Lie algebras
Representations
and
Analytic Semigroups
through
Dual Vector Fields

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Part IV. Polynomials

Orthogonal families

Appell states

Canonical polynomials from Lie algebras
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Appell states

Canonical polynomials from Lie algebras
Orthogonal polynomials and Fourier transform

- **Orthogonal polynomials** may be described in terms of Fourier-Laplace transforms.

- **Measure** $p(dx)$ functions $\phi_n(x)$ are orthogonal to all polynomials of degree less than $n$ if and only if $V_n(s)$

$$V_n(s) = \int_{-\infty}^{\infty} e^{sx} \phi_n(x) p(dx)$$

has a zero of order $n$ at $s = 0$.

- **Follows by**

$$\left( \frac{d}{ds} \right)^k V_n(s) = \int_{-\infty}^{\infty} x^k \phi_n(x) p(dx)$$

- **If the $\phi_n(x)$ are polynomials** they form a sequence of orthogonal polynomials.
2 Orthogonal polynomials via kernels

• Kernels

\[ K(x, z, A) \]

forming a group under convolution

\[
\int_{-\infty}^{\infty} K(x-y, z, A) K(y, z', A') \, dy = K(x, z+z', A'')
\]

• Multiplicative family

\[ \hat{K}(s, z, A) = \int_{-\infty}^{\infty} e^{sy} K(y, z, A) \, dy \]

Then

\[
\hat{K}(s, z, A) \hat{K}(s, z', A') = \hat{K}(s, z+z', A'')
\]

• Form the product that integrates to \( K(x, 0, A'') \), independent of \( z \)

\[ K(x - y, -z, A) K(y, z, A') \]
• Generating function for the orthogonal functions

\[ K(x - y, -z, A)K(y, z, A') = \sum z^n H_n(x, y; A, A') \]

• By construction

\[ \int_{-\infty}^{\infty} H_n(x, y; A, A') \, dy = 0 \]

• To get orthogonality with respect to all polynomials of degree less than \( n \)

\[ \sum z^n \int_{-\infty}^{\infty} y^k H_n(x, y; A, A') \, dy \]

\[ = \int_{-\infty}^{\infty} y^k K(x - y, -z, A)K(y, z, A') \, dy \]

where the terms of the summation must vanish for \( k < n \).

I.e., this must reduce to a polynomial in \( z \) of degree \( k \).

• Or the Fourier-Laplace transform must have terms with zeros of the corresponding order.
3 Natural exponential families

- **Means form an additive group** for a convolution family of measures.

- **The densities** provide kernels of the form $K(x, z, A)$, where $z$ is the mean, and $A$, e.g., is the variance, or other parameters determining the distribution.

- **Gaussian distributions**

  $$K(x, z, A) = \frac{e^{-(x-z)^2/(2A)}}{\sqrt{2\pi A}}$$

  Note that the means and variances are additive.

- **Natural exponential families** allow for parametrization by the means.

Consider MGF $M(s) = \int_{\mathbb{R}} e^{sx} p(dx)$. The NEF

$$p_s(dx) = M(s)^{-1} e^{sx} p(dx)$$

has means $\mu(s) = M'(s)/M(s)$. 
4 Bernoulli systems

- **Bernoulli system** is a canonical Appell system such that the basis \( \psi_n = R^n \Omega \) is orthogonal.

- Define the generating function

\[
\omega^t(z, x) = \sum_{n \geq 0} \frac{z^n}{n!} \phi_n
\]

where \( \phi_n = n! \psi_n / \gamma_n \).

- Consider a Bernoulli system in \( d \geq 1 \) dimensions with canonical operator \( V \) and Hamiltonian \( H \).

\[
e^{z \mu x_\mu - t H(z)} = \sum_{n \geq 0} \frac{V(z)^n}{n!} \psi_n
\]

- Fourier-Laplace transform of \( \omega_t \) times the measure of orthogonality turns out to be

\[
\int e^{sy} \omega^t(z, y) p_t(dy) = e^{z V(s) + t H(s)}
\]
• **Expanding in powers of** $z$ **yields the relation**

$$\int_{-\infty}^{\infty} e^{sy} \phi_n(y) p_t(dy) = V(s)^n e^{tH(s)}$$

so that $V(0) = 0$ is all we need to conclude that the $\phi_n$ are an orthogonal family.

• **The function** $V(z)$ **is normalized to**

$V'(0) = V''(0) = 1$. And $V(0) = H(0) = 0$.

• **We take** $t$ **as our parameter** $A$ **and**

$$K(x, z, A) = \omega^A(z, x)p_A(x)$$

• **For** $\omega^A(z, x) \geq 0$, **these are a family of probability measures** with mean $z + \mu A$, and variance $z + \sigma^2 A$,

where $\mu$ and $\sigma^2$ are the mean and variance respectively of $p_1$.  

• From the basic construction

\[ K(x - y, -z, A)K(y, z, B) = \]
\[ \omega^A(-z, x - y)\omega^B(z, y) p_A(x - y)p_B(y) \]

• \( H_n(x, y; A, B) = \)
\[ \sum_{k=0}^{n} \binom{n}{k} (-1)^k \phi_k(x-y, A)\phi_{n-k}(y, B) p_A(x-y)p_B(y) \]

with corresponding orthogonal polynomials

\[ \phi_n(x, y; A, B) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \phi_k(x-y, A)\phi_{n-k}(y, B) \]

• Measure of orthogonality \( p_A(x - y)p_B(y)/p_{A+B}(x) \).

• Proof of orthogonality is based on an addition formula for \( V(s) \).
New families from old

For the Meixner classes, i.e., the Bernoulli systems in one variable corresponding to $\text{sl}(2)$,

we have the corresponding classes generated as follows:

- **Gaussian** $\rightarrow$ Gaussian
- **Poisson** $\rightarrow$ Krawtchouk
- **Laguerre** $\rightarrow$ Jacobi
- **Binomial** (3 types) $\rightarrow$ Hahn (3 types)

Observe that for the binomial types, this is essentially the construction of **Clebsch-Gordan coefficients** for real forms of $\text{sl}(2)$. This construction works for the multivariate case as well.

Probabilistically, we are looking at the distribution of $X_1$ given $X_1 + X_2$, where $X_2$ is an independent copy of $X_1$. 
Orthogonal families

▷ Appell states

Canonical polynomials from Lie algebras
Definition

- **Given a probability measure** \( p(dx) \) and a family of square-integrable functions \( F(s, x) \)
  \[
  M(s) = \langle F(s, X) \rangle
  \]

- **Appell states** with respect to the measure \( p \)
  
  and the family \( F \) are the functions
  \[
  \Psi_s(x) = \frac{F(s, x)}{M(s)}
  \]

  That is, the \( \Psi_s \) are the functions \( F \) normalized to have unit expectation.

- **States** comes from physics terminology denoting a function of unit norm in \( L^2 \) of \( p \).

- **Typical choices** of the family \( F \) are
  1. \( F(s, x) = e^{sx} \) giving Fourier-Laplace transforms
  2. \( F(s, x) = (1 - sx)^{-1} \) corresponding to Stieltjes transforms.
7 Expansion in orthogonal polynomials

- **The main feature** is that the family $F(s, x)$ are eigenfunctions of an operator $X_s$

\[ X_s F(s, x) = x F(s, x) \]

- **The family of orthogonal polynomials** is

\[ \{ \phi_n \} \text{ with squared norms } \gamma_n = \| \phi_n \|^2 \]

- **Transforms** are defined by

\[ \langle \phi_n, \Psi_s \rangle = V_n(s) \]

Thus, we have the expansion (assuming completeness)

\[ \Psi_s = \sum_{n \geq 0} V_n(s) \phi_n(x) / \gamma_n \]

- **In terms of the family** $F$

\[ F(s, x) = M(s) \sum_{n \geq 0} V_n(s) \phi_n(x) / \gamma_n \]
Recurrence relations for orthogonal polynomials

- **Three-term recurrence** is of the form
  \[ x \phi_n = c_n \phi_{n+1} + a_n \phi_n + b_n \phi_{n-1} \]
  with initial conditions \( \phi_{-1} = 0, \phi_0 = 1 \).

- **The recurrence relation** implies
  \[ \phi_1(x) = (x - a_0)/c_0 \]

- **Theorem**

Let \( F(0, x) = 1, X_s F(s, x) = x F(s, x) \). Then

\[ M(s)^{-1} X_s (M(s)V_n(s)) = c_n V_{n+1} + a_n V_n + b_n V_{n-1} \]

with \( V_0 = 1, V_1 = c_0^{-1}(M^{-1} X_s M - a_0) \).

We illustrate for the Meixner case.
9 Exponential families

- For $F(s, x) = e^{sx}$, we have $M(s) = \langle e^{sX} \rangle$, the MGF and
  
  $$X_s = \frac{d}{ds}$$

- The exponential function $e^{sx}$ has the expansion in orthogonal polynomials

  $$e^{sx} = M(s) \sum_{n \geq 0} V_n(s) \phi_n(x) / \gamma_n$$

  where the coefficients $V_n, n \geq 1$, satisfy the recurrence formula

  $$V'_n + c_0 V_1 V_n = c_n V_{n+1} + (a_n - a_0) V_n + b_n V_{n-1}$$

  with $V_0(s) = 1$ and

  $$V_1(s) = c_0^{-1} \left( \frac{M'(s)}{M(s)} - a_0 \right) .$$
9.1 Meixner systems

- These arise when we have the special form

\[ V_n(s) = V(s)^n \]

where, in particular, \( V_1(s) = V(s) \).

- We have the expansion

\[ e^{sx} = M(s) \sum_{n \geq 0} V(s)^n \phi_n(x) / \gamma_n \]

- with \( V(s) = c_0^{-1} \left( \frac{M'(s)}{M(s)} - a_0 \right) \)

- And \( V \) satisfies the **Riccati differential equation**

\[ V' = \gamma + 2\alpha V + \beta V^2 \]

- The recurrence formula for the orthogonal polynomials is

\[ x \phi_n = (c_0 + \beta n) \phi_{n+1} + (a_0 + 2\alpha n) \phi_n + \gamma n \phi_{n-1} \]
10 Canonical description of Meixner classes

• **Six families of orthogonal polynomials** that are canonical Appell systems.

• The $V$ and $H$ operators take the form

Meixner

\[
V(z) = \frac{\tanh qz}{q - \alpha \tanh qz} \quad H(z) = -\frac{\alpha}{\beta}z - \log \frac{qV(z)}{\sinh qz}
\]

Meixner-Pollaczek

\[
V(z) = \tan z \quad H(z) = \log \sec z
\]

Krawtchouk

\[
V(z) = \tanh z \quad H(z) = \log \cosh z
\]

Charlier

\[
V(z) = e^z - 1 \quad H(z) = e^z - 1 - z
\]

Laguerre

\[
V(z) = z/(1 - z) \quad H(z) = -\log(1 - z) - z
\]

Hermite

\[
V(z) = z \quad H(z) = z^2/2
\]

• **Parameters** are $\alpha, \beta$ with $q^2 = \alpha^2 - \beta$.

We will see how these arise by specialization from families of canonical polynomials for Lie algebras. They come from some basic Lie algebras, namely, sl(2), HW, and osc.
Orthogonal families

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▷ Canonical polynomials from Lie algebras
11 Flow of the group law

• **The left dual** vector field \( X^\dagger = \alpha_{\mu} \xi^\dagger_{\mu} \) generates the flow of the group law

\[
\exp(tX^\dagger) f(A) = f(A(\alpha t) \odot A)
\]

Setting \( t = 1 \) we have

\[
e^{X^\dagger} f(A) = f(A(\alpha) \odot A)
\]

• **Let** \( \hat{X} = \alpha_{\mu} \hat{\xi}_{\mu} \) be the double dual realization of \( X \).

In terms of \((x, D)\) variables, it is the dvf to \( X^\dagger \).

• **The Main Observation** for dvf’s gives

\[
e^{\hat{X}} e^{a x} = e^{(A(\alpha) \odot a) x}
\]

• **Compare with**

\[
e^{\alpha_{\mu} Y_{\mu}} e^{a x} = e^{x_{\mu} U_{\mu} (V(a)+\alpha)}
\]

our main formula for dvf’s in the abelian case.
11.1 Main theorem

Group elements generated by the double dual $\hat{X}$ and group elements generated by the canonical variable $\alpha_\mu Y_\mu$ give the same result on the vacuum state

$$ e^{\hat{X}} 1 = \exp(x \cdot A(\alpha)) = e^{\alpha_\mu Y_\mu} 1 = \exp(x \cdot U(\alpha)) $$

- **Correspondence** of the momentum variables with the $A$ coordinates is

  $$ D \leftrightarrow A, \quad V \leftrightarrow \alpha $$

Thus, the canonical operators $Y_i$ are given as

$$ Y_i = x_\mu W_{\mu i}(D) $$

where $W$ is the inverse Jacobian matrix of the coordinate map $A \rightarrow \alpha$.

- **Express** $\partial A/d\alpha$ in the $A$ variables, then replace every $A_i$ by the corresponding $D_i$.

We have a **Lie canonical system** of polynomials $\{y_n\}$

$$ e^{x \cdot A(\alpha)} = \sum_{n \geq 0} \frac{\alpha^n}{n!} y_n(x) $$
12 Cartan decomposition

- **Family of commuting self-adjoint operators** are the quantum observables for the system.
- We want the Lie algebra to be a **symmetric Lie algebra**
  
  Raising operators $\mathcal{P} \leftrightarrow$ Lowering operators $\mathcal{L}$

so that $\mathfrak{g}$ has the **Cartan decomposition**

$$\mathfrak{g} = \mathcal{L} \oplus \mathcal{K} \oplus \mathcal{P}$$

with the relations

$$[\mathcal{L}, \mathcal{P}] \subset \mathcal{K}, \quad [\mathcal{K}, \mathcal{L}] \subset \mathcal{L}, \quad [\mathcal{K}, \mathcal{P}] \subset \mathcal{P}$$

$\mathcal{L}$ and $\mathcal{P}$ are **abelian subalgebras** that generate $\mathfrak{g}$ as a Lie algebra.

- **Inner product** so that $L_i^* = R_i$ making $\mathfrak{g}$ a “Lie*-algebra”.

- **Self-adjoint operators** of interest have the general form

$$X_i = R_i + K_i + L_i$$

A commuting family of such $X_i$ will be the quantum observables.
The coordinate map is

\[ A_1 = \alpha_1, \quad A_2 = \alpha_2 + \frac{1}{2} \alpha_1 \alpha_3, \quad A_3 = \alpha_3 \]

From the double dual

\[
\exp(\alpha_1 R_1 + \alpha_2 R_2 + \alpha_3 (R_3 + R_2 V_1)) 1 = \\
\exp(\alpha_1 R_1 + (\alpha_2 + \frac{1}{2} \alpha_1 \alpha_3) R_2 + \alpha_3 R_3) 1
\]

Note that \( R_3 \) and \( \alpha_2 R_2 \) drop out.

Setting \( R_2 = t, \alpha_1 = \alpha_3 = z \), we get, using \( R = R_1 \) as our raising operator,

\[
\exp(z(R + tV)) 1 = e^{zR + z^2 t/2} 1
\]

Our quantum observable is \( X = R + tV \), with spectral variable \( x \). With \( v = z \),

\[ e^{vR} 1 = e^{vx - v^2 t/2} \]

the generating function for the Hermite polynomials for the corresponding Gaussian distribution.

We have recovered our example of Chapter 1.
The coordinate map is:

\[ A_1 = \frac{\alpha_1 \tanh \delta}{\delta - \alpha_2 \tanh \delta}, \quad A_2 = \log \frac{\delta \sech \delta}{\delta - \alpha_2 \tanh \delta}, \quad A_3 = \frac{\alpha_3 \tanh \delta}{\delta - \alpha_2 \tanh \delta} \]

where \( \delta = \sqrt{\alpha_2^2 - \alpha_1 \alpha_3} \).

The double dual is:

\[ \hat{\xi}_1 = R_1, \quad \hat{\xi}_2 = R_2 + 2R_1 V_1, \quad \hat{\xi}_3 = R_3 e^{2V_2} + R_2 V_1 + R_1 V_1^2 \]

Now take \( \alpha_1 \to z, \alpha_2 \to \alpha z, \alpha_3 \to \beta z, \) and \( \delta \to qz, q^2 = \alpha^2 - \beta \). Noting that \( R_3 \) drops out, send \( R_2 \to t \), and use \( R = R_1 \) as our raising operator to yield

\[ e^{zX_1} = \left( \frac{q \sech qz}{q - \alpha \tanh qz} \right)^t \exp \left( \frac{\tanh qz}{q - \alpha \tanh qz} R \right) \]

Our quantum random variable is

\[ X = R + \alpha t + 2\alpha RV + \beta(tV + RV^2) \]
• **With spectral variable** $x$, this is of the form

$$e^{zx} = e^{tH(z)}e^{V(z)R1}$$

and solving for $e^{vR1}$ gives the generating function for the corresponding class of polynomials as a canonical Appell system

$$e^{vR1} = e^{xU(v) - tH(U(v))}$$

• **Various specializations** lead to the Meixner classes for Bernoulli, negative binomial and continuous binomial (hyperbolic) distributions.

• **The gamma/exponential** family is an interesting limiting case where $q \to 0$. We get, then, with $\beta = \alpha^2$,

$$e^{zx}1 = (1 - \alpha z)^{-t} \exp \left( R \frac{z}{1 - \alpha z} \right) 1$$

and solving for $z = U(v) = \frac{v}{1 + \alpha v}$ yields the generating function for Laguerre polynomials in an appropriate normalization.

• The Poisson and Gaussian are limiting cases as well.
For the oscillator algebra we have the coordinate map

\[ A_1 = \frac{\alpha_1}{\alpha_4} (e^{\alpha_4} - 1), \quad A_2 = \alpha_2 + \frac{\alpha_1 \alpha_3}{\alpha_4^2} (e^{\alpha_4} - 1 - \alpha_4), \]

\[ A_3 = \frac{\alpha_3}{\alpha_4} (1 - e^{-\alpha_4}), \quad A_4 = \alpha_4. \]

The double dual is

\[ \hat{\xi}_1 = R_1, \quad \hat{\xi}_2 = R_2, \quad \hat{\xi}_3 = R_3 + R_2 V_1, \]

\[ \hat{\xi}_4 = R_4 + R_1 V_1 - R_3 V_3 \]

We take \( \alpha_4 \to \alpha z, \alpha_1 \to z, \alpha_3 \to \beta z, \)

with \( R_4 \) dropping out and get, setting \( R_2 = t, \)

\[ e^{zX+zY} = \exp(R_1 (e^{\alpha z} - 1)/\alpha) \cdot \exp(\beta t(e^{\alpha z} - 1 - \alpha z)/\alpha^2) \exp(R_3 \beta(1 - e^{-\alpha z})/\alpha) 1 \]
where

\[ X = R_1 + \alpha R_1 V_1 + \beta t V_1 \]

and

\[ Y = \beta R_3 - \alpha R_3 V_3 \]

- **The** \( Y \) **term** gives an independent \( \text{aff}(2) \).

- **The** \( X \) **term** gives, with \( R \to R_1 \),

\[
    e^{vR_1} = (1 + \alpha v)^{x/\alpha + \beta t/\alpha^2} \exp(-v\beta t/\alpha)
\]

which is the generating function for **Poisson-Charlier polynomials** for a scaled Poisson process with drift.

- **Observe** that in each case, the formula for \( X \) in terms of \( R \) and \( V \) gives the **three-term recurrence** relation for the corresponding orthogonal polynomials.
16 Procedure

1. Cartan decomposition.

2. Find coordinate mapping.

3. Exponentiate the double dual.

4. Arrange into commuting operators of the form
   \[ X = R + K + L. \]

5. Group elements generated by raising operators acting on the vacuum are generating functions for the basis of the representation.

6. Expectation values of group elements generated by \( X \) operators interpreted as moment generating functions yield the spectral measures.