Lie algebras Representations and Analytic Semigroups through Dual Vector Fields

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We want to present the basics of a new point of view in a variety of areas using the idea of *Dual Vector Fields.* These topics include operator calculus, representations of Lie algebras, analytic semigroups, and probability semigroups.

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Part I.

Coherent State Representations: Operators and Duality

Simple Fock Spaces

Coherent states and Coherent State Representations

Appell families

▷ Simple Fock Spaces

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1 Raising and lowering operators I

- **vector space** $\mathcal{H}$ with a basis $\{\psi_n\}_{n\geq 0}$
  - scalars: $\mathbf{C}$, $\mathbf{R}$

- **Dirac notation** writes $\psi_n = |\, n\,\rangle$, called "ket", where the label n is the eigenvalue of an operator on $\mathcal{H}$

- **vacuum state** ket $|\, 0\,\rangle$ is called the **vacuum state** Ω
  - Mapped to the zero vector by all lowering operators

- **raising** and **lowering** operators

$$\mathcal{R}\,|\, n\,\rangle = |\, n+1\,\rangle \qquad \mathcal{V}\,|\, n\,\rangle = n|\, n-1\,\rangle$$

Generally we work with d variables setting

$$\begin{aligned}
\mathcal{R}_i\,|\, n\,\rangle &= |\, n+\mathrm{e}_i\,\rangle = |\, n_1,\ldots,n_i+1,\ldots,n_d\,\rangle \\
\mathcal{V}_i\,|\, n\,\rangle &= n_i|\, n-\mathrm{e}_i\,\rangle
\end{aligned}$$

$\mathcal{V}$ is for **velocity**

2 Lie algebras

- **Lie algebra** $\mathfrak{g}$ with product $[a, b]$ satisfies $[a, a] = 0$

Jacobi identity: $[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$

commutator $[a, b] = ab - ba$ will be used throughout

- **Representation**
  - elements are linear maps on a vector space
  - Lie product maps to the commutator

- **adjoint representation:** $(\mathrm{ad}\, a)(b) = [a, b]$

Basis $\{ \xi_1, \xi_2, \ldots, \xi_d \}$ determines **structure constants**

$$(\mathrm{ad}\, \xi_k)(\xi_j) = [\xi_k, \xi_j] = \sum_i c^i_{kj} \xi_i$$

giving matrices of the adjoint representation

$$(\check{\xi}_k)_{ij} = c^i_{kj}$$

- **suitable functions** for realizations via differential

operators are **polynomials** and by extension

locally holomorphic functions

  - **formal power series** are included as suitable

functions

3 HW and representations

➧ **Heisenberg-Weyl** algebra is given by the commutation rule

$$[\xi_3, \xi_1] = \xi_2$$

A **matrix representation** of the HW algebra is

$$\xi_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \xi_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \xi_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The **adjoint representation** is

$$\check{\xi}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \check{\xi}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \check{\xi}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

➧ **Boson realization** $\mathcal{R}$ and $\mathcal{V}$ acting on the vectors $|\, n \,\rangle$

$$(\mathcal{V}\mathcal{R} - \mathcal{R}\mathcal{V})|\, n \,\rangle = (n + 1 - n)|\, n \,\rangle = |\, n \,\rangle$$

satisfy $[\mathcal{V}, \mathcal{R}] = I$, where I is the identity operator.

And I commutes with all operators.

So this is a representation of the HW algebra.

➡ On polynomials in d variables let

$$X_i = \text{ operator of multiplication by } x_i$$

$$D_i = \text{ differentiation with respect to } x_i$$

acting on the **basis** $|\, n \,\rangle = x^n = x_1^{n_1} \cdots x_d^{n_d}$
with vacuum state $|\, 0 \,\rangle = 1$

◗ **Commutation relations** prescribing HW(d)

$$[D_j, X_i] = \delta_{ij}\, I$$

◗ **Boson operators:** any family of operators $\{\, R_i, V_j \,\}$

$$[V_j, R_i] = \delta_{ij}\, I$$

◗ **Jordan map** Any Lie algebra may be realized as **vector fields**

$$\xi_i \leftrightarrow X_\lambda c_{i\mu}^\lambda D_\mu = x_\lambda c_{i\mu}^\lambda \frac{\partial}{\partial x_\mu}$$

Notation. Our summation convention is:

Greek indices are always summed.

4 Gaussian

- **Gaussian density** $p_t(dx) = \dfrac{e^{-x^2/(2t)}}{\sqrt{2\pi t}}\, dx$

- **Moment polynomials**

$$h_n(x) = \int_{-\infty}^{\infty} (x+y)^n \; p_t(dy)$$

In this case, we have

$$h_n(x) = \int_{-\infty}^{\infty} (x + y\sqrt{t})^n p_1(dy)$$

This may be written as an **expected value**

$$h_n(x) = \langle (x + X_t)^n \rangle$$

where X_t is the corresponding Gaussian variable.

Lowering operator: $V = D$, i.e., $Dh_n = n\, h_{n-1}$

Raising operator: $R = X + tD$

- **Recurrence formula** Write $X = R - tD = R - tV$

$$x\, h_n = h_{n+1} - tn\, h_{n-1}$$

➨ **Hermite polynomials** are orthogonal with respect to the Gaussian distribution

$$H_n(x) = \int_{-\infty}^{\infty} (x + iy)^n \; p_t(dy)$$

where $i = \sqrt{-1}$.

➨ For the Hermite polynomials

$$R = X - tD, \qquad V = D$$

The recurrence is thus

$$x\,\psi_n = h_{n+1} + tn\,h_{n-1}$$

which is the **three-term recurrence** a family of orthogonal polynomials must satisfy.

➨ $L = R^* = tV$, the operator **adjoint** to R with respect to the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(y)g(y)\,p_t(dy)$$

on polynomials or smooth functions with derivatives in $\mathrm{L}^2(\mathbf{R})$ of the corresponding Gaussian measure.

5 Poisson

➧ **Poisson distribution** $\quad p_t(x) = e^{-t}\,\dfrac{t^x}{x!}$

for integer $x \geq 0$.

➧ **Poisson-Charlier polynomials** are orthogonal with respect to this Poisson distribution. A generating function is

$$G(v) = G(v;x,t) = (1+v)^x\,e^{-vt} = \sum_{n\geq 0}\frac{v^n}{n!}\,P_n(x,t)$$

➨ **Difference operator** expressed in terms of D is

$$(e^D-1)f(x) = f(x+1)-f(x)$$

➧ **Duality**

"Multiplication by v" $\leftrightarrow$ lowering operator $VP_n = nP_{n-1}$

$$V = e^D - 1$$

"Differentiation wrt v" $\leftrightarrow$ raising operator $RP_n = P_{n+1}$

> The operators V, R are given by transferring
> *the HW representation*
> "multiplication by v, differentiation with respect to v"
> via *generating function* G to the *vector space* spanned by $\{\, P_n \,\}$

➨ To find R we must express $\partial/\partial v$ in terms of X and D.

Observe

$$\frac{1}{1+V} = e^{-D}$$

yielding the HW representation

$$R = Xe^{-D} - tI, \qquad V = e^{D} - I$$

Solving, we find

$$X = (R+t)(1+V) = t + R + RV + tV$$

Note that RV is the **number operator**: $RV\ P_n = nP_n$

And the form of X gives the recurrence formula

$$xP_n = P_{n+1} + (n+t)P_n + ntP_{n-1}$$

The Lie algebra generated by $\{\ R, V, RV\ \}$ is the **oscillator algebra.**

6 Analytic representations of HW

➧ **The HW relation** $[V, R] = I$ implies that for any polynomial $f(x)$

$$[V, f(R)] = f'(R)$$

acting on kets. Dually,

$$[f(V), R] = f'(V)$$

These extend to suitable functions f.

➧ **Canonical operators** $V(z)$ is a locally holomorphic function, $V(0) = 0, V'(0) \neq 0, W(D) = V'(D)^{-1}$.

$$Y = XW(D), \qquad V = V(D)$$

The **vacuum** for the representation is the function equal to 1. Acting on polynomials or exponential functions in x, $[V(D), XW(D)] = V'(D)W(D) = I$. Thus

$$[V, Y] = I$$

Simple Fock Spaces

▷ Coherent states and Coherent State Representations

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7 Raising and lowering operators II

➧ **Question** how to express various operators of the representation of a Lie algebra in terms of R's and V's ?

➧ **Adjoint** of R with respect to the inner product on the L^2 space of the underlying measure gives the lowering operator L

➧ **Squared norms** determine its properties. Let

$$L\psi_n = b_n \psi_{n-1}$$

Then the condition $L = R^*$ yields

$$\langle \psi_n, \psi_n \rangle = \gamma_n = b_n \gamma_{n-1}$$

With $\|\Omega\|^2 = 1$, $b_0 = 0$, we get

$$\gamma_n = b_1 b_2 \cdots b_n$$

♫ For the Gaussian, $R = X - tD$, $L = tD$ so the squared norms are

$$\|H_n\|^2 = t^n \, n!$$

8 Coherent states

➧ **Coherent state** $\psi_v = e^{vR}\Omega$

Let $\langle \psi_n, \psi_m \rangle = \delta_{mn} n!$ with squared norms $\gamma_n = n!$
In this case $L = V$: $\langle \psi_n, R\psi_m \rangle = \langle V\psi_n, \psi_m \rangle$

➧ **Duality** goes like this

$$V\psi_v = Ve^{vR}\Omega = e^{vR}V\Omega + [V, e^{vR}]\Omega = ve^{vR}\Omega = v\psi_v$$

Thus, on ψ_v we have

$$R\,\psi_v = \frac{\partial}{\partial v}\,\psi_v, \qquad V\psi_v = v\,\psi_v$$

➧ **Leibniz function** Multiplying by $v^m/m!$ and summing gives $\langle \psi_n, \psi_v \rangle = v^n$. Multiplying by $w^n/n!$ and summing gives the inner product of coherent states

$$\langle \psi_w, \psi_v \rangle = \Upsilon_{wv} = e^{wv}$$

➧ **Partial differential equation** for Υ

$$\frac{\partial \Upsilon}{\partial w} = \langle R\psi_w, \psi_v \rangle = v\,\Upsilon$$

compactly expresses the relation $R^* = V$.

9 Coherent states for HW

➧ **coherent state representation** of an operator Q

$$\langle Q\rangle_{wv} = \frac{\langle \psi_w, Q\psi_v\rangle}{\langle \psi_w, \psi_v\rangle}$$

➨ Gaussian case: $R = X - tD$, $L = tD$. We find

$$\Upsilon_{wv} = \langle e^{wR}\Omega, e^{vR}\Omega\rangle = e^{twv}$$

➨ Differentiating with respect to v yields the CSR of R, differentiating with respect to w yields the CSR of L :

$$\langle R\rangle_{wv} = tw, \qquad \langle L\rangle_{wv} = tv$$

➨ The Leibniz function satisfies the PDE

$$\frac{\partial \Upsilon}{\partial w} = tv\Upsilon$$

another way to see that $L = tV$.

10 sl(2)

➧ **sl(2)** is the Lie algebra of 2×2 matrices of trace zero.

The **standard basis** is

$$R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Delta = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

- ➨ Commutation relations

$$[\Delta, R] = \rho, \quad [\rho, R] = 2R, \quad [\Delta, \rho] = 2\Delta$$

On radial functions, Δ is **one-half times the Laplacian**

$$\Delta = \tfrac{1}{2} \sum \frac{\partial^2}{\partial x_j^2}, \quad R = \tfrac{1}{2} \sum x_j^2, \quad \rho = \sum x_j \frac{\partial}{\partial x_j} + \frac{d}{2}$$

➨ Representation space has basis

$$\psi_n = R^n \Omega, \quad \text{with} \quad \Delta\Omega = 0, \quad \rho\Omega = c\Omega$$

- ➨ Commutation rule corresponding to Leibniz' rule for differentiating a power function times another function

$$[\Delta, R^n] = n(\rho + n - 1)R^{n-1}$$

- ➨ On the vacuum state $\Delta\psi_n = n(c + n - 1)\psi_{n-1}$

11 Coherent states for sl(2)

- **Lowering operator** $L = \Delta = cV + RV^2$
  - squared norms $\gamma_n = n!(c)_n$

- **Coherent state** $\psi_v = e^{vR}\Omega$
  - $\langle \psi_n, \psi_v \rangle = (c)_n v^n$ and the **Leibniz function** is

$$\Upsilon_{wv} = (1 - wv)^{-c}$$

which satisfies $$\frac{\partial \Upsilon}{\partial w} = cv\Upsilon + v^2 \frac{\partial \Upsilon}{\partial v}.$$

  - Operators $\hat{R} = R$, $\hat{\Delta} = cV + RV^2$ and

$$\hat{\rho} = [\hat{\Delta}, \hat{R}] = c + 2RV$$

Then $$(c + 2RV)\psi_v = (c + 2v\frac{\partial}{\partial v})\psi_v$$

gives $$\langle \rho \rangle_{wv} = c + \frac{2v}{\Upsilon}\frac{\partial \Upsilon}{\partial v}.$$ The CSR's are

$$\langle R \rangle_{wv} = \frac{cw}{1 - wv}, \quad \langle \rho \rangle_{wv} = c\,\frac{1 + wv}{1 - wv}, \quad \langle L \rangle_{wv} = \frac{cv}{1 - wv}$$

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12 Definition

- **Appell systems** $\{h_n\}$ are defined by these properties:

① $h_n(x)$ is a polynomial of degree n, $\quad n \geq 0$

② $Dh_n(x) = n\, h_{n-1}(x)$

For $N \geq 1$, we have analogously

- $h_n(x)$ is a polynomial of degree $|n|$, $\quad n \geq 0$
- $D_j h_n(x) = n_j\, h_{n-e_j}(x)$

- **Generating function** $F(z,x) = \sum_{n \geq 0} z^n\, h_n(x)/n!$

satisfies

$$\frac{\partial F}{\partial x_i} = z_i\, F$$

In general we have the form $F(z,x) = e^{z \cdot x}\, G(z)$.

The expansion $G(z) = \sum_{n \geq 0} z^n\, c_n/n!$ yields

$$h_n(x) = \sum_{m \geq 0} \binom{n}{m} c_m x^{n-m}$$

The condition on the degree means $c_0 \neq 0$, i.e., $G(0) \neq 0$.

➡ Multiplication by z_i acts as differentiation D_i while

$$\frac{\partial F}{\partial z_j} = \sum_{n \geq 0} z^n \, h_{n+\mathrm{e}_j}(x)/n!$$

i.e., $\partial/\partial z_j$ acts as a raising operator $R_j h_n = h_{n+\mathrm{e}_j}$.
With $G(0) \neq 0$ we can locally express $G(z) = e^{H(z)}$

$$F(z,x) = e^{z \cdot x + H(z)}$$

➡ The operators D_j and $\partial/\partial z_j$ satisfy

$$D_j \, F = z_j \, F, \qquad \frac{\partial F}{\partial z_j} = \left(x_j + \frac{\partial H}{\partial z_j} \right) \, F$$

Thus, X_j denoting the operator of multiplication by x_j,

$$h_{n+\mathrm{e}_j} = \left(X_j + \frac{\partial H}{\partial D_j} \right) \, h_n$$

with $\partial H/\partial D_j$ a function of $D = (D_1, \ldots, D_d)$.

Theorem 12.1 *For Appell systems, given $H(z)$ an arbitrary function holomorphic in a neighborhood of 0, the boson calculus is given by*

$$R_i = X_i + \frac{\partial H}{\partial D_i}, \qquad V_i = D_i$$

with states $|\, n\, \rangle = h_n$.

The corresponding coherent state is

$$e^{z\cdot R}\,|\,0\,\rangle = e^{z\cdot x + H(z)} \; = \sum_{n\geq 0} \frac{z^n}{n!}\, h_n(x)$$

13 Evolution equation

➧ **Evolution equation** $\dfrac{\partial u}{\partial t} = H(D)u,\ u(x,0) = e^{z\cdot x}$
has solution

$$u(x,t) = e^{tH(D)}\, e^{z\cdot x} = e^{z\cdot x + tH(z)}$$

➨ Appell system $h_n(x,t) = e^{tH(D)}\, x^n$ satisfies

$$\frac{\partial u}{\partial t} = H(D)u, \text{ with } u(x,0) = x^n$$

so we see Appell systems as evolved powers.

➧ **HW formulation and boson variables**

Monomials x^n are built by $\quad X_j x^n = x^{n+\mathrm{e}_j}$

➨ **Conjugate by the flow e^{tH}**

$$h_{n+\mathrm{e}_j} = (e^{tH}\, X_j e^{-tH})e^{tH}\, x^n = e^{tH}\, x^{n+\mathrm{e}_j}$$

So raising operators are $\quad R_j = e^{tH}\, X_j e^{-tH}$.

➨ By the holomorphic operator calculus we have

$$[e^{tH}\,, X_j] = t\frac{\partial H}{\partial D_j}e^{tH}$$

so that

$$R_j = X_j + t\frac{\partial H}{\partial D_j}$$

Heisenberg-Hamiltonian flow

maps $(X, D) \to (R, V)$ given by

$$R = e^{tH}\, Xe^{-tH}\ , \qquad V = e^{tH}\, De^{-tH}$$

Heisenberg-Hamiltonian equations of motion

$$\dot{X} = [H, X] = \frac{\partial H}{\partial D}\,, \qquad \dot{D} = [H, D] = -\frac{\partial H}{\partial X}$$

For $H = H(D)$, D remains constant so $V = D$.

14 Stochastic formulation

➧ **Convolution family of probability measures** p_t with corresponding random variables X_t satisfies

$$\langle e^{z\cdot X_t}\rangle = \int e^{z\cdot x}\, p_t(dx) = e^{tH(z)}$$

here $H(0) = 0$ for probability measures.

$$e^{z\cdot x + tH(z)} = \int e^{z\cdot(x+u)}\, p_t(du)$$

and

$$h_n(x,t) = \int (x+u)^n p_t(du) = \langle (x+X_t)^n\rangle$$

are corresponding **moment polynomials**.

Proposition 14.1 *In the stochastic case,*

$$h_n(x,t) = \sum_{m\geq 0} \binom{n}{m} \mu_m(t) x^{n-m}$$

where $\mu_m(t)$ are moments of the probability measure p_t.

➨ infinitely divisible laws required for continuous t ?

15 Canonical systems

- **Evolve any canonical pair** (Y, V) via H.-H. flow

$$\dot{Y} = [H, Y], \qquad \dot{V} = [H, V]$$

$V = V(D)$ is invariant. Let $H' = (\frac{\partial H}{\partial D_1}, \ldots, \frac{\partial H}{\partial D_N})$.

- **Raising operator** is now

$$\begin{aligned} R &= e^{tH} Y e^{-tH} = e^{tH} X e^{-tH} W(D) \\ &= (X + tH')W = Y + tH'W \end{aligned}$$

- **Canonical polynomials** $y_n(x) = Y^n 1$
- **Canonical Appell system** $h_n(x, t) = e^{tH} y_n(x)$

Theorem 15.1 *For canonical Appell systems, we have:*

1. *The generating function*

$$e^{v \cdot R} \,|\, 0 \,\rangle = e^{x \cdot U(v) + tH(U(v))} = \sum_{n \geq 0} \frac{v^n}{n!} h_n(x, t)$$

2. *The relation* $e^{V(z) \cdot R} \,|\, 0 \,\rangle = e^{z \cdot x + tH(z)}$

3. *The* **recursion operator** $X = RV' - tH'$

   $U(v)$ is the functional inverse to V: $V(U(v)) = v$.