# Lie algebras Representations and Analytic Semigroups through Dual Vector Fields

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## Part II. Lie Algebras: Representations and Groups

### Enveloping algebras

### Dual Representations

### Matrix Elements

# 1 Group elements

## ○ Universal enveloping algebra

Lie algebra $\mathfrak{g}$ has basis $\{\, \xi_1, \ldots, \xi_d \,\}$.
We consider the algebra generated by the elements $\xi_i$ modulo the commutation relations defining $\mathfrak{g}$.
This is $\mathcal{U}(\mathfrak{g})$.
We will find the action of $\mathfrak{g}$ on that space.

The **Poincaré-Birkhoff-Witt theorem** says that $\mathcal{U}(\mathfrak{g})$ is an associative algebra with basis vectors given by ordered monomials

$$| \, n \, \rangle = \xi^n = \xi_1^{n_1} \cdots \xi_d^{n_d}$$

on which $\mathfrak{g}$ acts. These monomials are the **Poincaré-Birkhoff-Witt basis**.

## ○ Boson operators $R_i, V_j$ act on the basis as

$$R_i | \, n \, \rangle = | \, n + \mathrm{e}_i \, \rangle, \qquad V_i | \, n \, \rangle = n_i | \, n - \mathrm{e}_i \, \rangle$$

○ The idea is to

**express the elements of $\mathfrak{g}$ in terms of $R$'s and $V$'s**.

○ Now $X$ will denote a general element of $\mathfrak{g}$, with coefficients $\{\alpha_i\}$,

$$X = \alpha_\mu \xi_\mu$$

○ The operator of multiplication by $x$ we will identify with $x$.

$\boxed{♪}$ **HW algebra** with the basis $\{Q, H, P\}$. We may take

$Q = X$, $P = tD$, $H = tI$.

$\mathcal{U}(\mathfrak{g})$ has basis

$$|\, l, m, n\,\rangle = Q^l H^m P^n$$

By induction: $[P, Q^l] = lQ^{l-1}H$. Thus, the representation

$$\begin{aligned}
\hat{Q}|\, l, m, n\,\rangle &= |\, l+1, m, n\,\rangle \\
\hat{H}|\, l, m, n\,\rangle &= |\, l, m+1, n\,\rangle \\
\hat{P}|\, l, m, n\,\rangle &= |\, l, m, n+1\,\rangle + l|\, l-1, m+1, n\,\rangle
\end{aligned}$$

○ **Duality techniques** will show how multiplication by the basis elements $\xi_i$ on $\mathcal{U}(\mathfrak{g})$ looks.

Form the generating function $g(A) = \sum_{n \geq 0} \frac{A^n}{n!}\, \xi^n$.

So

$$
\begin{aligned}
g(A) &= \sum_{n_1, n_2, \ldots, n_d} \frac{(A_1\xi_1)^{n_1}}{n_1!} \cdots \frac{(A_d\xi_d)^{n_d}}{n_d!} \\
&= e^{A_1\xi_1} \cdots e^{A_d\xi_d}
\end{aligned}
$$

This is an element of the group $\mathcal{G}$ generated by $\mathfrak{g}$, as it is a product of one-parameter subgroups of $\mathcal{G}$.

○ **The group law** is

$$g(A)g(A') = g(A \odot A')$$

♪ For the HW group we have, using the $3 \times 3$ matrix representation

$$e^{A_1\xi_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & A_1 \\ 0 & 0 & 1 \end{pmatrix}, e^{A_2\xi_2} = \begin{pmatrix} 1 & 0 & A_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$e^{A_3\xi_3} = \begin{pmatrix} 1 & A_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiplying these gives $g(A) = \begin{pmatrix} 1 & A_3 & A_2 \\ 0 & 1 & A_1 \\ 0 & 0 & 1 \end{pmatrix}$. Multiplying

$$g(A)g(B) = \begin{pmatrix} 1 & A_3 + B_3 & A_2 + B_2 + A_3 B_1 \\ 0 & 1 & A_1 + B_1 \\ 0 & 0 & 1 \end{pmatrix}$$

Now compare with the form of $g(A)$ to find the group law

$$\begin{aligned} (A \odot B)_1 &= A_1 + B_1 \\ (A \odot B)_2 &= A_2 + B_2 + A_3 B_1 \\ (A \odot B)_3 &= A_3 + B_3 \end{aligned}$$

## 2 Adjoint representation of the group

○ **Adjoint representation** extends to the group by exponentiation. We have $u = e^{AY} X e^{-AY}$ satisfying

$$\frac{\partial u}{\partial A} = Yu - uY = (\mathrm{ad}\, Y)u$$

with initial condition $u(0) = X$. Or

$$e^{AY} X e^{-AY} = e^{A\,\mathrm{ad}\,Y} X$$

As a series expansion

$$e^{AY} X e^{-AY} = X + A[Y, X] + \frac{A^2}{2}[Y, [Y, X]] + \cdots$$

○ **Adjoint group matrices** are exponentials of the adjoint representation of $\mathfrak{g}$. The adjoint group action of $\xi_k$ on $\xi_j$ is

$$\begin{aligned} e^{A\xi_k} \xi_j e^{-A\xi_k} &= C^1_{kj}(A)\xi_1 + C^2_{kj}(A)\xi_2 + \cdots + C^d_{kj}(A)\xi_d \\ &= C^\mu_{kj}(A)\xi_\mu \end{aligned}$$

Thus, the matrices

$$(\check{C}_k(A))_{ij} = C^i_{kj}(A)$$

Note that $\check{C}_k(0)$ is the identity matrix for every $k$.

# 3 Examples

♩ For the HW algebra, we have

$$(\mathrm{ad}\,P)(Q) = H\,, \quad (\mathrm{ad}\,P)^2(Q) = [P, H] = 0$$

So

$$e^{AP} Q e^{-AP} = Q + AH$$

For any suitable $f$,

$$e^{AP} f(Q) e^{-AP} = f(Q + AH)$$

Acting on the vacuum with

$$P\Omega = 0\,, \quad H\Omega = 1\,, \quad Q\Omega = x$$

Since $Q$ and $H$ commute, we may iteratively calculate

$$e^{AP} Q^n \Omega = (x + A)^n \Omega$$

which shows that $P$ generates the **translation group**

$$e^{AP} f(x) = f(x + A)$$

♪ The **affine algebra**, aff(2) has basis elements $\xi_1, \xi_2$ satisfying

$$[\xi_2, \xi_1] = \xi_1$$

We may take

$$\xi_1 = x \text{ [multiplication by x]}, \xi_2 = xD \text{ [the number operator]}$$

The adjoint action is

$$e^{A\xi_2}\xi_1 e^{-A\xi_2} = \xi_1 + A\xi_1 + \frac{A^2}{2}\xi_1 + \cdots = e^A \xi_1$$

I.e., we have the formula

$$e^{A\,xD} x\, e^{-A\,xD} = e^A x$$

Raising both sides to the $n^{\text{th}}$ power, for suitable functions $f$,

$$e^{A\,xD} f(x) e^{-A\,xD} = f(e^A x)$$

Applying this to the vacuum, $1$, yields

$$e^{A\,xD} f(x) = f(e^A x)$$

With $\lambda = e^A$, this shows that $xD$ generates the dilation group

$$\lambda^{xD} f(x) = f(\lambda x)$$

♫ For sl(2), we have

$$[\Delta, R] = \rho\,, \quad (\mathrm{ad}\,\Delta)^2(R) = 2\Delta$$

Thus

$$e^{A\Delta} R e^{-A\Delta} = R + A\rho + A^2\Delta$$

On the vacuum with

$$\Delta\Omega = 0\,, \quad R\Omega = x\,, \quad \rho\Omega = c\Omega$$

we get

$$e^{A\Delta} f(x) = f(R + A\rho + A^2\Delta)\Omega$$

○ The action is not immediate as these elements do not commute.

This is one of the motivations behind the splitting technique that has been developed.

## 4 Method of characteristics

○ **The flow of a vector field** Write

$$X = \pi_\mu(x)\frac{\partial}{\partial x_\mu}$$

where $\pi_i(x)$ are locally analytic functions.

Note that $X1 = 0$.

Let

$$x_i(t) = e^{tX} x_i \, e^{-tX}$$

Then for suitable functions $f$,

$$f(x(t)) = e^{tX} f(x) e^{-tX}$$

Thus the solution to

$$\frac{\partial u}{\partial t} = Xu, \qquad\qquad u(0) = f(x)$$

is given by

$$u = e^{tX} f(x) = f(x(t))1$$

- Observe that

$$(\mathrm{ad}\, X) f(x) = [X, f(x)] = \pi_\mu(x)[D_\mu, f(x)] = Xf(x)$$

is a function, i.e., no derivative operators are involved.

Iterating yields

$$f(x(t)) = e^{tX} f(x) = f(x(t))1$$

as a function of $x$ and $t$.

Now

$$\dot{x}_i(t) = e^{tX}[X, x_i]e^{-tX} = e^{tX}\pi_i(x)e^{-tX} = \pi_i(x(t))$$

holds for $x_i(t)$ as functions of $x$ and $t$.

The equations

$$\dot{x} = \pi(x)$$

are the **characteristic equations** for the flow generated by $X$. They are solved with initial conditions

$$x_i(0) = x_i$$

## 5 Pi-matrices

◦ **Left and right multiplication operators**

$$\xi_i g = \xi_i^{\ddagger} g, \qquad g\xi_i = \xi_i^* g$$

where now $\xi_i^{\ddagger}$ and $\xi_i^*$ are **vector fields** in the $A$ variables.

◦ **Operator $\partial_i$** brings down $\xi_i$ in the product $g(A)$. First, $\partial_1 g = \xi_1^{\ddagger}$. Next,

$$\begin{aligned} \partial_2 g &= e^{A_1\xi_1}\xi_2 e^{A_2\xi_2}\dots e^{A_d\xi_d} \\ &= e^{A_1\xi_1}\xi_2 e^{-A_1\xi_1}e^{A_1\xi_1}e^{A_2\xi_2}\dots e^{A_d\xi_d} \\ &= (\xi^{\ddagger}\check{C}_1(A_1))_2\, g \end{aligned}$$

For $\partial_3$ we find

$$\partial_3 g = (\xi^{\ddagger}\check{C}_1(A_1)\check{C}_2(A_2))_3$$

And so on. We get

$$\begin{aligned} \partial_i &= (\xi^{\ddagger}\check{C}_1(A_1)\check{C}_2(A_2)\check{C}_3(A_3)\dots\check{C}_{k-1}(A_{k-1}))_i \\ &= \Pi^{\ddagger}_{i\mu}(A)\xi^{\ddagger}_{\mu} \end{aligned}$$

We can write these in terms of column vectors $\partial = (\partial_1, \partial_2, \dots, \partial_d)$ and $\xi^{\ddagger}$ as

$$\partial = \Pi^{\ddagger}(A)\xi^{\ddagger}$$

○ **Pi-matrices** are inverse to the $\Pi$'s.

We have **left-dual vector fields**

$$\xi_i^{\ddagger} = \pi^{\ddagger}(A)_{i\mu}\partial_\mu$$

○ **Right action** is found by converting $\partial_A$'s pulling $\xi_i$'s to the right.

○ **right-dual vector fields**

$$\xi_i^{*} = \pi^{*}(A)_{i\mu}\partial_\mu$$

○ **Dual maps** Right dual $\xi \longrightarrow \xi^*$ is a Lie homomorphism, i.e.,

$$[\xi_i, \xi_j]^* = [\xi_i^*, \xi_j^*]$$

Left dual reverses the order, so is a Lie antihomomorphism

$$[\xi_i, \xi_j]^{\ddagger} = [\xi_j^{\ddagger}, \xi_i^{\ddagger}]$$

As vector fields, every $\xi_i^{\ddagger}$ commutes with every $\xi_j^*$.

## 6 Coordinates of the second kind

As a vector space with basis $\{\xi_1, \ldots, \xi_d\}$, a typical element of $\mathfrak{g}$ has the form $X = \alpha_\mu \xi_\mu$.

For the one-parameter subgroup generated by $X$ we have

$$\begin{aligned} e^{tX} &= e^{A_1(t)\xi_1} e^{A_2(t)\xi_2} \ldots e^{A_d(t)\xi_d} \\ &= g(A(t)) \end{aligned}$$

For group elements

- **Coordinates of the first kind** are the $\{\ \alpha_i\ \}$.
- **Coordinates of the second kind** are the $\{\ A_i\ \}$.
- **Coordinate mapping** When $t = 1$,

$$\alpha \longrightarrow A(\alpha) \qquad \text{and} \qquad A \longrightarrow \alpha(A)$$

corresponding to the relation

$$e^{\alpha_\mu \xi_\mu} = e^{A_1(\alpha)\xi_1} e^{A_2(\alpha)\xi_2} \ldots e^{A_d(\alpha)\xi_d}$$

We have effectively factorized, “split”, the exponential into a product of one-parameter subgroups. Relating the two types of coordinates is the **splitting lemma**.

## 7 Flow of the group law

- **Left dual**

$$X\,g(A) = X^{\ddagger} g(A) = \alpha_\lambda \pi^{\ddagger}_{\lambda\mu} \partial_\mu \, g(A)$$

  - $\alpha = (\alpha_1, \ldots, \alpha_d)$
  - $t\alpha = (t\alpha_1, \ldots, t\alpha_d)$ for a real parameter $t$.
  - $A(t) = A(t\alpha)$, as $X \rightarrow tX$ maps $\alpha \rightarrow t\alpha$.

Since $X$ and $X^{\ddagger}$ commute, we can iteratively generate the exponentials to get

$$g(x(t)) = e^{tX^{\ddagger}} g(A) = g(A(t\alpha) \odot A)$$

The characteristics for the flow generated by $X^{\ddagger}$ are given by

$$\dot{x}_i = \alpha_\lambda \pi^{\ddagger}_{\lambda i}(x)$$

with solution $x(t) = A(t\alpha) \odot A$.

- **Right dual** multiplying on the right yields the equations

$$\dot{x}_i = \alpha_\lambda \pi^{*}_{\lambda i}(x)$$

for $x(t) = A \odot A(t\alpha)$.

# 8 Splitting Lemma

Let $X = \alpha_\mu \xi_\mu$. Factor

$$\exp(X) = g(A) = e^{A_1(\alpha)\xi_1} \ldots e^{A_d(\alpha)\xi_d}$$

Let $\tilde{\pi}$ denote the coefficient matrix (pi-matrix) of **either** the left or the right dual representation.

Then the coordinate map

$$\alpha \to (A_1(\alpha), \ldots, A_d(\alpha))$$

is determined by solving the differential equations

$$\dot{A}_j = \alpha_\lambda \tilde{\pi}_{\lambda j}(A)$$

$j = 1, \ldots, d$, for $A_i$ as functions of $t$ with the initial conditions

$$A_1(0) = \cdots = A_d(0) = 0$$

Then

$$A_i(\alpha) = A_i(t)\big|_{t=1}$$

for $1 \leq i \leq d$.

## 9 Finding pi-matrices

The splitting lemma is useful for finding the pi-matrices.

Here's the procedure:

1. Write $X = \alpha_\mu \xi_\mu$.

2. Calculate $g(A)$. Formally differentiate with respect to $t$.

3. Equate the result of step 2 with $Xg(A)$. Solve for $\dot{A}_i$.

4. Express the formulas for $\dot{A}_i$ as $\alpha_\mu \pi^\ddagger_{\mu i}(A)$.

5. Similarly, use $g(A)X$ to find $\pi^*(A)$.

## 10 Examples

♩ For HW, $X = \begin{pmatrix} 0 & \alpha_3 & \alpha_2 \\ 0 & 0 & \alpha_1 \\ 0 & 0 & 0 \end{pmatrix}$. From our result for $g(A)$, we find

$$\dot{g} = \begin{pmatrix} 0 & \dot{A}_3 & \dot{A}_2 \\ 0 & 0 & \dot{A}_1 \\ 0 & 0 & 0 \end{pmatrix} = Xg = \begin{pmatrix} 0 & \alpha_3 & \alpha_2 + A_1\alpha_3 \\ 0 & 0 & \alpha_1 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus,

$$\begin{aligned} \dot{A}_1 &= \alpha_1 \\ \dot{A}_2 &= \alpha_2 + A_1\alpha_3 \\ \dot{A}_3 &= \alpha_3 \end{aligned}$$

We read off

$$\pi^{\ddagger}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & A_1 & 1 \end{pmatrix}$$

Similarly, we find

$$\pi^{*}(A) = \begin{pmatrix} 1 & A_3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## 10.1 HW coordinate map

Solving for the left flow with initial conditions $A(0) = A$ :

$$\begin{aligned} A_1(t) &= A_1 + \alpha_1 t \\ A_2(t) &= A_2 + \alpha_2 t + A_1 \alpha_3 t + \alpha_1 \alpha_3 t^2/2 \\ A_3(t) &= A_3 + \alpha_3 t \end{aligned}$$

Setting $t = 1$, $A = 0$, gives the coordinate map

$$\begin{aligned} A_1(\alpha) &= \alpha_1 \\ A_2(\alpha) &= \alpha_2 + \alpha_1 \alpha_3/2 \\ A_3(\alpha) &= \alpha_3 \end{aligned}$$

- Now we can verify that

$$A(t)\Big|_{t=1} = A(\alpha) \odot A$$

Similar properties hold for the right flow.

♪ A matrix realization of aff(2) is given by

$$X = \begin{pmatrix} \alpha_2 & \alpha_1 \\ 0 & 0 \end{pmatrix}$$

The corresponding group element is

$$g(A) = \begin{pmatrix} e^{A_2} & A_1 \\ 0 & 1 \end{pmatrix}$$

The group law is

$$\begin{aligned} (A \odot B)_1 &= A_1 + B_1 e^{A_2} \\ (A \odot B)_2 &= A_2 + B_2 \end{aligned}$$

Equating $\dot{g} = Xg$ and $\dot{g} = gX$ we find the pi-matrices

$$\pi^{\ddagger} = \begin{pmatrix} 1 & 0 \\ A_1 & 1 \end{pmatrix}$$

and

$$\pi^{*} = \begin{pmatrix} e^{A_2} & 0 \\ 0 & 1 \end{pmatrix}$$

## 10.2 Aff2 coordinate map

- **Left flow** $\dot{A}_1 = \alpha_1 + \alpha_2 A_1$, $\dot{A}_2 = \alpha_2$ and

$$A_1(t) = A_1 e^{\alpha_2 t} + \frac{\alpha_1}{\alpha_2}\left(e^{\alpha_2 t} - 1\right), \quad A_2(t) = A_2 + \alpha_2 t$$

- **Right flow** $\dot{A}_1 = \alpha_1 e^{A_2}$, $\dot{A}_2 = \alpha_2$ so

$$A_1(t) = A_1 + \frac{\alpha_1}{\alpha_2}\left(e^{\alpha_2 t} - 1\right)e^{A_2}, \quad A_2(t) = A_2 + \alpha_2 t$$

- Now, setting $t = 1$ yields $A(\alpha) \odot A$ and $A \odot A(\alpha)$.
- Further, setting $A = 0$, gives the coordinate map

$$A_1(\alpha) = \frac{\alpha_1}{\alpha_2}\left(e^{\alpha_2} - 1\right), \quad A_2(\alpha) = \alpha_2$$

And from this we can check consistency with the flow of the group.

## 11 Double dual

The **right dual vector fields** $\xi_i^*$ give a Lie homomorphism.
To get a Lie homomorphism from the left dual,
we must dualize it.

○ Rewrite the left dual in terms of boson operators
$R$'s and $V$'s, exchanging

$$A \leftrightarrow V, \quad \partial \leftrightarrow R$$

ordering with all $R$'s on the left.

Thus, the **double dual** representation

$$\hat{\xi}_i = R_\mu \pi^{\ddagger}_{i\mu}(V)$$

Originally this is the action of multiplication on the left by $\xi_i$.
It is now expressed in terms of $R$ and $V$ acting on the basis $|\, n\, \rangle$. So

**we have calculated the action of $\mathfrak{g}$ on $\mathcal{U}(\mathfrak{g})$**

Note that since $R$ and $V$ are boson variables, we may conveniently replace them by $R \to x$, $V \to D$ to get a realization of $\mathfrak{g}$ in terms of operators acting on functions of $x$.

## 11.1 Examples

$\boxed{♩}$ HW. Let's find $\xi^*$, $\xi^\ddagger$, and $\hat{\xi}$.

The action on $\mathcal{U}(\mathfrak{g})$ indicates the double dual should be

$$\hat{Q} = R_1, \quad \hat{H} = R_2, \quad \hat{P} = R_3 + R_2 V_1$$

Now let's use the pi-matrices and write the dual vector fields.

$$\xi_1^* = \partial_1 + A_3 \partial_2, \quad \xi_2^* = \partial_2, \quad \xi_3^* = \partial_3$$

And

$$\xi_1^\ddagger = \partial_1, \quad \xi_2^\ddagger = \partial_2, \quad \xi_3^\ddagger = A_1 \partial_2 + \partial_3$$

which gives the double dual

$$\hat{\xi}_1 = R_1, \quad \hat{\xi}_2 = R_2, \quad \hat{\xi}_3 = R_2 V_1 + R_3$$

We may write the double dual in terms of $(x, D)$ as

$$\hat{\xi}_1 = x_1, \quad \hat{\xi}_2 = x_2, \quad \hat{\xi}_3 = x_2 D_1 + x_3$$

♪ Affine. Using our pi-matrices we have

$$\xi_1^* = e^{A_2}\partial_1, \quad \xi_2^* = \partial_2$$

And

$$\xi_1^\ddagger = \partial_1, \quad \xi_2^\ddagger = A_1\partial_1 + \partial_2$$

which gives the double dual

$$\hat{\xi}_1 = R_1, \quad \hat{\xi}_2 = R_1 V_1 + R_2$$

which we may write as

$$\hat{\xi}_1 = x_1, \quad \hat{\xi}_2 = x_1 D_1 + x_2$$

which recovers our original formulation of aff(2) if we ignore $x_2$.

Enveloping algebras

Dual Representations

▷ Matrix Elements

# 12 Principal formula

○ **Matrix elements** of the group acting on $\mathcal{U}(\mathfrak{g})$ are defined by

$$g(A)|\, n\,\rangle = \sum_m \left\langle {m \atop n} \right\rangle_A |\, m\,\rangle$$

○ These are **special functions** and typically can be expressed in terms of generalized hypergeometric functions.

○ We use as a basis for polynomials in $A$

$$c_m(A) = A^m/m! = (A_1^{m_1}/m_1!)\cdots(A_d^{m_d}/m_d!)$$

For finding the matrix elements, we have

○ **Principal formula** The matrix elements are given by

$$\left\langle {m \atop n} \right\rangle_A = (\xi^*)^n A^m/m!$$

where $(\xi^*)^n = (\xi_1^*)^{n_1}\cdots(\xi_d^*)^{n_d}$, basis monomials in terms of the right dual representation.

## 12.1 Proof

Write the product of group elements $g(A)$ and $g(B)$ as

$$
\begin{aligned}
g(A)g(B) &= g(A,\xi)\sum_n c_n(B)|\,n\,\rangle \\
&= \sum_n c_n(B)g(A)|\,n\,\rangle \\
&= \sum_{m,n} c_n(B)\left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_A |\,m\,\rangle
\end{aligned}
$$

since the $A$'s and $B$'s commute.

On the other hand, pulling exponentials in $B$ across $g(A)$ one at a time reconstitutes the group element $g(B)$ with $\xi$ replaced by $\xi^*$. Denoting this by $g(B)^*$ we have

$$
\begin{aligned}
g(A)g(B) &= g(B)^* g(A) \\
&= \sum_{n,m} c_n(B)(\xi^*)^n c_m(A)|\,m\,\rangle
\end{aligned}
$$

Comparing these two expressions leads to the desired formula.

♫ An immediate consequence of this formula is that **the right dual pi-matrices are matrix elements for transitions between basis elements.**

That is,

$$\pi^*_{ij} = \left\langle \begin{matrix} \mathrm{e}_j \\ \mathrm{e}_i \end{matrix} \right\rangle$$

*Proof:* This follows from the principal formula thus

$$\left\langle \begin{matrix} \mathrm{e}_j \\ \mathrm{e}_i \end{matrix} \right\rangle = \xi^*_i A_j = \pi^*_{i\lambda} \partial_\lambda A_j = \pi^*_{ij}$$

Now we will look at some of the many interesting relations for the matrix elements that can be deduced from the group law and the relations of the operators $\xi^*$. This approach to special functions is in the spirit of the classic work of Vilenkin, see Klimyk & Vilenkin’s four-volume opus.

# 13 Addition theorems

Write the group law (as in the above proof)

$$g(A)g(B) = \sum_{m,n} c_n(B) \left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_A |\, m \,\rangle$$

and as

$$g(A \odot B) = \sum_m c_m(A \odot B) |\, m \,\rangle,$$

we read off the transformation formula

$$c_m(A \odot B) = \sum_n \left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_A c_n(B)$$

○ the coefficients $c_n$ transform as a **vector** for the representation.

○ **Addition theorem** follows:

$$g(A)g(B) |\, n \,\rangle = g(A \odot B) |\, n \,\rangle$$

so

$$\left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_{A \odot B} = \left\langle \begin{matrix} m \\ \lambda \end{matrix} \right\rangle_A \left\langle \begin{matrix} \lambda \\ n \end{matrix} \right\rangle_B$$

So these are indeed a **matrix representation of the group** acting on $\mathcal{U}(\mathfrak{g})$.

○ **Left multiplication** by $\xi_i$ on $|\, n\,\rangle$ has matrix elements

$$\xi_i |\, n\,\rangle = \sum_r M_{rn}(\xi_i) |\, r\,\rangle$$

The right dual representation is a homomorphism, so

$$\begin{aligned}
\xi_i^* \left\langle {m \atop n} \right\rangle_A &= \xi_i^* (\xi^*)^n c_m(A) \\
&= \sum_r M_{rn}(\xi_i)(\xi^*)^r c_m(A) \\
&= \sum_r \left\langle {m \atop r} \right\rangle_A M_{rn}(\xi_i)
\end{aligned}$$

Now, recall that this action is the same as the **double dual** $\hat{\xi}_i = R_\mu \pi^\ddagger_{i\mu}(V)$ acting on the $n$-indices. In other words,

$$\xi_i^* \left\langle {m \atop n} \right\rangle_A = \hat{\xi}_i \left\langle {m \atop \mathbf{n}} \right\rangle_A$$

the boldface indicating that the multi-index $n$ is varied.

This is a **differential recurrence** as on one side we have a vector field and on the other a function of shift operators.

# 15 Example

♪ For the affine group, the principal formula gives the matrix elements

$$\left\langle \begin{matrix} m_1, m_2 \\ n_1, n_2 \end{matrix} \right\rangle_{A_1, A_2} = (\xi_1^*)^{n_1} (\xi_2^*)^{n_2} (A_1^{m_1}/m_1!)(A_2^{m_2}/m_2!)$$

- Difference indices $\Delta = m - n = (m_1 - n_1, m_2 - n_2)$.
- Using the right dual we find

$$\left\langle \begin{matrix} m_1, m_2 \\ n_1, n_2 \end{matrix} \right\rangle_{A_1, A_2}$$

$$= \quad (e^{A_2}\partial_1)^{n_1} (\partial_2)^{n_2} (A_1^{m_1}/m_1!)(A_2^{m_2}/m_2!)$$

$$= \quad e^{n_1 A_2} \frac{A_1^{\Delta_1}}{\Delta_1!} \frac{A_2^{\Delta_2}}{\Delta_2!}$$

- Bringing in the double dual

$$\hat{\xi}_1 = R_1, \quad \hat{\xi}_2 = R_2 + R_1 V_1$$

we find the following differential recurrence relations

$$(e^{A_2}\partial_1)\left\langle \begin{matrix} m_1, m_2 \\ n_1, n_2 \end{matrix} \right\rangle_{A_1,A_2} \quad = \quad \left\langle \begin{matrix} m_1, m_2 \\ n_1+1, n_2 \end{matrix} \right\rangle_{A_1,A_2}$$

$$\partial_2 \left\langle \begin{matrix} m_1, m_2 \\ n_1, n_2 \end{matrix} \right\rangle_{A_1,A_2}$$

$$= \quad \left\langle \begin{matrix} m_1, m_2 \\ n_1, n_2+1 \end{matrix} \right\rangle_{A_1,A_2} + n_1 \left\langle \begin{matrix} m_1, m_2 \\ n_1, n_2 \end{matrix} \right\rangle_{A_1,A_2}$$

Our approach provides a canonical formalism for expressing and discovering the properties these matrix elements have that qualifies them as “special” functions. Developing our approach further we find pure recurrence relations, not involving derivatives, that generalize the well-known ‘contiguous relations’ satisfied by classical hypergeometric functions.