \$ \times \$ [ [0, 4, 0, 0, 4, 4], [0, 0, 0, 0, 4, 8], [0, 0, 0, 0, 8, 4] ] \$$

Omega Rank for B : cycles: {{3, 4}}, net cycles: 0 . order: 2

$$[y_1, 0, y_2, y_3, 0, 0]$$

$B = \$ [ [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 1/6, -1/12], [0, -1/12, 1/6], [0, -1/12, 1/6], [0, 1/6, -1/12], [1/2, -1/12, -1/3], [0, -1/12, 1/6] ] \$ \times \$ [ [2, 0, 4, 6, 0, 0], [0, 0, 8, 4, 0, 0], [0, 0, 4, 8, 0, 0] ] \$$

Â» SYNC'D 1/4 , 0.2500000000

8 . Coloring, {2, 4}

**R:** [2, 6, 4, 3, 6, 5] **B:** [3, 4, 5, 2, 1, 4]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

' [ '2' (' - 1 + τ ')'' (' 3 + τ ')', 12' (' - 1 + τ ')', 4' (' - 3 + τ <sup>2</sup> '), 2' (' - 9 + 4τ + τ <sup>2</sup> '), -4' (' 3 + τ ')', 4' (' 1 + τ ')'' (' - 3 + τ ')'

For τ=1/2, [-7, -12, -22, -27, -28, -30] . FixedPtCheck, [7, 12, 22, 27, 28, 30]



$$\det(A + \tau \Delta) = (1 - \tau)^2 (1 + \tau)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	6 vs 6	6 vs 6	3 vs 5	4 vs 5

bi =

$$\begin{aligned} & \$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 3/4, 0, 1/4], [0, 0, 0, 1/4, 3/4, 0], [0, 3/4, 1/4, 0, 0, 0], [3/4, 0, 0, 0, 0, 0, \\ & 1/4], [0, 0, 0, 3/4, 1/4, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, \\ & 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \end{aligned}$$

$$\begin{aligned} & \$ [ [-77/3096, 4879/3096, 1241/1161, -1969/1161, -2048/1161, 1072/1161], [-247/1548, -400/387, \\ & -502/1161, -64/1161, 2944/1161, -896/1161], [-7/1548, 20/387, 2618/1161, 2480/1161, -2624/1161, \\ & -2432/1161], [-677/3096, 679/3096, -1111/1161, -505/1161, -1280/1161, 2992/1161], [631/3096, \\ & 1579/3096, 425/1161, 1835/1161, 1504/1161, -4496/1161], [2479/3096, -1997/3096, -1495/1161, \\ & -2509/1161, 1120/1161, 2800/1161] ] \times \$ [ [3/2, 5/2, 3/2, 7/2, 2, 1], [3/2, 3, 2, 3, 11/8, 9/8], [33/32, \\ & 21/8, 15/8, 115/32, 57/32, 35/32], [171/128, 189/64, 107/64, 417/128, 215/128, 141/128], [645/512, \\ & 711/256, 465/256, 1771/512, 783/512, 593/512], [2349/2048, 2979/1024, 1853/1024, 6975/2048, \\ & 3383/2048, 2205/2048] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{5, 6}, {3, 4}}, net cycles: 1 . order: 2

$$\begin{aligned} & \$ [ [0, 1, 3, 2, 2, 4], [0, 0, 2, 3, 4, 3], [0, 0, 3, 2, 3, 4], [0, 0, 2, 3, 4, 3], [0, 0, 3, 2, 3, 4] ] \$ \\ & [0, -7 y_1 - y_3 + 6 y_2, y_1, -6 y_1 + 5 y_2, y_3, y_2] \end{aligned}$$

$$p' = -s^2 + s^4 \quad p = -s^2 + s^4$$

Omega Rank for B : cycles: {{2, 4}, {1, 3, 5}}, net cycles: 2 . order: 6

$$\begin{aligned} & \$ [ [2, 3, 1, 4, 2, 0], [2, 4, 2, 3, 1, 0], [1, 3, 2, 4, 2, 0], [2, 4, 1, 3, 2, 0], [2, 3, 2, 4, 1, 0] ] \$ \\ & [5 y_3, 5 y_4, 5 y_1, 7 y_3 - 5 y_4 + 7 y_1 + 7 y_2, 5 y_2, 0] \end{aligned}$$

$$p = -s - s^2 + s^4 + s^5$$

Â» SYNC'D 5/256 , 0.01953125000

9. Coloring, {2, 5}

**R:** [2, 6, 4, 2, 1, 5]    **B:** [3, 4, 5, 3, 6, 4]

' See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

$$\left[ \begin{array}{c} 3(1+\tau)^2(3-\tau+5\tau^2+\tau^3), 6(1+\tau)^2(3+\tau^2), -6(-1+\tau)^2(3+\tau^2) \\ -3(-1+\tau)^2(9+7\tau+7\tau^2+\tau^3), 6(3-\tau+5\tau^2+\tau^3), -6(-3-\tau-5\tau^2+\tau^3) \end{array} \right]$$

For  $\tau=1/2$ , [93, 156, 52, 115, 124, 148] . FixedPtCheck, [93, 156, 52, 115, 124, 148]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	5 vs 5	5 vs 5	4 vs 5	3 vs 4

bi =

$$\begin{aligned} & \$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 3/4, 0, 1/4], [0, 0, 0, 1/4, 3/4, 0], [0, 1/4, 3/4, 0, 0, 0], [1/4, 0, 0, 0, 0, 0, \\ & 3/4], [0, 0, 0, 3/4, 1/4, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1/10, 3/10, 0, 0, 0], [0, 3/10, 9/10, 0, 0, 0], [0, 0, 0, \\ & 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [0, -5/8, 1/24, -4, 14/3], [3/4, 5/12, 1/4, 8/3, -4], [-1/4, 17/12, 1/4, 8/3, -4], [0, -5/8, 1/24, -4, \\ & 14/3], [0, -35/24, 7/8, -4/3, 2], [0, 7/8, -35/24, 4, -10/3] ] \times \$ [ [1/2, 1, 3, 7/2, 2, 2], [1/2, 1, 3, 3, 11/4, \\ & 7/4], [11/16, 7/8, 21/8, 45/16, 43/16, 37/16], [43/64, 7/8, 21/8, 195/64, 163/64, 143/64], [163/256, \\ & 119/128, 357/128, 765/256, 647/256, 545/256] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 2, 5, 6}}, net cycles: 0 . order: 4

$$\begin{aligned} & \$ [ [2, 4, 0, 2, 2, 2], [2, 4, 0, 0, 2, 4], [2, 2, 0, 0, 4, 4], [4, 2, 0, 0, 4, 2], [4, 4, 0, 0, 2, 2] ] \$ \\ & [y_1 - y_2 + y_3 - y_4, y_1, 0, y_2, y_3, y_4] \end{aligned}$$

$$p = -s^2 + s^3 - s^4 + s^5$$

Omega Rank for B : cycles: {{3, 4, 5, 6}}, net cycles: 1 . order: 4

$$\begin{aligned} & \$ [ [0, 0, 4, 4, 2, 2], [0, 0, 4, 2, 4, 2], [0, 0, 2, 2, 4, 4], [0, 0, 2, 4, 2, 4] ] \$ \\ & [0, 0, y_1, y_1 + y_2 - y_3, y_3, y_2] \end{aligned}$$

$$p = -s + s^2 - s^3 + s^4$$

Â» SYNC'D 3/32 , 0.09375000000

10 . Coloring, {2, 6}

**R:** [2, 6, 4, 2, 6, 4] **B:** [3, 4, 5, 3, 1, 5]

' See graph

' ' See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

' [ '1' (' - 1 +  $\tau$  ')<sup>2</sup> , 2' (' 1 +  $\tau$  ')<sup>2</sup> , -2' (' - 1 +  $\tau$  ')<sup>2</sup> , -1' (' 1 +  $\tau$  ')<sup>2</sup> (' - 3 +  $\tau$  ')<sup>2</sup> , -2' (' - 1 +  $\tau$  ')<sup>2</sup> , 2' (' 1 +  $\tau$  ')<sup>2</sup> ]'

For  $\tau=1/2$ , [1, 12, 4, 15, 4, 12] . FixedPtCheck, [1, 12, 4, 15, 4, 12]

$\det(A + \tau \Delta) = 0$

Delta Range : [- $y_2 - y_3 - y_4$ , - $y_1$ ,  $y_1$ ,  $y_2$ ,  $y_3$ ,  $y_4$ ]

[1, 2, 2, 3, 2, 2]

+ \ ; - \ ;  $\Delta$

\$ [ [0, 4, 0, 4, 0, 4] , [2, 2, 2, 2, 2, 2] , [1, 2, 2, 3, 2, 2] , [1, 2, 2, 3, 2, 2] ] \$ \$ [ [2, 0, 4, 2, 4, 0] , [0, 2, 2, 4, 2, 2] , [1, 2, 2, 3, 2, 2] , [1, 2, 2, 3, 2, 2] ] \$ \$ [ [-1, 2, -2, 1, -2, 2] , [1, 0, 0, -1, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] ] \$

[ $y_1$ ,  $y_2$ , - $y_2$ , - $y_1$ , - $y_2$ ,  $y_2$ ]

$p = s^3$

S+ \ ; S- \ ; NM

\$ [ [0, 0, 2, 1, 0, 1] , [1, 1, 0, 2, 0, 0] , [0, 1, 0, 0, 2, 1] , [0, 0, 2, 1, 0, 1] , [1, 1, 0, 2, 0, 0] , [0, 1, 0, 0, 2, 1] ]

\$ \$ [ [1, 2, 0, 0, 1, 0] , [0, 0, 1, 0, 1, 2] , [0, 0, 1, 3, 0, 0] , [1, 2, 0, 0, 1, 0] , [0, 0, 1, 0, 1, 2] , [0, 0, 1, 3, 0, 0] ] \$

\$ \$ [ [2, 2, 2, 6, 2, 2] , [1, 4, 2, 3, 4, 2] , [1, 2, 4, 3, 2, 4] , [2, 2, 2, 6, 2, 2] , [1, 4, 2, 3, 4, 2] , [1, 2, 4, 3, 2, 4] ] \$

CmmCk true, true, true

$p' = s^3$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
2 vs 4	2 vs 4	2 vs 4	1 vs 3	2 vs 4

Omega Rank for R : cycles: {{2, 4, 6}}, net cycles: 1 . order: 3

\$ [ [0, 4, 0, 4, 0, 4] , [0, 4, 0, 4, 0, 4] , [0, 4, 0, 4, 0, 4] ] \$

$$[0, y_1, 0, y_1, 0, y_1]$$

$$p = -s + s^3 \quad p = -s + s^2$$

Omega Rank for B : cycles:  $\{\{1, 3, 5\}\}$ , net cycles: 0 . order: 3

$$\$ [ [2, 0, 4, 2, 4, 0], [4, 0, 4, 0, 4, 0], [4, 0, 4, 0, 4, 0], [4, 0, 4, 0, 4, 0] ] \$$$

$$[y_2 - y_1, 0, y_2, y_1, y_2, 0]$$

$$p = -s^2 + s^3 \quad p = -s^2 + s^4$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[0, 0, x_1, x_2]$$

$$\text{For } A+2\Delta : [y_2, y_1, -3y_1 - 4y_2 - y_3 - 3y_4, y_2, y_3, y_4]$$

$$\text{For } A-2\Delta : [y_2, -4y_2 - 3y_1 - 3y_3 - y_4, y_1, y_2, y_3, y_4]$$

Range of  $\{\Omega\Delta^i\}$ :  $[-\mu_1, \mu_2, -\mu_2, \mu_1, -\mu_2, \mu_2]$

rank of M is 6 , rank of N is 3

M            N

$$\$ [ [0, 0, 1, 0, 1, 0], [0, 0, 0, 2, 0, 2], [1, 0, 0, 1, 2, 0], [0, 2, 1, 0, 1, 2], [1, 0, 2, 1, 0, 0], [0, 2, 0, 2, 0, 0] ] \$$$

$$\$ [ [0, 1, 1, 0, 1, 1], [1, 0, 1, 1, 0, 1], [1, 1, 0, 1, 1, 0], [0, 1, 1, 0, 1, 1], [1, 0, 1, 1, 0, 1], [1, 1, 0, 1, 1, 0] ] \$$$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [-1, 2, -2, 1, -2, 2]$

$$\ker M, [0, 0, 0, 0, 0, 0]$$

$$\text{Range } M, [x_1, x_2, x_3, x_4, x_5, x_6]$$

$$\tau = 12, r' = 2/3$$

Ranges

$$\text{Action of } R \text{ on ranges, } [[2], [2], [2]]$$

$$\text{Action of } B \text{ on ranges, } [[1], [3], [1]]$$

$$\beta(\{1, 3, 5\}) = 1/4$$

$$\beta(\{2, 4, 6\}) = 1/2$$

$$\beta(\{3, 4, 5\}) = 1/4$$

$$\ker N, [\mu_2, \mu_3, \mu_1, -\mu_2, -\mu_3, -\mu_1]$$

Range of N

$$[y_3, y_1, y_2, y_3, y_1, y_2]$$

Partitions

$$\alpha(\{1, 4\}, \{2, 5\}, \{3, 6\}) = 1/1$$

$$b1 = \{1, 4\}, b2 = \{2, 5\}, b3 = \{3, 6\}$$

Action of R and B on the blocks of the partitions:  $\$ [ [0, 1, 1], [1, 0, 1], [1, 1, 0] ] \$ = \$ [ [0, 0, 1], [1, 0, 0], [0, 1, 0] ] \$ + \$ [ [0, 1, 0], [0, 0, 1], [1, 0, 0] ] \$$   
 $['3', '1', '2'], ['2', '3', '1']$  with invariant measure  $[1, 1, 1]$

N by blocks, check: true . ' See partition graph.

' ' See level-3 partition graph.

'

Right Group	
<b>Coloring</b>	{2, 6}
<b>Rank</b>	3
<b>R,B</b>	[2, 6, 4, 2, 6, 4], [3, 4, 5, 3, 1, 5]
$\pi_2$	[0, 1, 0, 1, 0, 0, 2, 0, 2, 1, 2, 0, 1, 2, 0]
$u_2$	[1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1] (dim 1)
<b>wpp</b>	[2, 2, 2, 2, 2, 2]
$\pi_3$	[0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 2, 0, 1, 0, 0, 0]
$u_3$	[1, 0, 0, 1, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1]

11 . Coloring, {3, 4}

$$\mathbf{R}: [2, 4, 5, 3, 6, 5] \quad \mathbf{B}: [3, 6, 4, 2, 1, 4]$$

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$$[ '6' ('1 + \tau')' ('-1 + \tau')', 12' ('-1 + \tau')', 12' ('1 + \tau')' ('-1 + \tau')', 6' ('3 + \tau')' ('-1 + \tau')', -12' ('1 + \tau')', -12' ('1 + \tau^2')' ]'$$

For  $\tau=1/2$ ,  $[-3, -4, -6, -7, -12, -10]$  . FixedPtCheck,  $[3, 4, 6, 7, 12, 10]$

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	5 vs 5	5 vs 5	4 vs 5	5 vs 5

bi =

$$\begin{aligned} & \$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 1/4, 0, 3/4], [0, 0, 0, 3/4, 1/4, 0], [0, 3/4, 1/4, 0, 0, 0], [3/4, 0, 0, 0, 0, 0, \\ & 1/4], [0, 0, 0, 3/4, 1/4, 0] ] \$ \times \$ [ [819/820, 0, 0, -9/820, 27/820, 3/820], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], \\ & , [-9/820, 0, 0, 739/820, 243/820, 27/820], [27/820, 0, 0, 243/820, 91/820, -81/820], [3/820, 0, 0, 27/820, \\ & -81/820, 811/820] ] \$ = \\ & \$ [ [-181/3172, 72/61, 98/793, -1904/2379, -288/793], [-9/3172, -19/61, -258/793, -1048/793, \\ & 4864/2379], [3/244, 1/61, 86/61, 24/61, -320/183], [-725/3172, 8/61, -694/793, 2608/2379, -32/793], \\ & [2695/3172, -31/61, -986/793, -1688/2379, 1344/793], [3/244, 1/61, 86/61, 24/61, -320/183] ] \$ \times \$ [ \\ & [3/2, 5/2, 3/2, 7/2, 1, 2], [3/4, 3, 2, 13/4, 7/8, 17/8], [21/32, 21/8, 11/8, 123/32, 33/32, 79/32], [99/128, \\ & 195/64, 93/64, 453/128, 123/128, 285/128], [369/512, 729/256, 375/256, 1803/512, 471/512, 1293/512] ] \$ \\ & \$ \end{aligned}$$

Check x AllOnes:  $[1, 1, 1, 1, 1, 1]$

Omega Rank for R : cycles:  $\{\{5, 6\}\}$ , net cycles: 0 . order: 4

$$\begin{aligned} & \$ [ [0, 1, 3, 2, 4, 2], [0, 0, 2, 1, 5, 4], [0, 0, 1, 0, 6, 5], [0, 0, 0, 0, 6, 6], [0, 0, 0, 0, 6, 6] ] \$ \\ & [0, -y_1 + y_2 + y_3 - y_4, y_1, y_2, y_3, y_4] \end{aligned}$$

$$p = -s^4 + s^5$$

Omega Rank for B : cycles:  $\{\{2, 4, 6\}\}$ , net cycles: 0 . order: 3

$$[y_1, y_2, y_3, y_4, 0, y_5]$$

$$\begin{aligned} \mathbf{B} = & \$ [ [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, \\ & 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, \\ & 0, 0, 1] ] \$ = & \$ [ [0, 1/2, 1/36, -11/36, -5/36], [0, 0, 1/36, -11/36, 13/36], [0, 0, 13/36, 1/36, -11/36], [0, 0, \\ & -11/36, 13/36, 1/36], [1/2, -1/4, -11/36, -5/36, 5/18], [0, 0, 13/36, 1/36, -11/36] ] \$ \times \$ [ [2, 3, 1, 4, 0, 2], \\ & [0, 4, 2, 3, 0, 3], [0, 3, 0, 5, 0, 4], [0, 5, 0, 4, 0, 3], [0, 4, 0, 3, 0, 5] ] \$ \end{aligned}$$

$\hat{A}$  » SYNC'D 27/256 , 0.1054687500

12 . Coloring,  $\{3, 5\}$

**R:** [2, 4, 5, 2, 1, 5]    **B:** [3, 6, 4, 3, 6, 4]

' See graph

' ' See pair graph

'

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ '3' ('3 + \tau')'' ('1 + \tau')' ^2 ('-1 + \tau')', -6' ('1 + \tau')'' ('3 + \tau^2')', 6' ('3 + \tau^2')'' ('-1 + \tau')', \\ & -3' ('9 - 4\tau + 6\tau^2 + 4\tau^3 + \tau^4')', 6' ('3 + \tau')'' ('1 + \tau')'' ('-1 + \tau')', -6' ('1 + \tau')'' ('-3 + \tau')'' ('-1 + \tau')'' ]' \end{aligned}$$

For  $\tau=1/2$ , [-63, -156, -52, -145, -84, -60] . FixedPtCheck, [63, 156, 52, 145, 84, 60]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	4 vs 4	4 vs 4	4 vs 4	3 vs 3

bi =

$$\begin{aligned} & \$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 1/4, 0, 3/4], [0, 0, 0, 3/4, 1/4, 0], [0, 1/4, 3/4, 0, 0, 0], [1/4, 0, 0, 0, 0, 3/4], \\ & [0, 0, 0, 3/4, 1/4, 0] ] \$ \times \$ [ [91/100, 0, 0, -9/100, 27/100, 3/100], [0, 1/10, 3/10, 0, 0, 0], [0, 3/10, 9/10, 0, 0, 0], \\ & [-9/100, 0, 0, 91/100, 27/100, 3/100], [27/100, 0, 0, 27/100, 19/100, -9/100], [3/100, 0, 0, 3/100, -9/100, 99/100] ] \$ = \\ & \$ [ [-3/4, -5/2, 2/3, 8/3], [-3/4, -11/2, -1/3, 20/3], [1/4, 3/2, -1/3, -4/3], [-3/4, -5/2, 2/3, 8/3], [9/4, 15/2, -1/3, -28/3], \\ & [1/4, 3/2, -1/3, -4/3] ] \$ \times \$ [ [1/2, 1, 3, 7/2, 1, 3], [1/4, 1, 3, 19/4, 3/2, 3/2], [3/8, 5/4, 15/4, 29/8, 9/8, 15/8], \\ & [9/32, 1, 3, 145/32, 45/32, 57/32] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{2, 4}}, net cycles: 0 . order: 4

$$[y_1, y_2, 0, y_3, y_4, 0]$$

$$\begin{aligned} R = & \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], \\ & [0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1], \\ & [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 0, 1/6, -1/12], [0, 0, -1/12, 1/6], [1/4, -1/8, -1/12, 1/24], [0, 0, 1/6, -1/12], [0, 1/4, -1/12, -1/12], \\ & [1/4, -1/8, -1/12, 1/24] ] \$ \times \$ [ [2, 4, 0, 2, 4, 0], [4, 4, 0, 4, 0, 0], [0, 8, 0, 4, 0, 0], [0, 4, 0, 8, 0, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{3, 4}}, net cycles: 0 . order: 2

$$[0, 0, y_1, y_3, 0, y_2]$$

$$B = \$ [ [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, -1/12, 1/6], [1/4, -1/12, -1/12], [0, 1/6, -1/12], [0, -1/12, 1/6], [1/4, -1/12, -1/12], [0, 1/6, -1/12] ] \$ x \$ [ [0, 0, 4, 4, 0, 4], [0, 0, 4, 8, 0, 0], [0, 0, 8, 4, 0, 0] ] \$$$

Â» SYNC'D 1/4 , 0.2500000000

13 . Coloring, {3, 6}

**R:** [2, 4, 5, 2, 6, 4]    **B:** [3, 6, 4, 3, 1, 5]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[ '3' ( ' - 1 + \tau ' ) ^ 2 ( ' 1 + \tau ' ) ^ 2 ( ' - 3 + \tau ' ) ^ 2 , -6' ( ' 1 + \tau ' ) ^ 2 ( ' 3 + \tau ^ 2 ' ) ^ 2 , 6' ( ' - 1 + \tau ' ) ^ 2 ( ' 3 + \tau ^ 2 ' ) ^ 2 , -3' ( ' 9 + 4\tau + 6\tau ^ 2 - 4\tau ^ 3 + \tau ^ 4 ' ) ^ 2 , -6' ( ' - 1 + \tau ' ) ^ 2 ( ' 1 + \tau ' ) ^ 2 ( ' - 3 + \tau ' ) ^ 2 , 6' ( ' - 1 + \tau ' ) ^ 2 ( ' 1 + \tau ' ) ^ 2 ( ' 3 + \tau ' ) ^ 2 ]$$

For τ=1/2, [-15, -156, -52, -193, -60, -84] . FixedPtCheck, [15, 156, 52, 193, 60, 84]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	5 vs 5	5 vs 5	3 vs 4	4 vs 5

bi =

$$\$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 1/4, 0, 3/4], [0, 0, 0, 3/4, 1/4, 0], [0, 1/4, 3/4, 0, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 0, 0, 1/4, 3/4, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0], [0, 1/10, 3/10, 0, 0, 0], [0, 3/10, 9/10, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ =$$

$$\$ [ [0, -3/8, 9/8, 4, -14/3], [3/4, -3/4, -5/4, -8/3, 4], [-1/4, 1/4, -5/4, -8/3, 4], [0, -3/8, 9/8, 4, -14/3], [0, 1/8, 5/8, -4, 10/3], [0, 9/8, -3/8, 4/3, -2] ] \$ x \$ [ [3/2, 1, 3, 5/2, 2, 2], [3/2, 1, 3, 3, 9/4, 5/4], [27/16, 9/8, 27/8, 45/16, 27/16, 21/16], [81/64, 9/8, 27/8, 201/64, 117/64, 81/64], [351/256, 141/128, 423/128, 801/256, 459/256, 333/256] ] \$$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{2, 4}}, net cycles: 0 . order: 4

$$\$ [ [0, 4, 0, 4, 2, 2], [0, 4, 0, 6, 0, 2], [0, 6, 0, 6, 0, 0], [0, 6, 0, 6, 0, 0] ] \$$$



$$[0, y_1 + y_2 - y_3, 0, y_1, y_2, y_3]$$

$$p = -s^3 + s^4$$

Omega Rank for B : cycles: {{3, 4}}, net cycles: 0 . order: 4

$$\$ [ [2, 0, 4, 2, 2, 2], [2, 0, 4, 4, 2, 0], [2, 0, 6, 4, 0, 0], [0, 0, 6, 6, 0, 0], [0, 0, 6, 6, 0, 0] ] \$$$

$$[y_4, 0, y_4 + y_1 - y_2 + y_3, y_1, y_2, y_3]$$

$$p = s^4 - s^5$$

Â» SYNC'D 3/32 , 0.09375000000

14 . Coloring, {4, 5}

**R:** [2, 4, 4, 3, 1, 5] **B:** [3, 6, 5, 2, 6, 4]

' See graph

' ' See pair graph

Ω for A+τΔ :

$$[ '6' ( ' - 1 + \tau ' ) ' ( ' 1 + \tau ' ) ' ^ 2 ' ( ' - 3 + \tau ' ) ' , -36' ( ' - 1 + \tau ' ) ' ( ' 3 + 2\tau + 3\tau ^ 2 ' ) ' , 12' ( ' 1 + \tau ' ) ' ( ' 3 - \tau + \tau ^ 2 + \tau ^ 3 ' ) ' , 6' ( ' 3 + \tau ^ 2 ' ) ' ^ 2 , 12' ( ' - 1 + \tau ' ) ' ( ' 1 + \tau ' ) ' ( ' - 3 + \tau ' ) ' , 12' ( ' - 1 + \tau ' ) ' ^ 2 ' ( ' 3 + 2\tau + \tau ^ 2 ' ) ' ] '$$

For τ=1/2, [45, 76, 138, 169, 60, 34] . FixedPtCheck, [45, 76, 138, 169, 60, 34]

$$\det(A + \tau \Delta) = 1' ( ' \tau ' ) ' ^ 2 ' ( ' - 1 + \tau ' ) ' ( ' 1 + \tau ' ) '$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	6 vs 6	6 vs 6	4 vs 5	5 vs 5

bi =

$$\$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 1/4, 0, 3/4], [0, 0, 0, 1/4, 3/4, 0], [0, 3/4, 1/4, 0, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 0, 0, 3/4, 1/4, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ =$$

$$\$ [ [-1979/936, -215/936, 1189/117, 2375/117, 1984/351, -11824/351], [-593/468, 160/117, 110/117, 880/117, -928/351, -2048/351], [91/36, -20/9, -34/9, -128/9, 224/27, 256/27], [589/936, -671/936, -203/117, -337/117, -704/351, 2384/351], [-751/936, 3101/936, -175/117, 355/117, -4480/351, 3088/351], [145/936, -1187/936, 217/117, -253/117, 2560/351, -2032/351] ] \$ x \$ [ [1/2, 5/2, 3/2, 5/2, 2, 3], [1/2, 2, 1, 13/4, 15/8, 27/8], [15/32, 41/16, 19/16, 105/32, 51/32, 93/32], [51/128, 165/64, 75/64, 399/128, 207/128, 399/128], [207/512, 39/16, 69/64, 1677/512, 849/512, 1611/512], [849/2048,$$

2619/1024, 1149/1024, 6633/2048, 3267/2048, 6291/2048 ] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 4}}, net cycles: 0 . order: 4

\$ [ [2, 1, 3, 4, 2, 0] , [2, 2, 4, 4, 0, 0] , [0, 2, 4, 6, 0, 0] , [0, 0, 6, 6, 0, 0] , [0, 0, 6, 6, 0, 0] ] \$

[y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>, -y<sub>2</sub> + y<sub>3</sub> + y<sub>4</sub> + y<sub>1</sub>, y<sub>1</sub>, 0]

$$p = -s^4 + s^5$$

Omega Rank for B : cycles: {{2, 4, 6}}, net cycles: 0 . order: 3

[0, y<sub>1</sub>, y<sub>5</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>]

B = \$ [ [0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 1, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1] , [0, 0, 0, 1, 0, 0] ] \$ x \$ [ [0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 1, 0] , [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [1, -2, 13/36, -47/36, 73/36] , [0, 0, -11/36, 1/36, 13/36] , [0, 1, 1/36, 13/36, -47/36] , [0, 0, 1/36, 13/36, -11/36] , [0, 0, -11/36, 1/36, 13/36] , [0, 0, 13/36, -11/36, 1/36] ] \$ x \$ [ [0, 3, 1, 2, 2, 4] , [0, 2, 0, 4, 1, 5] , [0, 4, 0, 5, 0, 3] , [0, 5, 0, 3, 0, 4] , [0, 3, 0, 4, 0, 5] ] \$

Â» SYNC'D 63/512 , 0.1230468750

15 . Coloring, {4, 6}

**R:** [2, 4, 4, 3, 6, 4] **B:** [3, 6, 5, 2, 1, 5]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

' [ '6' ('3 + τ<sup>2</sup> ' )'' (' - 1 + τ ' )'<sup>2</sup> , 12' (' - 1 + τ ' )'' (' 1 + τ ' )'' (' - 3 + τ ' )' , 12' ('3 + 2τ + 4τ<sup>2</sup> - 2τ<sup>3</sup> + τ<sup>4</sup> ' )' , 6' ('9 - τ - τ<sup>2</sup> + τ<sup>3</sup> ' )'' (' 1 + τ ' )' , -12' ('3 + τ<sup>2</sup> ' )'' (' - 1 + τ ' )' , -12' ('3 - 2τ + τ<sup>2</sup> ' )'' (' - 1 + τ ' )'' (' 1 + τ ' )'' ]'

For τ=1/2, [13, 60, 154, 201, 52, 54] . FixedPtCheck, [13, 60, 154, 201, 52, 54]

det(A + τ Δ) = 0

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	5 vs 5	5 vs 5	2 vs 4	5 vs 5

bi =

$$\begin{aligned} & \$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 1/4, 0, 3/4], [0, 0, 0, 1/4, 3/4, 0], [0, 3/4, 1/4, 0, 0, 0], [3/4, 0, 0, 0, 0, 0, \\ & 1/4], [0, 0, 0, 1/4, 3/4, 0] ] \$ \times \$ [ [99/100, 0, 0, -9/100, 3/100, 3/100], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], \\ & [-9/100, 0, 0, 19/100, 27/100, 27/100], [3/100, 0, 0, 27/100, 91/100, -9/100], [3/100, 0, 0, 27/100, -9/100, \\ & 91/100] ] \$ = \\ & \$ [ [7/148, 6/37, -142/111, -96/37, 416/111], [-183/1036, 340/259, 138/259, -1520/777, 96/259], \\ & [-303/1036, -6/259, 738/259, 880/777, -928/259], [801/1036, -138/259, -638/259, -160/259, 2272/777], \\ & [81/1036, -142/259, -1474/777, 496/259, 416/777], [-303/1036, -6/259, 738/259, 880/777, -928/259] ] \$ \\ & \times \$ [ [3/2, 5/2, 3/2, 3/2, 3, 2], [9/4, 3/2, 3/2, 3/2, 21/8, 21/8], [63/32, 27/16, 33/16, 45/32, 99/32, 57/32], \\ & [297/128, 99/64, 117/64, 177/128, 369/128, 261/128], [1107/512, 207/128, 267/128, 693/512, 1485/512, \\ & 963/512] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 4}}, net cycles: -1 . order: 2

$$\begin{aligned} & \$ [ [0, 1, 3, 6, 0, 2], [0, 0, 6, 6, 0, 0], [0, 0, 6, 6, 0, 0], [0, 0, 6, 6, 0, 0] ] \$ \\ & [0, y_2, y_1, 3y_2 + y_1, 0, 2y_2] \end{aligned}$$

$$p = s^2 - s^4 \quad p' = s^2 - s^3$$

Omega Rank for B : cycles: {{1, 3, 5}}, net cycles: 0 . order: 3

$$[y_1, y_2, y_3, 0, y_4, y_5]$$

$$\begin{aligned} B = & \$ [ [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0], [0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, \\ & 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, \\ & 0, 0, 1] ] \$ = \$ [ [0, 0, 1/36, -11/36, 13/36], [0, 1/3, 1/36, -11/36, 1/36], [0, 0, 13/36, 1/36, -11/36], [1/3, \\ & -2/9, -11/36, 1/36, 1/4], [0, 0, -11/36, 13/36, 1/36], [0, 0, 13/36, 1/36, -11/36] ] \$ \times \$ [ [2, 3, 1, 0, 4, 2], \\ & [4, 0, 2, 0, 3, 3], [3, 0, 4, 0, 5, 0], [5, 0, 3, 0, 4, 0], [4, 0, 5, 0, 3, 0] ] \$ \end{aligned}$$

Â» SYNC'D 3/64 , 0.04687500000

16 . Coloring, {5, 6}

**R:** [2, 4, 4, 2, 1, 4]    **B:** [3, 6, 5, 3, 6, 5]

‘ See graph

‘ ‘ See pair graph

‘

Ω for A+τΔ :

$$\begin{aligned} & [ [ -3' ( ' 3 + \tau ' ) ' ( ' 1 + \tau ' ) ' ( ' - 1 + \tau ' ) ' ^ 2 , 6' ( ' 1 + \tau ' ) ' ^ 2 ' ( ' - 3 + \tau ' ) ' , -6' ( ' 1 + \tau ' ) ' ( ' - 1 \\ & + \tau ' ) ' ( ' - 3 + \tau ' ) ' , 3' ( ' 1 + \tau ' ) ' ( ' - 9 - \tau + \tau ^ 2 + \tau ^ 3 ' ) ' , -6' ( ' 3 + \tau ' ) ' ( ' - 1 + \tau ' ) ' ^ 2 , 6' ( ' 3 + \tau \end{aligned}$$

$$2 \cdot (\tau - 1 + \tau^2)$$

For  $\tau=1/2$ ,  $[-21, -180, -60, -219, -28, -52]$  . FixedPtCheck,  $[21, 180, 60, 219, 28, 52]$

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	4 vs 4	4 vs 4	3 vs 3	3 vs 3

bi =

$$\begin{aligned} & \$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 1/4, 0, 3/4], [0, 0, 0, 1/4, 3/4, 0], [0, 1/4, 3/4, 0, 0, 0], [1/4, 0, 0, 0, 0, 0, \\ & 3/4], [0, 0, 0, 1/4, 3/4, 0] ] \$ \times \$ [ [11/20, 0, 0, -9/20, 3/20, 3/20], [0, 1/10, 3/10, 0, 0, 0], [0, 3/10, 9/10, 0, \\ & 0, 0], [-9/20, 0, 0, 11/20, 3/20, 3/20], [3/20, 0, 0, 3/20, 19/20, -1/20], [3/20, 0, 0, 3/20, -1/20, 19/20] ] \$ \\ & = \end{aligned}$$

$$\begin{aligned} & \$ [ [1/4, -7/6, -1/3, 4/3], [1/4, 11/6, 2/3, -8/3], [-1/12, 1/18, -1/3, 4/9], [1/4, -7/6, -1/3, 4/3], [-1/12, \\ & 7/18, 2/3, -8/9], [-1/12, 1/18, -1/3, 4/9] ] \$ \times \$ [ [1/2, 1, 3, 3/2, 3, 3], [3/4, 1/2, 3/2, 7/4, 9/2, 3], [9/8, 5/8, \\ & 15/8, 5/4, 27/8, 15/4], [27/32, 19/32, 57/32, 25/16, 135/32, 3] ] \$ \end{aligned}$$

Check x AllOnes:  $[1, 1, 1, 1, 1, 1]$

Omega Rank for R : cycles:  $\{\{2, 4\}\}$ , net cycles: 0 . order: 2

$$[y_1, y_2, 0, y_3, 0, 0]$$

$$\begin{aligned} \mathbf{R} = & \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, \\ & 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, \\ & 0, 0, 0] ] \$ = \$ [ [0, 1/6, -1/12], [0, -1/12, 1/6], [0, -1/12, 1/6], [0, 1/6, -1/12], [1/2, -1/12, -1/3], [0, \\ & -1/12, 1/6] ] \$ \times \$ [ [2, 4, 0, 6, 0, 0], [0, 8, 0, 4, 0, 0], [0, 4, 0, 8, 0, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles:  $\{\{5, 6\}\}$ , net cycles: 0 . order: 2

$$[0, 0, y_1, 0, y_2, y_3]$$

$$\begin{aligned} \mathbf{B} = & \$ [ [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, \\ & 0] ] \$ \times \$ [ [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, \\ & 0, 0, 1] ] \$ = \$ [ [1/4, -1/12, -1/12], [0, -1/12, 1/6], [0, 1/6, -1/12], [1/4, -1/12, -1/12], [0, -1/12, 1/6], [0, \\ & 1/6, -1/12] ] \$ \times \$ [ [0, 0, 4, 0, 4, 4], [0, 0, 0, 0, 8, 4], [0, 0, 0, 0, 4, 8] ] \$ \end{aligned}$$

$\hat{A} \gg \text{SYNC'D } 1/4, 0.2500000000$

17 . Coloring,  $\{2, 3, 4\}$

**R:**  $[2, 6, 5, 3, 6, 5]$  **B:**  $[3, 4, 4, 2, 1, 4]$

' See graph

' ' See pair graph

Ω for A+τΔ :

$$\begin{aligned} & [ ' 6 ' ( ' 1 + \tau ' ) ' ^ 2 ( ' - 1 + \tau ' ) ' ( ' - 3 + \tau ' ) ' , -12 ' ( ' 3 + \tau ^ 2 ' ) ' ( ' - 1 + \tau ' ) ' , 12 ' ( ' 1 + \tau ' ) ' ( ' - 3 + \tau ^ 2 ' ) ' ( ' - 1 + \tau ' ) ' , 6 ' ( ' - 1 + \tau ' ) ' ( ' - 9 - \tau + \tau ^ 2 + \tau ^ 3 ' ) ' , -12 ' ( ' 1 + \tau ' ) ' ^ 2 ( ' - 3 + \tau ' ) ' , \\ & -12 ' ( ' 1 + \tau ' ) ' ( ' - 3 - \tau - \tau ^ 2 + \tau ^ 3 ' ) ' ] ' \end{aligned}$$

For τ=1/2, [45, 52, 66, 73, 180, 174] . FixedPtCheck, [45, 52, 66, 73, 180, 174]

$$\det(A + \tau \Delta) = 0$$

Delta Range : [-y<sub>2</sub> - y<sub>3</sub> - y<sub>4</sub>, -y<sub>1</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>]

$$[1, 2, 2, 3, 2, 2]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

$$\begin{aligned} \$ [ [0, 1, 3, 0, 4, 4] , [0, 6, 2, 4, 7, 5] , [1, 8, 8, 11, 7, 13] , [9, 14, 18, 19, 21, 15] ] \$ \quad \$ [ [2, 3, 1, 6, 0, 0] , \\ [4, 2, 6, 8, 1, 3] , [7, 8, 8, 13, 9, 3] , [7, 18, 14, 29, 11, 17] ] \$ \quad \$ [ [-1, -1, 1, -3, 2, 2] , [-2, 2, -2, -2, 3, 1] , \\ [-3, 0, 0, -1, -1, 5] , [1, -2, 2, -5, 5, -1] ] \$ \end{aligned}$$

$$[y_2, y_1, -y_1, 7y_2 + 4y_1 + 4y_3, -8y_2 - 4y_1 - 5y_3, y_3]$$

$$p = s - 2s^3 - 4s^4$$

$$S+ \quad \backslash ; \quad S- \quad \backslash ; \quad NM$$

$$\begin{aligned} \$ [ [1, 3, 3, 3, 2, 2] , [1, 2, 2, 4, 2, 3] , [1, 2, 2, 4, 3, 2] , [1, 3, 3, 3, 2, 2] , [2, 2, 2, 3, 2, 3] , [1, 2, 2, 4, 3, 2] ] \\ \$ \quad \$ [ [1, 3, 3, 3, 2, 2] , [1, 2, 2, 4, 2, 3] , [1, 2, 2, 4, 3, 2] , [1, 3, 3, 3, 2, 2] , [2, 2, 2, 3, 2, 3] , [1, 2, 2, 4, 3, \\ 2] ] \$ \quad \$ [ [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, \\ 0, 0, 0] ] \$ \end{aligned}$$

CmmCk true, true, true

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
3 vs 4	5 vs 5	5 vs 5	3 vs 4	4 vs 4

bi =

$$\begin{aligned} \$ [ [0, 1/4, 3/4, 0, 0, 0] , [0, 0, 0, 3/4, 0, 1/4] , [0, 0, 0, 3/4, 1/4, 0] , [0, 3/4, 1/4, 0, 0, 0] , [3/4, 0, 0, 0, 0, \\ 1/4] , [0, 0, 0, 3/4, 1/4, 0] ] \$ \times \$ [ [19/20, 0, 0, -1/20, 3/20, 3/20] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , \\ [-1/20, 0, 0, 19/20, 3/20, 3/20] , [3/20, 0, 0, 3/20, 11/20, -9/20] , [3/20, 0, 0, 3/20, -9/20, 11/20] ] \$ = \\ \$ [ [-415/1964, 480/491, 158/491, -1160/1473, -320/1473] , [69/3928, -195/3928, -219/491, \\ -785/1473, 1612/1473] , [57/3928, -1015/3928, 673/491, 547/1473, -2084/1473] , [-211/1964, 37/491, \\ -166/1473, 688/1473, -352/1473] , [2829/3928, -139/3928, -1123/491, -761/1473, 3244/1473] , [57/3928, \\ -1015/3928, 673/491, 547/1473, -2084/1473] ] \$ \quad \times \$ [ [3/2, 5/2, 3/2, 9/2, 1, 1] , [3/4, 15/4, 9/4, 15/4, 5/8, \end{aligned}$$

7/8], [15/32, 3, 3/2, 165/32, 25/32, 35/32], [75/128, 255/64, 105/64, 537/128, 83/128, 121/128], [249/512, 843/256, 381/256, 2523/512, 331/512, 593/512] ] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{5, 6}}, net cycles: -1 . order: 2

\$ [ [0, 1, 3, 0, 4, 4], [0, 0, 0, 0, 7, 5], [0, 0, 0, 0, 5, 7], [0, 0, 0, 0, 7, 5] ] \$

[0, y<sub>3</sub>, 3 y<sub>3</sub>, 0, y<sub>1</sub>, y<sub>2</sub>]

$$p = -s^2 + s^4$$

Omega Rank for B : cycles: {{2, 4}}, net cycles: 0 . order: 4

[y<sub>2</sub>, y<sub>1</sub>, y<sub>3</sub>, y<sub>4</sub>, 0, 0]

B = \$ [ [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0] ] \$ = \$ [ [0, 1/2, -1/12, -1/3], [0, 0, 1/6, -1/12], [0, 0, 1/6, -1/12], [0, 0, -1/12, 1/6], [1/2, -1/4, -1/3, 1/6], [0, 0, 1/6, -1/12] ] \$ x \$ [ [2, 3, 1, 6, 0, 0], [0, 6, 2, 4, 0, 0], [0, 4, 0, 8, 0, 0], [0, 8, 0, 4, 0, 0] ] \$

Â» SYNC'D 1/8 , 0.1250000000

18 . Coloring, {2, 3, 5}

**R:** [2, 6, 5, 2, 1, 5] **B:** [3, 4, 4, 3, 6, 4]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

' [ '3' ('1 + τ')<sup>2</sup>, 6' ('1 + τ')', -6' ('-1 + τ')', -3' ('3 + τ')' ('-1 + τ')', 6' ('1 + τ')', 6' ('1 + τ')' ]'

For τ=1/2, [9, 12, 4, 7, 12, 12] . FixedPtCheck, [9, 12, 4, 7, 12, 12]

det(A + τ Δ) = 0

Delta Range : [-y<sub>2</sub> - y<sub>3</sub> - y<sub>4</sub>, -y<sub>1</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>]

[1, 2, 2, 3, 2, 2]

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

$$\begin{aligned} & \$ [ [2, 4, 0, 0, 4, 2], [2, 1, 3, 3, 1, 2], [1, 5, 3, 6, 5, 4], [5, 7, 9, 12, 7, 8] ] \$ \quad \$ [ [0, 0, 4, 6, 0, 2], [0, 3, 1, \\ & 3, 3, 2], [3, 3, 5, 6, 3, 4], [3, 9, 7, 12, 9, 8] ] \$ \quad \$ [ [1, 2, -2, -3, 2, 0], [1, -1, 1, 0, -1, 0], [-1, 1, -1, 0, 1, 0] \\ & , [1, -1, 1, 0, -1, 0] ] \$ \end{aligned}$$

$$[-y_2 - y_1, y_2, -y_2, y_1, y_2, 0]$$

$$p = s^2 - 4s^4$$

$$S+ \quad \backslash ; \quad S- \quad \backslash ; \quad NM$$

$$\begin{aligned} & \$ [ [0, 2, 0, 4, 1, 3], [0, 1, 3, 1, 3, 2], [2, 1, 3, 1, 2, 1], [0, 2, 0, 4, 1, 3], [1, 3, 1, 4, 1, 0], [2, 1, 3, 1, 2, 1] ] \\ & \$ \quad \$ [ [1, 0, 2, 3, 3, 1], [1, 3, 1, 4, 1, 0], [1, 3, 1, 2, 0, 3], [1, 0, 2, 3, 3, 1], [0, 1, 3, 1, 3, 2], [1, 3, 1, 2, 0, \\ & 3] ] \$ \quad \$ [ [3, 2, 0, 9, 4, 0], [1, 6, 4, 3, 0, 4], [0, 4, 6, 0, 2, 6], [3, 2, 0, 9, 4, 0], [2, 0, 2, 6, 6, 2], [0, 4, 6, \\ & 0, 2, 6] ] \$ \end{aligned}$$

CmmCk *true, true, true*

$$p' = s^2 + 2s^3$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
2 vs 4	2 vs 4	2 vs 4	2 vs 4	2 vs 3

Omega Rank for R : cycles:  $\{\{1, 2, 5, 6\}\}$ , net cycles: 1 . order: 4

$$\begin{aligned} & \$ [ [2, 4, 0, 0, 4, 2], [4, 2, 0, 0, 2, 4], [2, 4, 0, 0, 4, 2], [4, 2, 0, 0, 2, 4] ] \$ \\ & [y_2, y_1, 0, 0, y_1, y_2] \end{aligned}$$

$$p = -s + s^3 \quad p' = -s + s^3$$

Omega Rank for B : cycles:  $\{\{3, 4\}\}$ , net cycles: 0 . order: 2

$$\begin{aligned} & \$ [ [0, 0, 4, 6, 0, 2], [0, 0, 6, 6, 0, 0], [0, 0, 6, 6, 0, 0] ] \$ \\ & [0, 0, y_1 - y_2, y_1, 0, y_2] \end{aligned}$$

$$p = -s^2 + s^3$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[0, x_1, x_2, -4x_1 + 2x_2]$$

$$\text{For } A+2\Delta: [y_4, y_3, 3y_4 - 3y_3 + 4y_1 - 3y_2, -3y_4 - 4y_1, y_2, y_1]$$

$$\text{For } A-2\Delta: [-3y_2 - 4y_1, y_3, y_4, y_2, -y_3 - 3y_4 - 3y_2, y_1]$$

Range of  $\{\Omega\Delta^i\}$ :  $[\mu_1, \mu_2, -\mu_2, -\mu_1 - \mu_2, \mu_2, 0]$

rank of M is 6 , rank of N is 3

M N

\$ [ [0, 0, 0, 0, 0, 1] , [0, 0, 0, 0, 2, 0] , [0, 0, 0, 2, 0, 0] , [0, 0, 2, 0, 0, 1] , [0, 2, 0, 0, 0, 0] , [1, 0, 0, 1, 0, 0] ]  
 \$ \$ [ [0, 2, 3, 0, 1, 3] , [2, 0, 1, 2, 3, 1] , [3, 1, 0, 3, 2, 0] , [0, 2, 3, 0, 1, 3] , [1, 3, 2, 1, 0, 2] , [3, 1, 0, 3, 2, 0] ] \$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [1, 2, -2, -3, 2, 0]$

ker M, [0, 0, 0, 0, 0, 0]

Range M, [ $x_1, x_4, x_3, x_2, x_6, x_5$ ]

$\tau = 18$  ,  $r' = 1/2$

Ranges

Action of R on ranges, [[2], [1], [2], [2]]

Action of B on ranges, [[3], [4], [3], [3]]

$\beta(\{1, 6\}) = 1/6$

$\beta(\{2, 5\}) = 1/3$

$\beta(\{3, 4\}) = 1/3$

$\beta(\{4, 6\}) = 1/6$

ker N, [ $\mu_3, -\mu_3 - \mu_1, \mu_2, \mu_1, -\mu_3 - \mu_1, \mu_3 + \mu_1 - \mu_2$ ]

Range of N

[ $y_2, y_2 - y_3 + y_1, y_1, y_2, y_3, y_1$ ]

Partitions

Action of R on partitions, [[2], [1]]

Action of B on partitions, [[2], [2]]

$\alpha(\{\{1, 2, 4\}, \{3, 5, 6\}\}) = 1/3$

$\alpha(\{\{2, 3, 6\}, \{1, 4, 5\}\}) = 2/3$

$b_1 = \{2, 3, 6\}$  , ,  $b_2 = \{1, 4, 5\}$  , ,  $b_3 = \{1, 2, 4\}$  , ,  $b_4 = \{3, 5, 6\}$

Action of R and B on the blocks of the partitions: \$ [ [0, 1, 1, 0] , [1, 0, 0, 1] , [1, 1, 0, 0] , [1, 1, 0, 0] ] \$ =  
 \$ [ [0, 0, 1, 0] , [0, 0, 0, 1] , [0, 1, 0, 0] , [1, 0, 0, 0] ] \$ + \$ [ [0, 1, 0, 0] , [1, 0, 0, 0] , [1, 0, 0, 0] , [0, 1, 0, 0] ] \$

[‘3’, ‘4’, ‘2’, ‘1’], [‘2’, ‘1’, ‘1’, ‘2’] with invariant measure [2, 2, 1, 1]

N by blocks, check: true . ‘ See partition graph.

‘ ‘ See level-2 partition graph.



Sandwich	
<b>Coloring</b>	{2, 3, 5}
<b>Rank</b>	2
<b>R,B</b>	[2, 6, 5, 2, 1, 5], [3, 4, 4, 3, 6, 4]
$\pi_2$	[0, 0, 0, 0, 1, 0, 0, 2, 0, 2, 0, 0, 1, 0]
$u_2$	[2, 3, 0, 1, 3, 1, 2, 3, 1, 3, 2, 0, 1, 3, 2] (dim 1)
<b>wpp</b>	[3, 3, 3, 3, 3, 3]

19 . Coloring, {2, 3, 6}

**R:** [2, 6, 5, 2, 6, 4]    **B:** [3, 4, 4, 3, 1, 5]

‘ See graph

‘ ‘ See pair graph

$\Omega$  for  $A+\tau\Delta$  :

$[-3(1+\tau)^2(3+\tau)^2(-1+\tau)^2, -6(1+\tau)^2(3+\tau)^2, 6(-1+\tau)^2(3+\tau)^2, 3(-9-2\tau-8\tau^2+2\tau^3+\tau^4), 6(1+\tau)^2(3+\tau)^2(-1+\tau)^2, 6(1+\tau)^2(-3+\tau)^2]$

For  $\tau=1/2$ , [-21, -156, -52, -187, -84, -180] . FixedPtCheck, [21, 156, 52, 187, 84, 180]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	5 vs 5	4 vs 5	4 vs 4	3 vs 4

Omega Rank for R : cycles: {{2, 4, 6}}, net cycles: 0 . order: 3

$[0, y_1, 0, y_4, y_2, y_3]$

$R = \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ \times \$ [ [0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, -5/36, 1/36, 7/36], [0, 7/36, -5/36, 1/36], [1/2, -5/36, 1/36, -11/36], [0, -5/36, 1/36,$

$$7/36], [0, 7/36, -5/36, 1/36], [0, 1/36, 7/36, -5/36]] \times \$ [ [0, 4, 0, 2, 2, 4], [0, 2, 0, 4, 0, 6], [0, 4, 0, 6, 0, 2], [0, 6, 0, 2, 0, 4]] \$$$

Omega Rank for B : cycles: {{3, 4}}, net cycles: 0 . order: 4

$$\$ [ [2, 0, 4, 4, 2, 0], [2, 0, 6, 4, 0, 0], [0, 0, 6, 6, 0, 0], [0, 0, 6, 6, 0, 0]] \$$$

$$[y_2 - y_3 + y_1, 0, y_1, y_3, y_2, 0]$$

$$p = -s^3 + s^4$$

Â» SYNC'D 1/4 , 0.2500000000

20 . Coloring, {2, 4, 5}

**R:** [2, 6, 4, 3, 1, 5] **B:** [3, 4, 5, 2, 6, 4]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[ '6' ( '3 + \tau' )^{ '2' } , 36' ( '3 + 2\tau + 3\tau^2' )^{ '2' } , 12' ( '1 + \tau' )^{ '2' } ( '3 + 2\tau + \tau^2' )^{ '2' } , 6' ( '9 + 7\tau + 7\tau^2 + \tau^3' )^{ '2' } , 12' ( '3 + \tau' )^{ '2' } ( '1 + \tau' )^{ '2' } , 12' ( '1 + \tau' )^{ '2' } ( '3 + \tau^2' )^{ '2' } ]^{ '2' }$$

For τ=1/2, [63, 76, 102, 115, 84, 78] . FixedPtCheck, [63, 76, 102, 115, 84, 78]

$$\det(A + \tau \Delta) = 1' ( ' \tau ' )^{ '2' } ( ' 1 + \tau ' )^{ '2' }$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	6 vs 6	6 vs 6	4 vs 6	4 vs 5

bi =

$$\$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 3/4, 0, 1/4], [0, 0, 0, 1/4, 3/4, 0], [0, 3/4, 1/4, 0, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 0, 0, 3/4, 1/4, 0]] \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \$ =$$

$$\$ [ [865/144, 463/144, -49/12, -377/6, -688/9, 1208/9], [-7/72, -103/72, 13/6, -7/3, 128/9, -112/9], [-199/72, -295/72, 13/6, 137/3, 512/9, -880/9], [193/144, 367/144, -49/12, -89/6, -304/9, 440/9], [229/144, 667/144, -37/12, -173/6, -304/9, 536/9], [-467/144, -653/144, 83/12, 235/6, 464/9, -808/9]] \$ \times \$ [ [1/2, 5/2, 3/2, 7/2, 2, 2], [1/2, 11/4, 5/4, 15/4, 13/8, 17/8], [13/32, 47/16, 21/16, 127/32, 47/32, 61/32], [47/128, 197/64, 83/64, 507/128, 187/128, 235/128], [187/512, 49/16, 81/64, 2053/512, 733/512, 955/512], [733/2048, 3173/1024, 1307/1024, 8217/2048, 2899/2048, 3767/2048]] \$$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 4}, {1, 2, 5, 6}}, net cycles: 2 . order: 4

\$ [ [2, 1, 3, 2, 2, 2], [2, 2, 2, 3, 2, 1], [2, 2, 3, 2, 1, 2], [1, 2, 2, 3, 2, 2], [2, 1, 3, 2, 2, 2], [2, 2, 2, 3, 2, 1] ] \$

$$[y_1, 6y_1 - 7y_2 - y_3 + 6y_4, y_2, 5y_1 - 6y_2 + 5y_4, y_3, y_4]$$

$$p' = -1 + s^4 \quad p' = -s + s^5$$

Omega Rank for B : cycles: {{2, 4}}, net cycles: 0 . order: 4

\$ [ [0, 3, 1, 4, 2, 2], [0, 4, 0, 5, 1, 2], [0, 5, 0, 6, 0, 1], [0, 6, 0, 6, 0, 0], [0, 6, 0, 6, 0, 0] ] \$

$$[0, -y_1 + y_4 + y_2 - y_3, y_1, y_4, y_2, y_3]$$

$$p = -s^4 + s^5$$

Â» SYNC'D 3/128 , 0.02343750000

21 . Coloring, {2, 4, 6}

**R:** [2, 6, 4, 3, 6, 4] **B:** [3, 4, 5, 2, 1, 5]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$[ '6' (' - 1 + \tau ' ) ^ 2 , -12' (' - 1 + \tau ' ) ^ 4 , 12' (' 1 + \tau ^ 2 ' ) ^ 4 , 6' (' 3 + \tau ^ 2 ' ) ^ 4 , -12' (' - 1 + \tau ' ) ^ 4 , -12' (' - 1 + \tau ' ) ^ 4 (' 1 + \tau ' ) ^ 4 ] ^ 4$$

For τ=1/2, [1, 4, 10, 13, 4, 6] . FixedPtCheck, [1, 4, 10, 13, 4, 6]

$$\det(A + \tau \Delta) = 0$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	5 vs 5	5 vs 5	4 vs 4	4 vs 5

bi =

\$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 3/4, 0, 1/4], [0, 0, 0, 1/4, 3/4, 0], [0, 3/4, 1/4, 0, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 0, 0, 1/4, 3/4, 0] ] \$ x \$ [ [91/100, 0, 0, -9/100, 3/100, 27/100], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [-9/100, 0, 0, 91/100, 3/100, 27/100], [3/100, 0, 0, 3/100, 99/100, -9/100], [27/100, 0, 0, 27/100, -9/100,

19/100] ] \$ =

\$ [ [-493/308, -661/231, -722/693, 128/77, 2720/693] , [843/308, 547/77, 1090/231, -944/231, -800/77] , [-621/308, -375/77, -226/77, 976/231, 1312/231] , [1107/308, 481/77, 914/231, -1472/231, -1696/231] , [-123/44, -175/33, -422/99, 48/11, 800/99] , [-621/308, -375/77, -226/77, 976/231, 1312/231] ] \$ x \$ [ [3/2, 5/2, 3/2, 5/2, 3, 1] , [9/4, 9/4, 7/4, 5/2, 15/8, 11/8] , [45/32, 39/16, 37/16, 79/32, 75/32, 33/32] , [225/128, 141/64, 107/64, 341/128, 321/128, 153/128] , [963/512, 39/16, 127/64, 1213/512, 1101/512, 603/512] ] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 4}}, net cycles: 0 . order: 4

[0, y<sub>3</sub>, y<sub>2</sub>, y<sub>1</sub>, 0, y<sub>4</sub>]

R = \$ [ [0, 1, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1] , [0, 0, 0, 1, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 0, 0, 1] , [0, 0, 0, 1, 0, 0] ] \$ x \$ [ [0, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [1, -4, -29/24, 103/24] , [0, 1, 7/24, -29/24] , [0, 0, -5/24, 7/24] , [0, 0, 7/24, -5/24] , [0, 1, 7/24, -29/24] , [0, 0, -5/24, 7/24] ] \$ x \$ [ [0, 1, 3, 4, 0, 4] , [0, 0, 4, 7, 0, 1] , [0, 0, 7, 5, 0, 0] , [0, 0, 5, 7, 0, 0] ] \$

Omega Rank for B : cycles: {{2, 4}, {1, 3, 5}}, net cycles: 2 . order: 6

\$ [ [2, 3, 1, 2, 4, 0] , [4, 2, 2, 3, 1, 0] , [1, 3, 4, 2, 2, 0] , [2, 2, 1, 3, 4, 0] , [4, 3, 2, 2, 1, 0] ] \$

[7 y<sub>1</sub> - 5 y<sub>2</sub> + 7 y<sub>3</sub> - 5 y<sub>4</sub>, 5 y<sub>1</sub>, 5 y<sub>2</sub>, 5 y<sub>3</sub>, 5 y<sub>4</sub>, 0]

p = -s - s<sup>2</sup> + s<sup>4</sup> + s<sup>5</sup>

Â» SYNC'D 63/512 , 0.1230468750

22 . Coloring, {2, 5, 6}

**R:** [2, 6, 4, 2, 1, 4] **B:** [3, 4, 5, 3, 6, 5]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

' [ '3' ('1 + τ')' ('3 + τ<sup>2</sup>')' ('-1 + τ')' , 6' ('1 + τ')'<sup>2</sup> ' ('-3 + τ')' , -6' ('1 + τ')' ('-1 + τ')' ('-3 + τ')' , -3' ('1 + τ')' ('9 - τ - τ<sup>2</sup> + τ<sup>3</sup>')' , 6' ('3 + τ<sup>2</sup>')' ('-1 + τ')' , 6' ('-3 - τ - 5τ<sup>2</sup> + τ<sup>3</sup>')' ]'

For τ=1/2, [-39, -180, -60, -201, -52, -148] . FixedPtCheck, [39, 180, 60, 201, 52, 148]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	4 vs 4	3 vs 4	4 vs 4	3 vs 4

Omega Rank for R : cycles: {{2, 4, 6}}, net cycles: 0 . order: 3

$$[y_3, y_4, 0, y_1, 0, y_2]$$

$$R = \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 7/36, -5/36, 1/36], [0, 1/36, 7/36, -5/36], [0, -5/36, 1/36, 7/36], [0, 7/36, -5/36, 1/36], [1/2, -5/36, 1/36, -11/36], [0, -5/36, 1/36, 7/36] ] \$ \times \$ [ [2, 4, 0, 4, 0, 2], [0, 6, 0, 2, 0, 4], [0, 2, 0, 4, 0, 6], [0, 4, 0, 6, 0, 2] ] \$$$

Omega Rank for B : cycles: {{5, 6}}, net cycles: 0 . order: 4

$$\$ [ [0, 0, 4, 2, 4, 2], [0, 0, 2, 0, 6, 4], [0, 0, 0, 0, 6, 6], [0, 0, 0, 0, 6, 6] ] \$$$

$$[0, 0, y_1 + y_3 - y_2, y_1, y_3, y_2]$$

$$p = s^3 - s^4$$

$\hat{A}$ » SYNC'D 1/4 , 0.2500000000

23 . Coloring, {3, 4, 5}

**R:** [2, 4, 5, 3, 1, 5] **B:** [3, 6, 4, 2, 6, 4]

' See graph

' ' See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

$$[ '6' ( '1 + \tau' ) ^ 2 ' ( '3 + \tau ^ 2' ) ^ , 12' ( '3 - \tau + 5\tau ^ 2 + \tau ^ 3' ) ^ , 12' ( '3 - \tau + \tau ^ 2 + \tau ^ 3' ) ^ ' ( '1 + \tau' ) ^ , 6' ( '9 - 4\tau + 6\tau ^ 2 + 4\tau ^ 3 + \tau ^ 4' ) ^ , 12' ( '1 + \tau' ) ^ ' ( '3 + \tau ^ 2' ) ^ , -12' ( '1 + \tau ^ 2' ) ^ ' ( '3 + \tau' ) ^ ' ( ' - 1 + \tau' ) ^ ' ] ^$$

For  $\tau=1/2$ , [117, 124, 138, 145, 156, 70] . FixedPtCheck, [117, 124, 138, 145, 156, 70]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	5 vs 5	5 vs 5	5 vs 5	4 vs 4

bi =

$$\begin{aligned} & \$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 1/4, 0, 3/4], [0, 0, 0, 3/4, 1/4, 0], [0, 3/4, 1/4, 0, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 0, 0, 3/4, 1/4, 0] ] \$ \times \$ [ [91/100, 0, 0, -9/100, 27/100, 3/100], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], \\ & [-9/100, 0, 0, 91/100, 27/100, 3/100], [27/100, 0, 0, 27/100, 19/100, -9/100], [3/100, 0, 0, 3/100, -9/100, 99/100] ] \$ = \end{aligned}$$

$$\begin{aligned} & \$ [ [-4127/5532, 7381/1383, 12358/1383, 3520/1383, -22112/1383], [-119/5532, -509/1383, 670/1383, 3088/1383, -3104/1383], [961/5532, -515/1383, -4202/1383, -1136/1383, 5728/1383], \\ & [-2495/5532, 1717/1383, 4750/1383, -896/1383, -4832/1383], [6769/5532, -4727/1383, -5570/1383, -1232/1383, 9952/1383], [961/5532, -515/1383, -4202/1383, -1136/1383, 5728/1383] ] \$ \times \$ [ [1/2, 5/2, 3/2, 7/2, 1, 3], [1/4, 11/4, 5/4, 4, 9/8, 21/8], [9/32, 49/16, 19/16, 115/32, 31/32, 93/32], [31/128, 177/64, 71/64, 491/128, 131/128, 387/128], [131/512, 47/16, 73/64, 1941/512, 529/512, 1455/512] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 2, 3, 4, 5}}, net cycles: 1 . order: 5

$$[y_1, y_2, y_3, y_4, y_5, 0]$$

$$\begin{aligned} R = & \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], \\ & [0, 0, 0, 0, 0, 1] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [-17/132, -29/132, 67/132, -41/132, 31/132], [31/132, -17/132, -29/132, 67/132, -41/132], \\ & [67/132, -41/132, 31/132, -17/132, -29/132], [-41/132, 31/132, -17/132, -29/132, 67/132], [-29/132, 67/132, -41/132, 31/132, -17/132], \\ & [67/132, -41/132, 31/132, -17/132, -29/132] ] \$ \times \$ [ [2, 1, 3, 2, 4, 0], [4, 2, 2, 1, 3, 0], [3, 4, 1, 2, 2, 0], [2, 3, 2, 4, 1, 0], [1, 2, 4, 3, 2, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{2, 4, 6}}, net cycles: 0 . order: 3

$$[0, y_4, y_3, y_2, 0, y_1]$$

$$\begin{aligned} B = & \$ [ [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], \\ & [0, 0, 0, 0, 0, 0] ] \$ \times \$ [ [0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [1, -11/36, 1/36, -23/36], [0, -11/36, 1/36, 13/36], [0, 13/36, -11/36, 1/36], [0, 1/36, 13/36, -11/36], [0, -11/36, 1/36, 13/36], [0, 13/36, -11/36, 1/36] ] \$ \times \$ [ [0, 3, 1, 4, 0, 4], [0, 4, 0, 5, 0, 3], \\ & [0, 5, 0, 3, 0, 4], [0, 3, 0, 4, 0, 5] ] \$ \end{aligned}$$

Â» SYNC'D 75/512 , 0.1464843750

24 . Coloring, {3, 4, 6}

**R:** [2, 4, 5, 3, 6, 4] **B:** [3, 6, 4, 2, 1, 5]

‘ See graph

‘ ‘ See pair graph

$\Omega$  for  $A+\tau\Delta$  :

$$\left[ \begin{array}{c} 6(\tau - 1 + \tau^2) (\tau^3 - 3\tau - \tau^2 - 5\tau^2 + \tau^3), -36(\tau - 1 + \tau^2) (\tau^3 + 2\tau + 3\tau^2), 12(\tau^3 + 2\tau + 4\tau^2 - 2\tau^3 + \tau^4), \\ 6(9 + 4\tau + 6\tau^2 - 4\tau^3 + \tau^4), -12(\tau^3 - 3\tau - \tau^2 - 5\tau^2 + \tau^3), 12(\tau + \tau^2) (\tau^3 + \tau^2) \end{array} \right]$$

For  $\tau=1/2$ , [37, 76, 154, 193, 148, 130] . FixedPtCheck, [37, 76, 154, 193, 148, 130]

$$\det(A + \tau \Delta) = (\tau - 1 + \tau^2)^2$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	6 vs 6	6 vs 6	4 vs 5	5 vs 6

bi =

$$\begin{aligned} & \$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 1/4, 0, 3/4], [0, 0, 0, 3/4, 1/4, 0], [0, 3/4, 1/4, 0, 0, 0], [3/4, 0, 0, 0, 0, 1/4], \\ & [0, 0, 0, 1/4, 3/4, 0] ] \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, \\ & 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [379/528, -259/144, 373/1188, -1175/594, 2128/297, -1288/297], [-305/264, 203/72, -665/594, \\ & -89/297, -1600/297, 1552/297], [-113/264, 11/72, -1337/594, -329/297, -1216/297, 2320/297], \\ & [1051/528, -163/144, 1717/1188, -695/594, 1744/297, -2056/297], [-833/528, 137/144, -2975/1188, \\ & 1957/594, -1328/297, 1304/297], [167/528, -191/144, 4217/1188, 509/594, 464/297, -1448/297] ] \times \$ [ \\ & [3/2, 5/2, 3/2, 5/2, 2, 2], [3/2, 9/4, 7/4, 9/4, 15/8, 19/8], [45/32, 33/16, 27/16, 79/32, 71/32, 69/32], \\ & [213/128, 141/64, 107/64, 297/128, 261/128, 269/128], [783/512, 69/32, 117/64, 1193/512, 1021/512, \\ & 1107/512], [3063/2048, 2181/1024, 1771/1024, 5019/2048, 4257/2048, 4333/2048] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 4, 5, 6}}, net cycles: 0 . order: 4

$$\$ [ [0, 1, 3, 4, 2, 2], [0, 0, 4, 3, 3, 2], [0, 0, 3, 2, 4, 3], [0, 0, 2, 3, 3, 4], [0, 0, 3, 4, 2, 3] ] \$$$

$$[0, y_3, y_4, y_2, y_3 + y_4 - y_2 + y_1, y_1]$$

$$p = -s^2 + s^3 - s^4 + s^5$$

Omega Rank for B : cycles: {{1, 2, 3, 4, 5, 6}}, net cycles: 1 . order: 6

$$\$ [ [2, 3, 1, 2, 2, 2], [2, 2, 2, 1, 2, 3], [2, 1, 2, 2, 3, 2], [3, 2, 2, 2, 2, 1], [2, 2, 3, 2, 1, 2], [1, 2, 2, 3, 2, 2] ] \$$$

$$[y_3, y_4, y_5, -y_3 + y_4 + y_5 + y_1 - y_2, y_1, y_2]$$

$$p' = 1 - s + s^2 - s^3 + s^4 - s^5$$

Â» SYNC'D 9/256 , 0.03515625000

25 . Coloring, {3, 5, 6}

**R:** [2, 4, 5, 2, 1, 4] **B:** [3, 6, 4, 3, 6, 5]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

' [ '3' (' 1 + τ ')'' (' - 1 + τ ')', -6' (' 1 + τ ')', 6' (' - 1 + τ ')', -3' (' 3 + τ <sup>2</sup> ')', 6' (' - 1 + τ ')', 6' (' - 1 + τ ')'' ]'

For τ=1/2, [-3, -12, -4, -13, -4, -4] . FixedPtCheck, [3, 12, 4, 13, 4, 4]

det(A + τ Δ) = 0

Delta Range : [-y<sub>2</sub> - y<sub>3</sub> - y<sub>4</sub>, -y<sub>1</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>]

[1, 2, 2, 3, 2, 2]

+ \; - \; Δ

\$ [ [2, 4, 0, 4, 2, 0] , [1, 3, 1, 4, 2, 1] , [2, 5, 3, 7, 4, 3] , [4, 9, 7, 13, 8, 7] ] \$ \$ [ [0, 0, 4, 2, 2, 4] , [1, 1, 3, 2, 2, 3] , [2, 3, 5, 5, 4, 5] , [4, 7, 9, 11, 8, 9] ] \$ \$ [ [1, 2, -2, 1, 0, -2] , [0, 1, -1, 1, 0, -1] , [0, 1, -1, 1, 0, -1] , [0, 1, -1, 1, 0, -1] ] \$

[y<sub>2</sub>, -y<sub>1</sub>, y<sub>1</sub>, -y<sub>2</sub> - y<sub>1</sub>, 0, y<sub>1</sub>]

p = s<sup>2</sup> - 4s<sup>4</sup>

S+ \; S- \; NM

\$ [ [0, 2, 0, 2, 3, 3] , [1, 1, 3, 4, 1, 0] , [1, 3, 1, 0, 2, 3] , [0, 2, 0, 2, 3, 3] , [1, 1, 3, 4, 1, 0] , [2, 1, 3, 3, 0, 1] ] \$ \$ [ [1, 0, 2, 5, 1, 1] , [0, 3, 1, 1, 3, 2] , [2, 1, 3, 3, 0, 1] , [1, 0, 2, 5, 1, 1] , [0, 3, 1, 1, 3, 2] , [1, 3, 1, 0, 2, 3] ] \$ \$ [ [1, 0, 2, 3, 0, 0] , [0, 2, 0, 0, 2, 2] , [1, 0, 2, 3, 0, 0] , [1, 0, 2, 3, 0, 0] , [0, 2, 0, 0, 2, 2] , [0, 2, 0, 0, 2, 2] ] \$

CmmCk true, true, true

p' = s<sup>2</sup> - 2s<sup>3</sup>

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
2 vs 4	3 vs 5	3 vs 5	3 vs 4	2 vs 4



Omega Rank for R : cycles:  $\{\{2, 4\}\}$ , net cycles: 0 . order: 4

$$\$ [ [2, 4, 0, 4, 2, 0], [2, 6, 0, 4, 0, 0], [0, 6, 0, 6, 0, 0], [0, 6, 0, 6, 0, 0] ] \$$$

$$[y_1 - y_2 + y_3, y_1, 0, y_2, y_3, 0]$$

$$p = -s^3 + s^4$$

Omega Rank for B : cycles:  $\{\{5, 6\}, \{3, 4\}\}$ , net cycles: 2 . order: 2

$$\$ [ [0, 0, 4, 2, 2, 4], [0, 0, 2, 4, 4, 2], [0, 0, 4, 2, 2, 4], [0, 0, 2, 4, 4, 2] ] \$$$

$$[0, 0, y_2, y_1, y_1, y_2]$$

$$p' = -s + s^3 \quad p = -s + s^3$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[0, x_1, x_2, -4x_1 - 2x_2]$$

$$\text{For } A+2\Delta : [y_2, y_1, -3y_2 - 3y_1 - y_3, y_2, -y_2, y_3]$$

$$\text{For } A-2\Delta : [-y_2, -3y_1 + y_2 - 3y_3, y_1, -y_2, y_2, y_3]$$

$$\text{Range of } \{\Omega\Delta^i\} : [-\mu_1 - \mu_2, -\mu_2, \mu_2, \mu_1, 0, \mu_2]$$

rank of M is 6 , rank of N is 2

M            N

$$\$ [ [0, 1, 0, 0, 0, 0], [1, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 2], [0, 1, 0, 0, 2, 0], [0, 0, 0, 2, 0, 0], [0, 0, 2, 0, 0, 0] ] \$$$

$$\$ [ [0, 1, 0, 0, 1, 1], [1, 0, 1, 1, 0, 0], [0, 1, 0, 0, 1, 1], [0, 1, 0, 0, 1, 1], [1, 0, 1, 1, 0, 0], [1, 0, 1, 1, 0, 0] ] \$$$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [1, 2, -2, 1, 0, -2]$

ker M,  $[0, 0, 0, 0, 0, 0]$

Range M,  $[x_1, x_6, x_5, x_4, x_3, x_2]$

$$\tau = 18, r' = 1/2$$

Ranges

Action of R on ranges,  $[[2], [2], [4], [1]]$

Action of B on ranges,  $[[3], [3], [4], [3]]$

$$\beta(\{1, 2\}) = 1/6$$

$$\beta(\{2, 4\}) = 1/6$$

$$\beta(\{3, 6\}) = 1/3$$

$$\beta(\{4, 5\}) = 1/3$$

$$\ker N, [-\mu_1 - \mu_2, -\mu_3 - \mu_4, \mu_1, \mu_2, \mu_3, \mu_4]$$

Range of N

$$[y_2, y_1, y_2, y_2, y_1, y_1]$$

Partitions

$$\alpha(\{\{1, 3, 4\}, \{2, 5, 6\}\}) = 1/1$$

$$b_1 = \{1, 3, 4\}, b_2 = \{2, 5, 6\}$$

Action of R and B on the blocks of the partitions:  $\$ [ [1, 1], [1, 1] ] \$ = \$ [ [0, 1], [1, 0] ] \$ + \$ [ [1, 0], [0, 1] ] \$$

$['2', '1'], ['1', '2']$  with invariant measure  $[1, 1]$

N by blocks, check: true . ' See partition graph.

' ' See level-2 partition graph.

,

Right Group	
<b>Coloring</b>	{3, 5, 6}
<b>Rank</b>	2
<b>R,B</b>	[2, 4, 5, 2, 1, 4], [3, 6, 4, 3, 6, 5]
$\pi_2$	[1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 2, 2, 0, 0]
$u_2$	[1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 1, 1, 1, 1, 0] (dim 1)
<b>wpp</b>	[3, 3, 3, 3, 3, 3]

26 . Coloring, {4, 5, 6}

$$\mathbf{R}: [2, 4, 4, 3, 1, 4] \quad \mathbf{B}: [3, 6, 5, 2, 6, 5]$$

' See graph

' ' See pair graph

,

$\Omega$  for  $A+\tau\Delta$  :

$$\begin{aligned} & [ '6' ( '3 + \tau^2' ) ' ' ( ' - 1 + \tau ' ) ' ' ( ' 1 + \tau ' ) ' , 12' ( ' 3 + \tau ' ) ' ' ( ' - 1 + \tau ' ) ' ' ( ' 1 + \tau ' ) ' , 12' ( ' - 3 - \\ & \tau - \tau^2 + \tau^3 ' ) ' ' ( ' 1 + \tau ' ) ' , 6' ( ' 1 + \tau ' ) ' ' ( ' - 9 - \tau + \tau^2 + \tau^3 ' ) ' , 12' ( ' 3 + \tau^2 ' ) ' ' ( ' - 1 + \tau ' ) ' , \\ & -12' ( ' 3 + 2\tau + \tau^2 ' ) ' ' ( ' - 1 + \tau ' ) ' ^2 ' ] . \end{aligned}$$

For  $\tau=1/2$ , [-39, -84, -174, -219, -52, -34] . FixedPtCheck, [39, 84, 174, 219, 52, 34]

$$\det(A + \tau \Delta) = 0$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	5 vs 5	5 vs 5	4 vs 4	3 vs 4

bi =

$$\begin{aligned} & \$ [ [0, 1/4, 3/4, 0, 0, 0] , [0, 0, 0, 1/4, 0, 3/4] , [0, 0, 0, 1/4, 3/4, 0] , [0, 3/4, 1/4, 0, 0, 0] , [1/4, 0, 0, 0, 0, 0, \\ & 3/4] , [0, 0, 0, 1/4, 3/4, 0] ] \times \$ [ [11/20, 0, 0, -9/20, 3/20, 3/20] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , \\ & [-9/20, 0, 0, 11/20, 3/20, 3/20] , [3/20, 0, 0, 3/20, 19/20, -1/20] , [3/20, 0, 0, 3/20, -1/20, 19/20] ] \$ = \\ & \$ [ [1343/2596, -533/649, -2330/649, 1376/1947, 6368/1947] , [11/118, 139/236, -64/59, 46/177, \\ & 40/177] , [-173/1298, 485/2596, 320/649, -2590/1947, 1688/1947] , [983/2596, -163/649, 734/649, \\ & 752/1947, -3040/1947] , [-199/1298, -455/2596, 128/649, 2858/1947, -2440/1947] , [-173/1298, 485/2596, \\ & 320/649, -2590/1947, 1688/1947] ] \$ \times \$ [ [1/2, 5/2, 3/2, 3/2, 3, 3] , [3/4, 5/4, 3/4, 7/4, 27/8, 33/8] , \\ & [27/32, 3/2, 1, 49/32, 117/32, 111/32] , [117/128, 87/64, 65/64, 191/128, 429/128, 495/128] , [429/512, \\ & 345/256, 271/256, 799/512, 1875/512, 1809/512] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 4}}, net cycles: 0 . order: 4

$$[y_1, y_2, y_3, y_4, 0, 0]$$

$$\begin{aligned} R = & \$ [ [0, 1, 0, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 1, 0, 0, 0] , [1, 0, 0, 0, 0, 0] , [0, 0, 0, 1, 0, \\ & 0] ] \times \$ [ [1, 0, 0, 0, 0, 0] , [0, 1, 0, 0, 0, 0] , [0, 0, 1, 0, 0, 0] , [0, 0, 0, 1, 0, 0] , [0, 0, 0, 0, 0, 0] , [0, 0, 0, \\ & 0, 0, 0] ] \$ = & \$ [ [0, 1/2, -1/12, -1/3] , [0, 0, 1/6, -1/12] , [0, 0, 1/6, -1/12] , [0, 0, -1/12, 1/6] , [1/2, -1/4, \\ & -1/3, 1/6] , [0, 0, 1/6, -1/12] ] \times \$ [ [2, 1, 3, 6, 0, 0] , [0, 2, 6, 4, 0, 0] , [0, 0, 4, 8, 0, 0] , [0, 0, 8, 4, 0, 0] ] \\ & \$ \end{aligned}$$

Omega Rank for B : cycles: {{5, 6}}, net cycles: -1 . order: 2

$$\$ [ [0, 3, 1, 0, 4, 4] , [0, 0, 0, 0, 5, 7] , [0, 0, 0, 0, 7, 5] , [0, 0, 0, 0, 5, 7] ] \$$$

$$[0, 3 y_1, y_1, 0, y_2, y_3]$$

$$p = -s^2 + s^4$$

$\hat{A}$ » SYNC'D 1/8 , 0.1250000000

27 . Coloring, {2, 3, 4, 5}

**R:** [2, 6, 5, 3, 1, 5]    **B:** [3, 4, 4, 2, 6, 4]

' See graph

' ' See pair graph

$\Omega$  for  $A+\tau\Delta$  :

' [ '6' ('1 +  $\tau$  ')<sup>3</sup> ('-3 +  $\tau$  ')<sup>2</sup>, -12' ('3 -  $\tau$  + 5 $\tau^2$  +  $\tau^3$  ')<sup>2</sup>, 12' ('-1 +  $\tau$  ')<sup>2</sup> ('3 + 2 $\tau$  +  $\tau^2$  ')<sup>2</sup> ('1 +  $\tau$  ')<sup>2</sup>, 6' ('-1 +  $\tau$  ')<sup>2</sup> ('3 +  $\tau$  ')<sup>2</sup> ('3 +  $\tau^2$  ')<sup>2</sup>, 12' ('1 +  $\tau$  ')<sup>2</sup> ('-3 +  $\tau$  ')<sup>2</sup>, -12' ('3 -  $\tau$  +  $\tau^2$  +  $\tau^3$  ')<sup>2</sup> ('1 +  $\tau$  ')<sup>2</sup> ]'

For  $\tau=1/2$ , [-135, -124, -102, -91, -180, -138] . FixedPtCheck, [135, 124, 102, 91, 180, 138]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	5 vs 5	5 vs 5	5 vs 5	2 vs 4

bi =

\$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 3/4, 0, 1/4], [0, 0, 0, 3/4, 1/4, 0], [0, 3/4, 1/4, 0, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 0, 0, 3/4, 1/4, 0] ] \$ x \$ [ [19/100, 0, 0, -9/100, 27/100, 27/100], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [-9/100, 0, 0, 99/100, 3/100, 3/100], [27/100, 0, 0, 3/100, 91/100, -9/100], [27/100, 0, 0, 3/100, -9/100, 91/100] ] \$ =

\$ [ [27/4, -6, -130/3, 32/3, 32], [3/4, 0, -6, -16/3, 32/3], [-5/4, 14/3, 2, -16, 32/3], [1/12, -2, 2, 32/3, -32/3], [-5/4, -10/3, 62/3, 16, -32], [-5/4, 14/3, 2, -16, 32/3] ] \$ x \$ [ [1/2, 5/2, 3/2, 9/2, 1, 2], [1/4, 7/2, 3/2, 9/2, 7/8, 11/8], [7/32, 55/16, 21/16, 153/32, 23/32, 49/32], [23/128, 233/64, 87/64, 603/128, 91/128, 179/128], [91/512, 229/64, 21/16, 2457/512, 353/512, 739/512] ] \$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 2, 5, 6}}, net cycles: 0 . order: 4

[ $y_2, y_3, y_4, 0, y_5, y_1$ ]

$R =$  \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0] ] \$ x \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 1/16, -1/48, 11/48, -3/16], [0, -3/16, 1/16, -1/48, 11/48], [0, 11/48, -3/16, 1/16, -1/48], [1/3, -3/16, 1/16, -1/48, -5/48], [0, -1/48, 11/48, -3/16, 1/16], [0, 11/48, -3/16, 1/16, -1/48] ] \$ x \$ [ [2, 1, 3, 0, 4, 2], [4, 2, 0, 0, 5, 1], [5, 4, 0, 0, 1, 2], [1, 5, 0, 0, 2, 4], [2, 1, 0, 0, 4, 5] ] \$

Omega Rank for B : cycles: {{2, 4}}, net cycles: -1 . order: 2

$$\$ [ [0, 3, 1, 6, 0, 2], [0, 6, 0, 6, 0, 0], [0, 6, 0, 6, 0, 0], [0, 6, 0, 6, 0, 0] ] \$$$

$$[0, -3y_1 + y_2, y_1, y_2, 0, 2y_1]$$

$$p = -s^2 + s^3 \quad p = -s^2 + s^4$$

Â» SYNC'D 15/64 , 0.2343750000

28 . Coloring, {2, 3, 4, 6}

**R:** [2, 6, 5, 3, 6, 4] **B:** [3, 4, 4, 2, 1, 5]

' See graph

' ' See pair graph

'

Ω for A+τΔ :

$$\begin{aligned} & [ '6' ('1 + \tau') ('-1 + \tau') ('3 + \tau^2') , 36' ('-1 + \tau') ('3 + 2\tau + 3\tau^2') , 12' ('1 + \tau') \\ & ) ('1 + \tau^2') ('-3 + \tau') , 6' ('-9 - 2\tau - 8\tau^2 + 2\tau^3 + \tau^4') , -12' ('1 + \tau') ('3 + \tau^2') , \\ & 12' ('1 + \tau') ('-3 - \tau - \tau^2 + \tau^3') ]' \end{aligned}$$

For τ=1/2, [-39, -76, -150, -187, -156, -174] . FixedPtCheck, [39, 76, 150, 187, 156, 174]

$$\det(A + \tau \Delta) = 1' ('1 + \tau') ('\tau')^2 ('-1 + \tau')$$

Δ-Rank	A+(1/2)Δ	A-(1/2)Δ	R	B
4 vs 4	6 vs 6	6 vs 6	5 vs 5	4 vs 5

bi =

$$\$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 3/4, 0, 1/4], [0, 0, 0, 3/4, 1/4, 0], [0, 3/4, 1/4, 0, 0, 0], [3/4, 0, 0, 0, 0, 1/4], [0, 0, 0, 1/4, 3/4, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ =$$

$$\begin{aligned} & \$ [ [-51371/102104, 108877/102104, 33367/38289, -53453/38289, -159104/38289, 160816/38289], \\ & [10375/51052, -17099/25526, -794/38289, -5060/38289, 167008/38289, -140096/38289], [9943/51052, \\ & -21475/25526, 32926/38289, 86380/38289, 74080/38289, -165440/38289], [-41971/102104, \\ & 64853/102104, -13001/38289, -10229/38289, -146432/38289, 164272/38289], [43193/102104, \\ & 9065/102104, -22637/38289, 38311/38289, 164416/38289, -196496/38289], [55865/102104, \\ & -6487/102104, -6677/38289, -77561/38289, -106304/38289, 175216/38289] ] \$ \times \$ [ [3/2, 5/2, 3/2, 7/2, \\ & 2, 1], [3/2, 3, 2, 13/4, 9/8, 9/8], [27/32, 45/16, 31/16, 129/32, 43/32, 33/32], [129/128, 207/64, 105/64, \\ & 489/128, 161/128, 133/128], [483/512, 399/128, 219/128, 2005/512, 609/512, 575/512], [1827/2048, \\ & 3249/1024, 1727/1024, 7991/2048, 2601/2048, 2205/2048] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 4, 5, 6}}, net cycles: 0 . order: 4

$$[0, y_1, y_2, y_3, y_4, y_5]$$

$$\begin{aligned} R = & \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], \\ & 0] ] \$ \times \$ [ [0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \\ & \$ = \$ [ [1, -5/48, -17/48, 19/48, -41/48], [0, 7/48, -5/48, -17/48, 19/48], [0, -5/48, -17/48, 19/48, 7/48], \\ & [0, -17/48, 19/48, 7/48, -5/48], [0, 7/48, -5/48, -17/48, 19/48], [0, 19/48, 7/48, -5/48, -17/48] ] \$ \times \\ & \$ [ [0, 1, 3, 2, 2, 4], [0, 0, 2, 4, 3, 3], [0, 0, 4, 3, 2, 3], [0, 0, 3, 3, 4, 2], [0, 0, 3, 2, 3, 4] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{2, 4}}, net cycles: 0 . order: 4

$$\$ [ [2, 3, 1, 4, 2, 0], [2, 4, 2, 4, 0, 0], [0, 4, 2, 6, 0, 0], [0, 6, 0, 6, 0, 0], [0, 6, 0, 6, 0, 0] ] \$$$

$$[y_1 + y_2 - y_4 + y_3, y_1, y_2, y_4, y_3, 0]$$

$$p = s^4 - s^5$$

Â» SYNC'D 29/512 , 0.05664062500

29 . Coloring, {2, 3, 5, 6}

**R:** [2, 6, 5, 2, 1, 4] **B:** [3, 4, 4, 3, 6, 5]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$$[ '1' ('1 + \tau')^{''} ('-1 + \tau')^{''} ('3 + \tau')', 2' ('1 + \tau')^{''} ('-3 + \tau')', -2' ('-1 + \tau')^{''} ('-3 + \tau')', -1' ('9 - 5\tau + 3\tau^2 + \tau^3')', 2' ('-1 + \tau')^{''} ('3 + \tau')', -2' ('3 + \tau^2')^{''} ]'$$

For τ=1/2, [-21, -60, -20, -59, -28, -52] . FixedPtCheck, [21, 60, 20, 59, 28, 52]

$$\det(A + \tau \Delta) = 0$$

Delta Range : [-y<sub>2</sub> - y<sub>3</sub> - y<sub>4</sub>, -y<sub>1</sub>, y<sub>1</sub>, y<sub>2</sub>, y<sub>3</sub>, y<sub>4</sub>]

$$[1, 2, 2, 3, 2, 2]$$

$$+ \quad \backslash ; \quad - \quad \backslash ; \quad \Delta$$

\$ [ [2, 4, 0, 2, 2, 2], [1, 2, 2, 3, 1, 3], [1, 4, 4, 7, 3, 5], [3, 8, 8, 13, 7, 9] ] \$ \$ [ [0, 0, 4, 4, 2, 2], [1, 2, 2, 3, 3, 1], [3, 4, 4, 5, 5, 3], [5, 8, 8, 11, 9, 7] ] \$ \$ [ [1, 2, -2, -1, 0, 0], [0, 0, 0, 0, -1, 1], [-1, 0, 0, 1, -1, 1], [-1, 0, 0, 1, -1, 1] ] \$

$$[-y_2, -y_1, y_1, y_2, -y_3, y_3]$$

$$p = s^3 - 2s^4$$

S+ \; S- \; NM

\$ [ [1, 2, 2, 3, 2, 2], [0, 3, 1, 3, 3, 2], [0, 3, 1, 3, 3, 2], [1, 2, 2, 3, 2, 2], [2, 1, 3, 3, 1, 2], [2, 1, 3, 3, 1, 2] ] \$ \$ [ [1, 2, 2, 3, 2, 2], [2, 1, 3, 3, 1, 2], [2, 1, 3, 3, 1, 2], [1, 2, 2, 3, 2, 2], [0, 3, 1, 3, 3, 2], [0, 3, 1, 3, 3, 2] ] \$ \$ [ [4, 4, 4, 12, 4, 4], [2, 8, 8, 6, 4, 4], [2, 8, 8, 6, 4, 4], [4, 4, 4, 12, 4, 4], [2, 4, 4, 6, 8, 8], [2, 4, 4, 6, 8, 8] ] \$

CmmCk true, true, true

$\Delta$ -Rank	A+(1/2) $\Delta$	A-(1/2) $\Delta$	R	B
3 vs 4	3 vs 5	3 vs 5	3 vs 5	1 vs 4

Omega Rank for R : cycles: {{2, 4, 6}}, net cycles: 0 . order: 3

\$ [ [2, 4, 0, 2, 2, 2], [2, 4, 0, 2, 0, 4], [0, 4, 0, 4, 0, 4], [0, 4, 0, 4, 0, 4], [0, 4, 0, 4, 0, 4] ] \$

$$[y_2 + y_3 - y_1, y_2 + y_3, 0, y_1, y_2, y_3]$$

$$p' = -s^3 + s^4 \quad p = s^3 - s^4$$

Omega Rank for B : cycles: {{5, 6}, {3, 4}}, net cycles: 2 . order: 2

\$ [ [0, 0, 4, 4, 2, 2], [0, 0, 4, 4, 2, 2], [0, 0, 4, 4, 2, 2], [0, 0, 4, 4, 2, 2] ] \$

$$[0, 0, 2y_1, 2y_1, y_1, y_1]$$

$$p = s - s^3 \quad p' = s - s^2 \quad p' = -s^2 + s^3$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[0, 0, x_1, -2x_1]$$

$$\text{For } A+2\Delta : [y_1, y_2, -4y_1 - 3y_2 - 4y_3, y_1, y_3, y_3]$$

$$\text{For } A-2\Delta : [y_1, -4y_1 - 3y_2 - 4y_3, y_2, y_1, y_3, y_3]$$

Range of  $\{\Omega\Delta^i\}$ :  $[-\mu_2, -\mu_1, \mu_1, \mu_2, -\mu_3, \mu_3]$

rank of M is 5 , rank of N is 3

M N

\$ [ [0, 2, 0, 0, 1, 1], [2, 0, 0, 2, 2, 2], [0, 0, 0, 4, 2, 2], [0, 2, 4, 0, 3, 3], [1, 2, 2, 3, 0, 0], [1, 2, 2, 3, 0, 0] ]  
\$ \$ [ [0, 1, 1, 0, 1, 1], [1, 0, 0, 1, 1, 1], [1, 0, 0, 1, 1, 1], [0, 1, 1, 0, 1, 1], [1, 1, 1, 1, 0, 0], [1, 1, 1, 1, 0, 0] ] \$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [1, 2, -2, -1, 0, 0]$

ker M,  $[0, 0, 0, 0, \lambda_1, -\lambda_1]$

Range M,  $[x_3, x_4, x_1, x_2, x_5, x_5]$

$\tau = 12$ ,  $r' = 2/3$

Ranges

Action of R on ranges,  $[[2], [4], [2], [4], [1], [3]]$

Action of B on ranges,  $[[6], [5], [6], [5], [6], [5]]$

$\beta(\{1, 2, 5\}) = 1/8$

$\beta(\{1, 2, 6\}) = 1/8$

$\beta(\{2, 4, 5\}) = 1/8$

$\beta(\{2, 4, 6\}) = 1/8$

$\beta(\{3, 4, 5\}) = 1/4$

$\beta(\{3, 4, 6\}) = 1/4$

ker N,  $[-\mu_2, -\mu_1, \mu_1, \mu_2, \mu_3, -\mu_3]$

Range of N

$[y_3, y_2, y_2, y_3, y_1, y_1]$

Partitions

$\alpha(\{\{5, 6\}, \{2, 3\}, \{1, 4\}\}) = 1/1$

$b_1 = \{5, 6\}$ ,  $b_2 = \{2, 3\}$ ,  $b_3 = \{1, 4\}$

Action of R and B on the blocks of the partitions:  $\$ [ [1, 1, 0], [0, 0, 2], [1, 1, 0] ] \$ = \$ [ [0, 1, 0], [0, 0, 1], [1, 0, 0] ] \$ + \$ [ [1, 0, 0], [0, 0, 1], [0, 1, 0] ] \$$   
 $['2', '3', '1'], ['1', '3', '2']$  with invariant measure  $[1, 1, 1]$

N by blocks, check: *true*. ' See partition graph.

' ' See level-3 partition graph.

,



<b>Right Group</b>	
<b>Coloring</b>	{2, 3, 5, 6}
<b>Rank</b>	3
<b>R,B</b>	[2, 6, 5, 2, 1, 4], [3, 4, 4, 3, 6, 5]
$\pi_2$	[2, 0, 0, 1, 1, 0, 2, 2, 2, 4, 2, 2, 3, 3, 0]
$u_2$	[1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0] (dim 1)
<b>wpp</b>	[2, 2, 2, 2, 2, 2]
$\pi_3$	[0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 2, 2, 0, 0]
$u_3$	[0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0]

30 . Coloring, {2, 4, 5, 6}

**R:** [2, 6, 4, 3, 1, 4]    **B:** [3, 4, 5, 2, 6, 5]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

‘ [ ‘6‘ (‘ 1 +  $\tau$  ‘)‘‘ (‘ - 1 +  $\tau$  ‘)‘‘ (‘ - 3 +  $\tau$  ‘)‘ , -12‘ (‘ - 1 +  $\tau$  ‘)‘‘ (‘ 3 +  $\tau$  ‘)‘ , 12‘ (‘ 1 +  $\tau$  ‘)‘‘ (‘ 3 - 2 $\tau$  +  $\tau^2$  ‘)‘ , 6‘ (‘ 9 -  $\tau$  -  $\tau^2$  +  $\tau^3$  ‘)‘ , 12‘ (‘ - 1 +  $\tau$  ‘)‘‘ (‘ - 3 +  $\tau$  ‘)‘ , -12‘ (‘ - 1 +  $\tau$  ‘)‘‘ (‘ 3 +  $\tau^2$  ‘)‘ ‘]‘

For  $\tau=1/2$ , [15, 28, 54, 67, 20, 26] . FixedPtCheck, [15, 28, 54, 67, 20, 26]

$\det(A + \tau \Delta) = 0$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	5 vs 5	5 vs 5	5 vs 5	3 vs 5

bi =

$\$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 3/4, 0, 1/4], [0, 0, 0, 1/4, 3/4, 0], [0, 3/4, 1/4, 0, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 0, 0, 1/4, 3/4, 0] ] \times \$ [ [91/820, 0, 0, -81/820, 27/820, 243/820], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [-81/820, 0, 0, 811/820, 3/820, 27/820], [27/820, 0, 0, 3/820, 819/820, -9/820], [243/820, 0, 0, 27/820, -9/820, 739/820] ] \$ =$

$\$ [ [51/4, 16, -66, -112/3, 224/3], [3/20, -1, 18/5, 88/15, -128/15], [-141/20, -13, 194/5, 424/15, -704/15], [63/20, 8, -82/5, -272/15, 352/15], [67/20, 7, -118/5, -248/15, 448/15], [-141/20, -13, 194/5, 424/15, -704/15] ] \$ \times \$ [ [1/2, 5/2, 3/2, 5/2, 3, 2], [3/4, 2, 1, 11/4, 21/8, 23/8], [21/32, 9/4, 5/4, 79/32, 93/32, 79/32], [93/128, 129/64, 71/64, 335/128, 357/128, 351/128], [357/512, 549/256, 307/256, 1267/512, 1479/512, 1329/512] ] \$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{3, 4}}, net cycles: 0 . order: 4

$$[y_4, y_1, y_2, y_3, 0, y_5]$$

$R = \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1], [0, 0, 0, 1, 0, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 1] ] \$ = \$ [ [0, 1/2, -1/4, -5/24, 1/24], [0, 0, 1/2, -5/24, -5/24], [0, 0, 0, 7/24, -5/24], [0, 0, 0, -5/24, 7/24], [1/2, -1/4, -3/8, 1/24, 1/6], [0, 0, 0, 7/24, -5/24] ] \$ \times \$ [ [2, 1, 3, 4, 0, 2], [0, 2, 4, 5, 0, 1], [0, 0, 5, 5, 0, 2], [0, 0, 5, 7, 0, 0], [0, 0, 7, 5, 0, 0] ] \$$

Omega Rank for B : cycles: {{5, 6}, {2, 4}}, net cycles: 1 . order: 2

$\$ [ [0, 3, 1, 2, 4, 2], [0, 2, 0, 3, 3, 4], [0, 3, 0, 2, 4, 3], [0, 2, 0, 3, 3, 4], [0, 3, 0, 2, 4, 3] ] \$$

$$[0, 5y_1 - 6y_2 + 5y_3, y_1, y_2, 6y_1 - 7y_2 + 6y_3, y_3]$$

$$p = -s^2 + s^4 \quad p' = -s^2 + s^4$$

Â» SYNC'D 3/128 , 0.02343750000

31 . Coloring, {3, 4, 5, 6}

**R:** [2, 4, 5, 3, 1, 4]    **B:** [3, 6, 4, 2, 6, 5]

' See graph

' ' See pair graph

,

Ω for A+τΔ :

$[ '6' ( '1 + \tau' ) ' ( ' - 3 - \tau - 5\tau^2 + \tau^3 ' ) ' , -36' ( '1 + \tau' ) ' ( '3 - 2\tau + 3\tau^2 ' ) ' , 12' ( ' - 3 - \tau - \tau^2 + \tau^3 ' ) ' ( '1 + \tau' ) ' , 6' ( '3 + \tau^2 ' ) ' ( '1 + \tau' ) ' ( ' - 3 + \tau ' ) ' , 12' ( ' - 3 - \tau - 5\tau^2 + \tau^3 ' ) ' , 12' ( ' - 1 + \tau ' ) ' ( '1 + \tau^2 ' ) ' ( '3 + \tau ' ) ' ] '$

For τ=1/2, [-111, -132, -174, -195, -148, -70] . FixedPtCheck, [111, 132, 174, 195, 148, 70]

$$\det(A + \tau \Delta) = 1' ( ' - 1 + \tau ' ) ' ( ' \tau ' ) ' ^2 ' ( '1 + \tau ' ) '$$

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
4 vs 4	6 vs 6	6 vs 6	5 vs 5	4 vs 5

bi =

$$\begin{aligned} & \$ [ [0, 1/4, 3/4, 0, 0, 0], [0, 0, 0, 1/4, 0, 3/4], [0, 0, 0, 3/4, 1/4, 0], [0, 3/4, 1/4, 0, 0, 0], [1/4, 0, 0, 0, 0, 3/4], [0, 0, 0, 1/4, 3/4, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [13091/7336, -62383/22008, -5183/2751, 3451/393, 12352/2751, -28208/2751], [289/3668, -811/5502, -866/2751, 1564/393, -7040/2751, -2624/2751], [-2447/3668, 13949/5502, -3938/2751, -2516/393, -704/2751, 17344/2751], [3707/7336, -21367/22008, 4993/2751, 67/393, 2368/2751, -6320/2751], [-2449/7336, 1037/22008, 1537/2751, -593/393, -12512/2751, 16144/2751], [-239/1048, 1379/3144, -233/393, -281/393, 1504/393, -1040/393] ] \$ \times \$ [ [1/2, 5/2, 3/2, 5/2, 2, 3], [1/2, 2, 1, 5/2, 21/8, 27/8], [21/32, 2, 1, 67/32, 89/32, 111/32], [89/128, 111/64, 65/64, 271/128, 365/128, 459/128], [365/512, 451/256, 269/256, 1071/512, 1507/512, 1761/512], [1507/2048, 1789/1024, 1083/1024, 4277/2048, 5821/2048, 7227/2048] ] \$ \end{aligned}$$

Check x AllOnes: [1, 1, 1, 1, 1, 1]

Omega Rank for R : cycles: {{1, 2, 3, 4, 5}}, net cycles: 1 . order: 5

$$[y_1, y_2, y_3, y_4, y_5, 0]$$

$$\begin{aligned} R = & \$ [ [0, 1, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 1, 0, 0, 0], [1, 0, 0, 0, 0, 0], [0, 0, 0, 1, 0, 0] ] \$ \times \$ [ [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [0, 0, 1, 0, 0, 0], [0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1] ] \$ = \\ & \$ [ [-107/492, 1/492, -23/492, 37/492, 133/492], [133/492, -107/492, 1/492, -23/492, 37/492], [-23/492, 37/492, 133/492, -107/492, 1/492], [37/492, 133/492, -107/492, 1/492, -23/492], [1/492, -23/492, 37/492, 133/492, -107/492], [133/492, -107/492, 1/492, -23/492, 37/492] ] \$ \times \$ [ [2, 1, 3, 4, 2, 0], [2, 2, 4, 1, 3, 0], [3, 2, 1, 2, 4, 0], [4, 3, 2, 2, 1, 0], [1, 4, 2, 3, 2, 0] ] \$ \end{aligned}$$

Omega Rank for B : cycles: {{5, 6}}, net cycles: 0 . order: 4

$$\$ [ [0, 3, 1, 2, 2, 4], [0, 2, 0, 1, 4, 5], [0, 1, 0, 0, 5, 6], [0, 0, 0, 0, 6, 6], [0, 0, 0, 0, 6, 6] ] \$$$

$$[0, y_1, -y_1 + y_4 - y_3 + y_2, y_4, y_3, y_2]$$

$$p = s^4 - s^5$$

Â» SYNC'D 35/256 , 0.1367187500

32 . Coloring, {2, 3, 4, 5, 6}

**R:** [2, 6, 5, 3, 1, 4] **B:** [3, 4, 4, 2, 6, 5]

‘ See graph

‘ ‘ See pair graph

‘

$\Omega$  for  $A+\tau\Delta$  :

$$[2^2(3+\tau^2)^2(1+\tau)^2, 12^2(3-2\tau+3\tau^2)^2, 4^2(1+\tau)^2(3-2\tau+\tau^2)^2, 2^2(9-5\tau+3\tau^2+\tau^3)^2, 4^2(3+\tau^2)^2, 4^2(3-\tau+\tau^2+\tau^3)^2]$$

For  $\tau=1/2$ , [39, 44, 54, 59, 52, 46] . FixedPtCheck, [39, 44, 54, 59, 52, 46]

$$\det(A + \tau \Delta) = 1^2 (\tau^2)^2 (1 + \tau^2)^2$$

Delta Range :  $[-y_2 - y_3 - y_4, -y_1, y_1, y_2, y_3, y_4]$

$$[1, 2, 2, 3, 2, 2]$$

$$+ \quad \backslash; \quad - \quad \backslash; \quad \Delta$$

$$[ [2, 1, 3, 2, 2, 2], [2, 6, 2, 6, 5, 3], [5, 8, 8, 11, 7, 9], [7, 18, 14, 25, 15, 17] ] \quad [ [0, 3, 1, 4, 2, 2], [2, 2, 6, 6, 3, 5], [3, 8, 8, 13, 9, 7], [9, 14, 18, 23, 17, 15] ] \quad [ [1, -1, 1, -1, 0, 0], [0, 2, -2, 0, 1, -1], [1, 0, 0, -1, -1, 1], [-1, 2, -2, 1, -1, 1] ]$$

$$[-y_2, -y_1, y_1, y_2, -y_3, y_3]$$

$$p = s - 2s^3 + 4s^4$$

$$S+ \quad \backslash; \quad S- \quad \backslash; \quad NM$$

$$[ [1, 3, 3, 3, 2, 2], [1, 2, 2, 4, 2, 3], [1, 2, 2, 4, 3, 2], [1, 3, 3, 3, 2, 2], [2, 2, 2, 3, 2, 3], [1, 2, 2, 4, 3, 2] ] \quad [ [1, 3, 3, 3, 2, 2], [1, 2, 2, 4, 2, 3], [1, 2, 2, 4, 3, 2], [1, 3, 3, 3, 2, 2], [2, 2, 2, 3, 2, 3], [1, 2, 2, 4, 3, 2] ] \quad [ [18, 18, 18, 54, 18, 18], [9, 36, 36, 27, 18, 18], [9, 36, 36, 27, 18, 18], [18, 18, 18, 54, 18, 18], [9, 18, 18, 27, 36, 36], [9, 18, 18, 27, 36, 36] ]$$

CmmCk true, true, true

$\Delta$ -Rank	$A+(1/2)\Delta$	$A-(1/2)\Delta$	<b>R</b>	<b>B</b>
3 vs 4	4 vs 6	4 vs 6	4 vs 6	2 vs 5

Omega Rank for R : cycles:  $\{\{1, 2, 3, 4, 5, 6\}\}$ , net cycles: 1 . order: 6

$$[ [2, 1, 3, 2, 2, 2], [2, 2, 2, 2, 3, 1], [3, 2, 2, 1, 2, 2], [2, 3, 1, 2, 2, 2], [2, 2, 2, 2, 1, 3], [1, 2, 2, 3, 2, 2] ]$$

$$[y_3, y_4, y_2, y_4 + y_2 - y_3, y_1, y_4 + y_2 - y_1]$$

$$p' = 1 - s^2 + s^3 - s^5 \quad p' = s - s^2 + s^4 - s^5$$

Omega Rank for B : cycles:  $\{\{5, 6\}, \{2, 4\}\}$ , net cycles: 1 . order: 2

$\$ [ [0, 3, 1, 4, 2, 2], [0, 4, 0, 4, 2, 2], [0, 4, 0, 4, 2, 2], [0, 4, 0, 4, 2, 2], [0, 4, 0, 4, 2, 2] ] \$$

$$[0, -y_1 + 2y_2, y_1, 2y_2, y_2, y_2]$$

$$p = -s^2 + s^4 \quad p = -s^2 + s^3 \quad p = -s^2 + s^5$$

Â« NOT SYNC'D Â»

Nullspace of  $\{\Omega\Delta^i\}$  :

$$[x_1, 0, -2x_1, 4x_1]$$

$$\text{For } A+2\Delta: [y_1, -y_1 - y_2, -y_1 - y_2, y_1, y_2, y_2]$$

$$\text{For } A-2\Delta: [-y_1 - y_2, y_2, y_2, -y_1 - y_2, y_1, y_1]$$

Range of  $\{\Omega\Delta^i\}$ :  $[\mu_2, \mu_1, -\mu_1, -\mu_2, \mu_3, -\mu_3]$

rank of M is 6 , rank of N is 3

M            N

$\$ [ [0, 2, 7, 0, 4, 5], [2, 0, 0, 16, 10, 8], [7, 0, 0, 11, 8, 10], [0, 16, 11, 0, 14, 13], [4, 10, 8, 14, 0, 0], [5, 8, 10, 13, 0, 0] ] \$$      $\$ [ [0, 1, 1, 0, 1, 1], [1, 0, 0, 1, 1, 1], [1, 0, 0, 1, 1, 1], [0, 1, 1, 0, 1, 1], [1, 1, 1, 1, 0, 0], [1, 1, 1, 1, 0, 0] ] \$$

Check is  $\Omega\Delta N$  zero? *true*,  $\pi\Delta = [1, -1, 1, -1, 0, 0]$

ker M,  $[0, 0, 0, 0, 0, 0]$

Range M,  $[x_1, x_2, x_3, x_4, x_5, x_6]$

$$\tau = 12, r' = 2/3$$

Ranges

Action of R on ranges,  $[[2], [6], [1], [5], [4], [8], [3], [7]]$

Action of B on ranges,  $[[8], [7], [8], [7], [6], [5], [6], [5]]$

$$\beta(\{1, 2, 5\}) = 1/27$$

$$\beta(\{1, 2, 6\}) = 1/54$$

$$\beta(\{1, 3, 5\}) = 2/27$$

$$\beta(\{1, 3, 6\}) = 13/108$$

$$\beta(\{2, 4, 5\}) = 13/54$$

$$\beta(\{2, 4, 6\}) = 11/54$$

$$\beta(\{3, 4, 5\}) = 4/27$$

$$\beta(\{3, 4, 6\}) = 17/108$$

ker N,  $[-\mu_2, -\mu_1, \mu_1, \mu_2, -\mu_3, \mu_3]$

Range of N

$$[y_1, y_2, y_2, y_1, y_3, y_3]$$

Partitions

$$\alpha(\{5, 6\}, \{2, 3\}, \{1, 4\}) = 1/1$$

$$b1 = \{5, 6\}, b2 = \{2, 3\}, b3 = \{1, 4\}$$

Action of R and B on the blocks of the partitions:  $\$ [ [1, 1, 0], [0, 0, 2], [1, 1, 0] ] \$ = \$ [ [0, 1, 0], [0, 0, 1], [1, 0, 0] ] \$ + \$ [ [1, 0, 0], [0, 0, 1], [0, 1, 0] ] \$$   
 $[ '2', '3', '1' ], [ '1', '3', '2' ]$  with invariant measure  $[1, 1, 1]$

N by blocks, check: true . ' See partition graph.

' ' See level-3 partition graph.

'

Right Group	
<b>Coloring</b>	{2, 3, 4, 5, 6}
<b>Rank</b>	3
<b>R,B</b>	[2, 6, 5, 3, 1, 4], [3, 4, 4, 2, 6, 5]
$\pi_2$	[2, 7, 0, 4, 5, 0, 16, 10, 8, 11, 8, 10, 14, 13, 0]
$u_2$	[1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0] (dim 1)
<b>wpp</b>	[2, 2, 2, 2, 2, 2]
$\pi_3$	[0, 0, 4, 2, 0, 8, 13, 0, 0, 0, 0, 0, 0, 26, 22, 0, 16, 17, 0, 0]
$u_3$	[0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 1, 0, 0]

SUMMARY	
<b>Graph Type</b>	CC
$v(A)$	2
$v(\Delta)$	2
$\pi$	[1, 2, 2, 3, 2, 2]
<b>Dbly Stoch</b>	false

<b>SANDWICH</b>		Total 2
<b>No .</b>	<b>Coloring</b>	<b>Rank</b>
<b>1</b>	{5}	2
<b>2</b>	{2, 3, 5}	2

<b>RT GROUPS</b>		Total 7	
<b>No .</b>	<b>Coloring</b>	<b>Rank</b>	<b>Solv</b>
<b>1</b>	{2, 6}	3	Solvable
<b>2</b>	{3}	2	Solvable
<b>3</b>	{}	3	Not Solvable
<b>4</b>	{3, 5, 6}	2	Solvable
<b>5</b>	{2, 3, 4, 5, 6}	3	Not Solvable
<b>6</b>	{4}	3	Not Solvable
<b>7</b>	{2, 3, 5, 6}	3	Not Solvable

<b><math>\Delta</math>-RANK'D</b>	<b>SC'D !RK'D</b>	<b><math>\tau</math>-RANK'D</b>	<b>R/B RANK'D</b>	<b>NOT SYNC'D</b>	<b>Total Runs</b>	<b><math>2^{n-1}</math></b>
22	0	21 , 21	12 , 10	9	32	32

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