Counting Spanning Trees

Diplomarbeit

zur Erlangung des akademischen Grades
Magister der Naturwissenschaften
an der
Formal- und Naturwissenschaftlichen Fakultät
der Universität Wien

Martin Rubey

Wien, Mai 2000

For my parents, who let me take this direction,
and for the patient: my wife, Christian and Markus

Contents

CHAPTER 1

Introduction

A *graph* consists of *edges* connecting *vertices*. A *tree* is a connected graph containing no cycles. A *spanning tree* is a subgraph of a graph that contains all its vertices and is itself a tree. It is the aim of this text to present various methods counting the number of spanning trees in special families of graphs as depicted below. The following table lists some families of graphs together with references to the sections where we calculate their number of spanning trees.

The material is organized as follows: In the second chapter, we give definitions and prove some very basic results of graph theory, which will be needed throughout the rest of the text.

Furthermore, many operations of graphs are presented which allow the composition of larger graphs from smaller ones, or transform complicated structures into less complicated. Especially in Chapter 5, Section 2 we will present formulas that calculate the spanning trees of a graph which can be obtained from smaller ones by some operation, if enough is known about the structure of the smaller graphs.

The third chapter is devoted to some other mathematical objects, which can be shown to be intimately related to the spanning trees of a graph. For example, given any planar graph G , i.e. a graph which can be drawn in the plane, we can construct another graph that has as many perfect matchings – spanning subgraphs in which every vertex is connected to exactly one other vertex – as G has spanning trees.

The last two chapters contain the main part of the text. In Chapter 4, we present some purely combinatorial methods. Although some of them are quite aesthetic, their application is limited to very few, special families of graphs.

Using the more algebraic methods of the last chapter – heavily relying on the famous Matrix-Tree-Theorem– we can compute the spanning tree number of many graphs in a very straightforward manner.

We will make use of the following notations: Given a subset S of a set $\mathcal{S}$ we will write S^c for the complement of S in $\mathcal{S}$. The entry in row i and column j of a matrix $\mathbf{M}$ will be denoted by $\mathbf{M}_{i,j}$. Similarly, $\mathbf{M}_{R,C}$ denotes the restriction of $\mathbf{M}$ to the rows indexed by R and the columns indexed by C. If $\mathbf{M}$ is a square matrix we will only write $\mathbf{M}_E$ for the minor of $\mathbf{M}$ given by the rows and columns indexed by R. In Table 1 of Chapter 2, Section 1 some more notational conventions are listed.

Name	Notation	References
Complete graph	K_p	Chapter 4, Corollary 3.2, page 42
		Chapter 5, Example 2.17, page 73
Complete multipartite graph	$K_{p_1,p_2,\ldots,p_n}$	Chapter 4, Example 3.10, page 50
		Chapter 5, Example 2.9, page 67
Circle	C_n	Chapter 5, Lemma 2.10, page 69
Ladder	L_n	Chapter 4, Example 1.1, page 34,
		Chapter 5, Example 2.19, page 74
		Chapter 5, Example 4.1, page 81
Cyclic Ladder	$K_2 \oplus C_n$	Chapter 4, Example 2.2, page 40
Möbius Ladder	M_n	Chapter 4, Example 2.3, page 41
Square of a Circle	C_n^2	Chapter 4, Example 2.4, page 41
Fan	F_n	Chapter 4, Example 1.2, page 35,
		Chapter 4, Example 3.15, page 52
		Chapter 5, Example 2.11, page 70
Wheel	W_n	Chapter 4, Example 1.3, page 35,
		Chapter 4, Example 3.16, page 52
		Chapter 5, Example 2.12, page 70
Even and Odd Aztec Rectangle	$ER_{m,n}, OR_{m,n}$	Chapter 3, Remark 5, page 33,
		Chapter 5, Example 2.23, page 78
		Chapter 5, Example 4.3, page 83

TABLE 1. Some special families of graphs

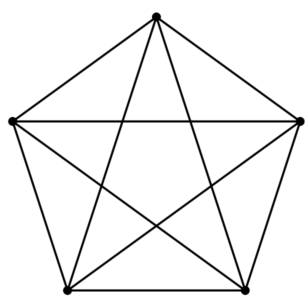


FIGURE 1
The complete graph K_5

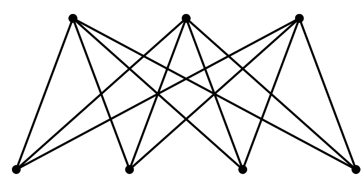


FIGURE 2
The complete bipartite graph $K_{3,4}$

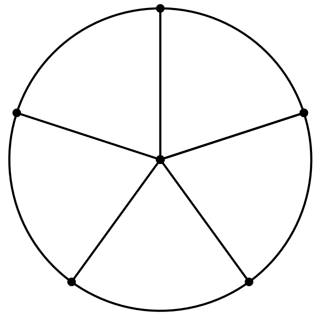


FIGURE 3
The wheel W_5

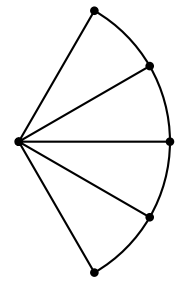


FIGURE 4
The fan F_5

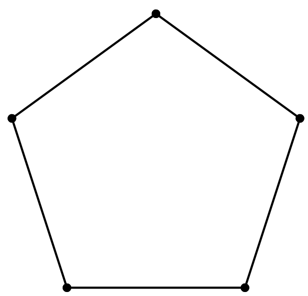


FIGURE 5
The circle C_5

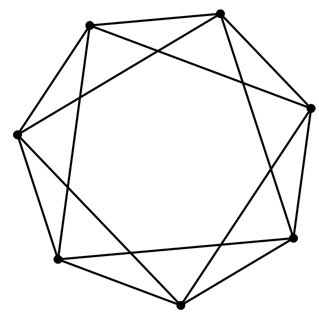


FIGURE 6
The square of a circle C_7^2

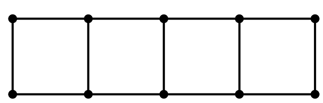


FIGURE 7
The ladder L_5

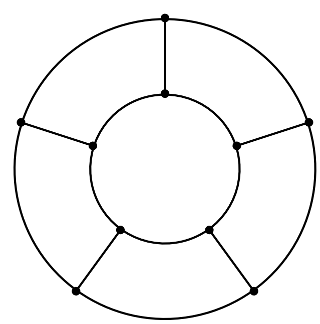


FIGURE 8
The cyclic ladder $K_2 \oplus C_5$

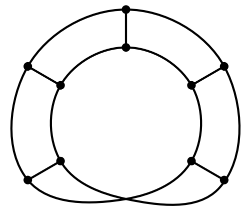


FIGURE 9. The Möbius ladder M_5

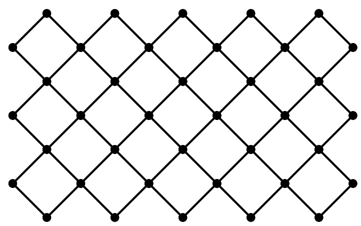


FIGURE 10
The even Aztec rectangle
$ER_{5,3}$

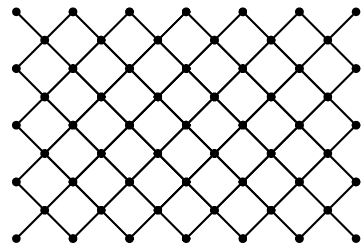


FIGURE 11
The odd Aztec rectangle
$OR_{5,3}$

CHAPTER 2

Preliminaries

1. Basic Graph Theory

In this section we will define what a graph is, and give definitions for the most important concepts in graph theory. As examples, we will use the graphs depicted in Figures 1 and 2 below.

DEFINITION 1.1. A *directed graph*, abbreviated *digraph*, G consists of a set of *vertices* $\mathcal{V}(G)$ and a set of *arcs* $\mathcal{E}(G)$. Every arc (u, v) joins two – possibly identical – vertices u and v. We say (u, v) is *incident from* u and *incident to* v, to indicate the direction of the arc. Alternatively, we will call u the *tail* and v the *head* of (u, v). Arcs of the form (u, u) are called *loops*. If there are two or more arcs incident from a vertex u and incident to a vertex v, these arcs are called *parallel* to each other.

digraph
graph
incidence
head, tail
adjacency
loop
parallel edges
weighted
(di)graph
underlying
graph
orientation of a
graph

An *undirected graph*, for short *graph*, is a digraph with symmetric incidence relation: if (u, v) is in the arc-set of G, then also (v, u). Hence, we call such a pair of arcs an *edge incident to* u and v. In the following we will use the term 'edge' for both edges and arcs.

Two vertices joined by an edge are called *adjacent*.

A *weighted (di)graph* is a (di)graph together with a weight-function on its edges $w : \mathcal{E}(G) \to \mathbb{R}$. The *weight* of a graph is the product of the weight of its edges. We will identify an unweighted (di)graph with a weighted (di)graph by giving each edge weight one.

We will usually denote the number of vertices by p and the (weighted) number of edges by q.

The *underlying graph* of a digraph G has the same vertex-set as G, and every arc of G is replaced by an edge connecting the same vertices as the arc.

An *orientation* of a graph G is a digraph $\vec{G}$, so that the underlying graph of $\vec{G}$ is equal to G.

EXAMPLE. The (unweighted) digraph $\vec{G}$ shown in Figure 1 has vertex set $\mathcal{V}(\vec{G}) = \{r, u, v, w\}$ and edge set $\mathcal{E}(\vec{G}) = \{(u, v), (v, u), (r, u), (v, r), (r, r)\}$. Hence $p = 4$ and $q = 5$. It has a loop attached to vertex r, but no parallel edges.

Its underlying graph $\tilde{G}$ is depicted in Figure 2. Note that in $\tilde{G}$, the two edges joining vertices u and v are parallel.

DEFINITION 1.2. The *neighbourhood* $\mathcal{N}_G(v)$ of a vertex is the set of vertices adjacent to it.

neighbourhood
successors
predecessors
degree
indegree
outdegree

The *degree* $d_G(v)$ of a vertex v in a graph G is the number of edges incident to v, where loops are counted twice. In weighted graphs the degree of a vertex is the sum of the weights of the edges incident to v.

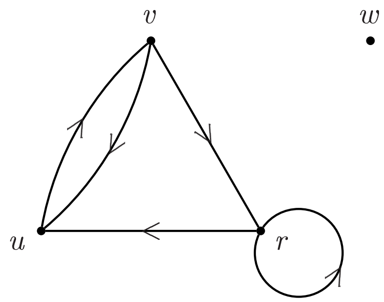


FIGURE 1
An example digraph $\vec{G}$

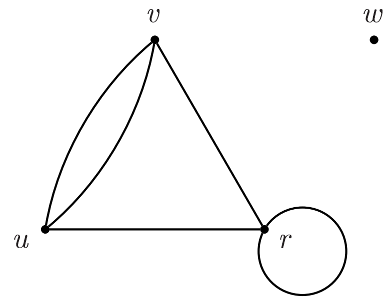


FIGURE 2
The underlying graph G of the digraph in Figure 2

Notation	Meaning
$\mathcal{V}(G)$, p	set and number of vertices of G
$\mathcal{E}(G)$, q	set and (weigthed) number of edges of G
$\mathcal{N}_G(v)$, $\mathcal{N}^o_G(v)$, $\mathcal{N}^i_G(v)$	neighbourhood, successors and predecessors of vertex v in G
$d_G(v)$, $d^o_G(v)$, $d^i_G(v)$	(weighted) degree, out- and indegree of vertex v in G
$\mathfrak{T}(G)$, $\boldsymbol{t}(G)$	set and number of spanning trees of G
$\mathfrak{F}_R(G)$, $\boldsymbol{f}_R(G)$	set and number of spanning forests of G with roots in R
$H \subseteq G$	H is a subgraph of G
$\mathcal{C}(G)$, $c(G)$	set and number of components of G

TABLE 1. Some graph theoretic notations

In a digraph the set of *successors (predecessors)* $\mathcal{N}^o_G(v)$ ($\mathcal{N}^i_G(v)$ is the set of vertices succeeding (preceding) a given vertex.

The *indegree* $d^i_G(v)$ *(outdegree* $d^o_G(v)$*)* of a vertex v in a digraph G is the number of arcs incident to (incident from) v. Directed loops are counted once. Again, in weighted digraphs the indegree (outdegree) of a vertex is the sum of the weights of the arcs incident to (incident from) to v. For undirected graphs it will be convenient to put $d^i_G(v) = d^o_G(v) = d_G(v)$.

adjacency matrix
degree matrix
Kirchhoff matrix
Laplacian

DEFINITION 1.3. Let G be any (weighted) (di)graph. Then its *adjacency matrix* $\mathbf{A}$ has rows and columns indexed by the vertices of G and entries $a_{u,v} = \sum_{e=(u,v)\in\mathcal{E}(G)} w(e)$ for $u, v \in \mathcal{V}(G)$, where $w : \mathcal{E}(G) \to \mathbb{R}$ denotes the weight function on G.

The *degree matrix* $\mathbf{D}$ of a (weighted) (di)graph G has nonzero entries only along its main diagonal. For any vertex $v \in \mathcal{V}(G)$ they are defined by

$$d_{v,v} = \begin{cases} d^i_G(v) & \text{for digraphs} \\ d_G(v) & \text{for graphs.} \end{cases}$$

The *Kirchhoff matrix* or *Laplacian* of a graph G is defined as $\mathbf{C} = \mathbf{D} - \mathbf{A}$.

Remark. Note that the indegrees of a digraph are given by the column sums of its adjacency matrix. Similarly, the outdegree of a particular vertex is the sum of all entries of the row it is in. Of course, we could also have defined the degree matrix of a digraph using the outdegree of its vertices.

Definition 1.4. The *weight matrix* $\mathbf{X}$ of a weighted (di)graph G is indexed by the edges of G and has nonzero entries only along its main diagonal: $x_{e,e}$ is the weight of the edge e. If G is not weighted, $\mathbf{X}$ is the identity matrix. weight matrix

Definition 1.5. Let G be a (loopless) digraph. Then its *incidence matrix* $\mathbf{B}$ has rows indexed by the vertices of G and columns indexed by the arcs of G. The entry corresponding to vertex u and arc e is defined to be incidence matrix

$$b_{u,e} = \begin{cases} -1 & e = (u,v) \\ +1 & e = (v,u) \\ 0 & \text{otherwise.} \end{cases}$$

Remark. By definition, $\mathbf{B}$ has exactly one positive and one negative entry in each column. The number of positive (negative) entries in each row equals the indegree (outdegree) of the corresponding vertex.

Proposition 1.6. *Let G be a loopless (weighted) graph. Let $\mathbf{B}$ be the incidence matrix of an arbitrary orientation of G. Let $\mathbf{X}$ be the weight matrix of G. Then we have*

$$\mathbf{C} = \mathbf{B} \cdot \mathbf{X} \cdot \mathbf{B}^t,$$

where $\mathbf{C}$ is the Kirchhoff matrix of G.

For an arbitrary (weighted) digraph let $\overline{\mathbf{B}}$ be the matrix obtained from its incidence matrix $\mathbf{B}$ by replacing all negative entries with zeros. Then

$$\mathbf{C} = \mathbf{B} \cdot \mathbf{X} \cdot \overline{\mathbf{B}}^t.$$

Proof. Clearly,

$$(\mathbf{B} \cdot \mathbf{X} \cdot \mathbf{B}^t)_{(u,v)} = \sum_{e \in \mathcal{E}(G)} b_{u,e} \cdot x_{e,e} \cdot b_{v,e}.$$

Furthermore we have

$$b_{u,e} \cdot b_{v,e} = \begin{cases} +1 & \text{if } u = v \text{ and } e \text{ is incident to } u \\ -1 & \text{if } e = (u,v) \text{ or } e = (v,u) \\ 0 & \text{otherwise,} \end{cases}$$

which yields the desired result.

Analogously, for digraphs we have

$$(\mathbf{B} \cdot \mathbf{X} \cdot \overline{\mathbf{B}}^t)_{(u,v)} = \sum_{e \in \mathcal{E}(G)} b_{u,e} \cdot x_{e,e} \cdot \overline{b}_{v,e},$$

and

$$b_{u,e} \cdot \overline{b}_{v,e} = \begin{cases} +1 & \text{if } u = v \text{ and } e \text{ is incident to } u \\ -1 & \text{if } e = (u,v) \\ 0 & \text{otherwise.} \end{cases}$$

□

REMARK. F. R. K. Chung and R. P. Langlands defined in [**13**] the weight of a *vertex* and *edge weighted (di)graph* G as

$$w(G) = \prod_{(u,v)\in\mathcal{E}(G)} w(u)w(u,v),$$

where the weight of vertex u is $w(u)$. Accordingly, the in-degree of a vertex is defined as

$$d^i_G(v) = \sum_{(u,v)\in\mathcal{E}(G)} w(u)w(u,v),$$

and the adjacency matrix as $\mathbf{A}$ with

$$a_{u,v} = \sum_{e=(u,v)\in\mathcal{E}(G)} w(e)\sqrt{w(u)w(v)}.$$

We will see in Chapter 5, Section 1 that this definition enables us to derive a generalized version of the famous Matrix-Tree-Theorem.

DEFINITION 1.7. A graph is called *Eulerian*, if all vertices have even degree, a (weighted) digraph is called *Eulerian*, if in all vertices in- and outdegree coincide. A (di)graph is r*-regular* if all of its vertices have (in)degree r.

Eulerian, regular, bipartite, semiregular graph

A graph is *bipartite*, if it is possible two separate its vertex set in two *parts*, so that edges join only vertices belonging to different parts.

A graph is *semiregular* of degrees r_1 and r_2 if it is bipartite and all vertices of one part have degree r_1, all vertices of the other part have degree r_2.

LEMMA 1.8. *For any (weighted) digraph G we have*

$$\sum_{v\in\mathcal{V}(G)} d^i_G(v) = \sum_{v\in\mathcal{V}(G)} d^o_G(v) = q,$$

where q is the (weighted) number of edges. For (weighted) graphs we have

$$\sum_{v\in\mathcal{V}(G)} d_G(v) = 2q.$$

Again, q is the (weighted) number of edges.

PROOF. This is because the arcs are in one to one correspondence to the vertices: In the first sum an arc corresponds to the vertex it is incident to, in the second to the vertex it is incident from. In a graph, every edge is counted exactly twice: Either it is incident to two vertices or it is a loop. □

(closed) chain
walk
path
circuit
cycle
Eulerian tour

Definition 1.9. Given a (di)graph G, a *chain* is a sequence of vertices and edges of the form $(v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_n)$, with $\{v_1, v_2, \ldots, v_n\} \subseteq \mathcal{V}(G)$ and with $\{e_1, e_2, \ldots, e_{n-1}\} \subseteq \mathcal{E}(G)$, so that e_i joins v_i and v_{i+1} for $i \in \{1, 2, \ldots, n-1\}$. Note that we do not pay attention to the direction of the arcs here. Furthermore, it is possible that some vertices or edges occur more than once. If $v_1 = v_n$ the chain is said to be *closed*.

Two *closed chains* are considered equal, if the succession of arcs is the same in both chains – we consider e_1 to be a successor of e_n.

A *walk* in a digraph is a chain with arcs of the form $e_i = (v_i, v_{i+1}) \in \mathcal{E}(G)$ only. For graphs we use *walk* as a synonym for *chain*.

A walk with all vertices distinct is called a *path*. In this case the edges are all distinct, too.

A *circuit* is a closed walk, a *cycle* is a closed path that contains at least one edge. Note that contrary to a circuit, a cycle cannot contain another *cycle* as a proper subset.

An *Eulerian tour* is a *circuit* containing every edge of G exactly once.

Figure 3. The path P_5

Remark. The path P_n depicted above and the circle C_n (see Figure 5 in the introduction) are probably the two most basic graphs. Many other families of graphs will be defined in terms of the path and the circle.

connected
strongly connected
unilaterally connected

Definition 1.10. A (di)graph is *connected* if any two vertices can be joined by a chain. It is *strongly connected* if any two vertices can be joined by a path. It is *unilaterally connected* if for any two vertices u and v there is either a path from u to v or a path from v to u.

Remark. In graphs the terms *connected* and *strongly connected* coincide.

subgraph
vertex-induced
edge-induced
graph,
restriction

Definition 1.11. A *subgraph* H of a (di)graph G is a (di)graph with edge-set $\mathcal{E}(H) \subseteq \mathcal{E}(G)$ and vertex-set $\mathcal{V}(H) \subseteq \mathcal{V}(G)$. A subgraph is *spanning* if it has the same vertex-set as the original (di)graph.

For a set of vertices $V \subseteq \mathcal{V}(G)$ the *vertex-induced subgraph* of a (di)graph G has vertex-set V and maximal edge-set $A \subseteq \mathcal{E}(G)$. In other words, an arc of G is in the induced subgraph if and only if both vertices it connects are in V. It is also called restriction of G to V and will be denoted by $G|_V$

For a set of edges $E \subseteq \mathcal{E}(G)$ the *edge-induced subgraph* of a (di)graph G has edge-set E and minimal vertex-set $V \subseteq \mathcal{V}(G)$. Hence a vertex is in the induced subgraph if and only if it is incident to or from some arc in E. This subgraph is also known as the *restriction* of G to E and is denoted by $G|_E$.

Remark. Digraphs are never considered as subgraphs of a graph.

DEFINITION 1.12. A *component* of a (di)graph is a maximal connected vertex-induced subgraph.

component
bridge

A *bridge* is an edge of a (di)graph whose removal increases the number of components.

REMARK. Sometimes maximal *strongly connected* vertex-induced subgraphs are considered. For our purposes, however, we do not need this concept.

DEFINITION 1.13. A graph T is a *tree* if one of the following equivalent conditions hold:

tree
arborescence
forest
roots
leaves

- T is connected and contains no cycles.
- T is connected and the number of edges is the number of vertices less one.
- Any two vertices of T are joined by a unique path.

One vertex can be chosen to be a *root*, we then speak of a *rooted tree*.

A digraph T is an *arborescence* or *out-tree*, if its underlying graph is a rooted tree and for every vertex v there is a unique path from the root to v. Similarly, we call a digraph T an *in-tree*, if its underlying graph is a rooted tree and for every vertex v there is a unique path from v to the root.

In the following we will use the term *tree* for both trees and arborescences.

A (di)graph F is a *forest* if each of its components is a (rooted) tree (an arborescence).

Vertices of a forest with degree one (outdegree zero), not roots, are called *leaves*.

PROOF OF EQUIVALENCE. The existence of two different paths joining any two vertices of T is equivalent to the existence of a cycle in T, because the concatenation of two different paths is a cycle and, conversely, in a cycle there are always two different paths joining distinct vertices.

Suppose that T is connected and contains no cycles. Then there must be a vertex with degree one. Deleting this vertex and its incident edge we obtain a graph which is still connected and does not contain a cycle, but has fewer vertices. Hence we can repeat this procedure, until we obtain the graph consisting of an isolated vertex, which is the only connected graph without edges. In every step we deleted one vertex and one edge, hence, after deleting all vertices but one, there is no edge left. Therefore, in a tree the number of edges is the number of vertices less one.

Now suppose that T is connected and $q = p-1$, where p is the number of vertices of T and q is the number of edges of T. Then T contains a spanning subgraph T' which contains no cycles and has $p' = p$ vertices and $q' \leq q$ edges. By the hypothesis and the preceding argument we have $q = p - 1 = p' - 1 = q'$, i.e., T' contains as many edges as T, which implies that $T' = T$. Therefore, T contains no cycles. $\square$

REMARK. Given any graph, we are interested in the number of spanning trees it contains as subgraphs. Note that in digraphs, the number of spanning trees rooted at a particular vertex is generally not the same as the number of spanning trees rooted at a different vertex. Therefore, we will write $\boldsymbol{t}_r(G)$ for the number of spanning trees of G rooted at vertex r.

In graphs, of course, the number of spanning trees does not depend on the root chosen. We will show in Chapter 3, Section 3 and in Chapter 5, Section 1, that this is also true for Eulerian digraphs. Therefore, we will often refer to the number of spanning trees of a graph or Eulerian digraph G as $\boldsymbol{t}(G)$.

Corollary 1.14. *In an unweighted tree T we have*

$$\sum_{v\in T}(d_T(v)-1)=p-2,$$

where p is the number of vertices of T. More generally, if we consider T as a bipartite unweighted graph with parts T_1 and T_2 with p_1 and p_2 vertices respectively, we have

$$\sum_{v\in T_1}(d_T(v)-1)=p_2-1$$

and the obvious counterpart for T_2. Analogously, in an unweighted forest F we have

$$\sum_{v\in F_1}(d_F(v)-1)=p_2-c(F).$$

Proof. Take T to be rooted at an arbitrary vertex $v\in T_1$. Then vertices with even distance to v must be in T_1, those with odd distance to v are in T_2. Think of T as a directed graph with all edges directed away from the root. Then the outdegree of any vertex $u\in T_1$ simply counts the number of vertices in the other part with predecessor u. Clearly, every vertex in T_2 has exactly one predecessor. Summing up over all vertices in T_1 we get

$$\sum_{u\in T_1} d_T^o(v)=p_2.$$

But clearly, the outdegree of the root is just its degree in T and the outdegree of any other vertex is its degree in T minus one. Now the result follows by substituting in the sum above. □

Definition 1.15. A *cutset* of a graph G is a set of edges whose removal from G increases the number of components of G. A *cocycle* is a minimal cutset. A *cotree* C of a graph G is a subgraph of G, so that $G|_{\mathcal{E}(C)^c}$ is a spanning tree of G. cutset cocycle cotree

2. Operations on Graphs

Below some operations on graphs are defined in terms of their Laplacian matrices. (See Definition 1.3) Furthermore, we will explain each operation by giving a description of the set of vertices $\mathcal{V}$ and the set of edges $\mathcal{E}$ of the resulting graph. Note, however, that these descriptions often apply only to simple, undirected graphs without loops or multiple edges.

Furthermore we define some families of graphs, also those depicted in the introduction.

Definition 2.1. For (weighted) (di)graphs G and H with identical vertex sets $\mathcal{V}(G)=\mathcal{V}(H)$ we define their *union* $G+H$ and their *product* $G\cdot H$ as follows: The edge set of the union is $\mathcal{E}(G)\cup\mathcal{E}(H)$, the edge set of the product is union, product

$$\{(u,w):(u,v)\in\mathcal{E}(G)\text{ and }(v,w)\in\mathcal{E}(H)\}.$$

Name	Notation	Reference
Union	$G + H$	
Product	$G \cdot H$	
Direct Sum	$G \dotplus H$	Chapter 5, Lemma 2.8, page 66
Complement	$\overline{G}$	Chapter 5, Lemma 2.8, page 66
Complete Product	$G \bigtriangledown H$	Chapter 5, Lemma 2.8, page 66
Kronecker sum	$G \oplus H$	Chapter 5, Theorem 2.18, page 73
Kronecker product	$G \otimes H$	Chapter 5, Theorem 2.18, page 73
Lexicographic Product	$G\,[H_1, H_2, \ldots, H_p]$	Chapter 4, Theorem 3.5, page 46
Line Graph	$\mathfrak{L}(G)$	Chapter 5, Proposition 2.16, page 72
Subgraph Deletion ($H \subseteq G$)	$G \setminus H$	
Contraction ($E \subseteq \mathcal{E}(G)$ or $V \subseteq \mathcal{V}(G)$)	$G{\bullet}_E,\ G{\bullet}_V$	Chapter 4, Section 1, page 34
Restriction ($E \subseteq \mathcal{E}(G)$ or $V \subseteq \mathcal{V}(G)$)	$G\vert_E,\ G\vert_V$	Chapter 4, Section 1, page 34

TABLE 2. Some operations on graphs

In terms of the Laplacian matrices of G and H we have

$$\mathbf{C}_{G+H} = \mathbf{C}_G + \mathbf{C}_H \text{ and}$$
$$\mathbf{C}_{G\cdot H} = \mathbf{C}_G \cdot \mathbf{C}_H.$$

EXAMPLE 2.2. The square of a circle C_n^2, see Figure 6 in the introduction, can be defined – nomen est omen – as $C_n \cdot C_n$.

DEFINITION 2.3. The *direct sum* $G \dotplus H$ of G and H is obtained by 'drawing' the two (weighted) (di)graphs side by side: direct sum

$$\mathcal{V}(G \dotplus H) = \mathcal{V}(G) \cup \mathcal{V}(H) \text{ and}$$
$$\mathcal{E}(G \dotplus H) = \mathcal{E}(G) \cup \mathcal{E}(H).$$

In terms of the Laplacian matrices of G and H we have

$$\mathbf{C}_{G\dotplus H} = \begin{pmatrix} \mathbf{C}_G & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_H \end{pmatrix}.$$

EXAMPLE 2.4. A graph for rather technical purposes is the graph O_n, which is the direct sum of n isolated vertices.

DEFINITION 2.5. The *complement* $\overline{G}$ of a (weighted) (di)graph can be defined by complement

$$\mathbf{C}_{\overline{G}} = p\mathbf{I} - \mathbf{J} - \mathbf{C}_G,$$

where $\mathbf{J}$ is the matrix which has all entries equal to one.

For a simple graph G, i.e., a graph without multiple edges (arcs) or loops, $\overline{G}$ is the graph with the same vertex set as G and edge set $\mathcal{E}(K_p) \setminus \mathcal{E}(G)$:

$$\overline{G} = K_p\big|_{(\mathcal{E}(G))^c}.$$

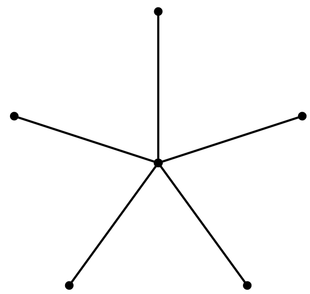


FIGURE 4. The star S_5

EXAMPLE 2.6. The complete graph K_n, see Figure 1 in the introduction, is the graph on n vertices, where every pair of distinct vertices is connected by exactly one edge. It can be also defined as the complement of the graph O_n.

DEFINITION 2.7. The *complete product* $G \bigtriangledown H$ of (weighted) (di)graphs G and H is obtained from the direct sum of G and H by additionally joining every vertex of G with every vertex of H. Hence

complete product

$$\mathcal{E}(G \bigtriangledown H) = \mathcal{E}(G) \cup \mathcal{E}(H) \cup \{(u, v) : u \in \mathcal{V}(G) \text{ and } v \in \mathcal{V}(H)\}$$

In terms of the Laplacian matrices of G and H we have

$$\mathbf{C}_{G \bigtriangledown H} = \begin{pmatrix} \mathbf{A}_G & \mathbf{J} \\ \mathbf{J} & \mathbf{A}_H \end{pmatrix} = \mathbf{C}_{\overline{\overline{G} + \overline{H}}}.$$

EXAMPLE 2.8. The complete product enables us to define many important families of graphs: The complete bipartite $K_{n,m}$, see Figure 2 in the introduction, or, more generally, the complete multipartite graph $K_{n_1,n_2,\ldots,n_m}$ can be defined as the complete product of the graphs O_{n_1}, O_{n_2}, ... O_{n_m}: $K_{n_1,n_2,\ldots,n_m} = \bigtriangledown_{k=1}^{m} O_{n_k}$.

The fan F_n, see Figure 4 in the introduction, can be defined as $K_1 \bigtriangledown P_n$. Similarly, the wheel W_n, see Figure 3 can be defined as $K_1 \bigtriangledown C_n$.

Another important graph, the star S_n, see Figure 4 above, is the complete product of a single vertex with the graph O_n consisting of n isolated vertices.

DEFINITION 2.9. The *Kronecker product* $G \otimes H$ and the *Kronecker sum* $G \oplus H$ of (weighted) (di)graphs G and H have as vertex sets the cartesian product $\mathcal{V}(G) \times \mathcal{V}(H)$ of the vertex sets of G and H. We will denote the vertex in $\mathcal{V}(G) \times \mathcal{V}(H)$ corresponding to the vertices $u \in \mathcal{V}(G)$ and $v \in \mathcal{V}(G)$ by uv.

Kronecker product
Kronecker sum

In terms of the Laplacian matrices of G and H we can define

$$\mathbf{C}_{G \otimes H} = \mathbf{C}_G \otimes \mathbf{C}_H \text{ and}$$
$$\mathbf{C}_{G \oplus H} = \mathbf{C}_G \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{C}_H.$$

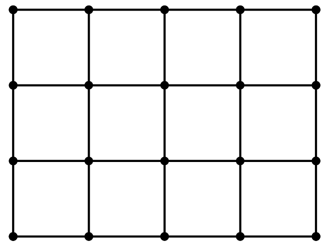


FIGURE 5. The grid graph $P_5 \oplus P_4$

If G and H are (di)graphs without parallel edges, the set of edges of the Kronecker product $\mathcal{E}(G \otimes H)$ is

$$\{(ux, vy) : (u, v) \in \mathcal{E}(G) \text{ and } (x, y) \in \mathcal{E}(H)\},$$

and the set of edges of the Kronecker sum $\mathcal{E}(G \oplus H)$ is

$$\{(ux, vy) : \big(u = v \text{ and } (x, y) \in \mathcal{E}(H)\big) \text{ or } \big(x = y \text{ and } (u, v) \in \mathcal{E}(G)\big)\}.$$

The preceding two operations are special cases of the so called NEPS of graphs, which is short for non-complete extended p-sum, a term coined by Cvetković, see [**15**]: The *NEPS* G with *basis* $\mathcal{B}$ of (weighted) (di)graphs $G_1,\ G_2, \ldots ,\ G_p$ has as its vertex set the cartesian product of the vertex sets of $G_1,\ G_2, \ldots ,\ G_p$. NEPS

In terms of the Laplacian matrices of the graphs $G_1,\ G_2, \ldots ,\ G_p$ we have

$$\mathbf{C}_G = \sum_{\beta \in \mathcal{B}} \mathbf{A}_1^{\beta_1} \otimes \cdots \otimes \mathbf{A}_p^{\beta_p}.$$

Therefore, if $G_1,\ G_2, \ldots ,\ G_p$ are (di)graphs without parallel edges, two vertices $(u_1, u_2, \ldots, u_p)$ and $(v_1, v_2, \ldots, v_p)$ of the NEPS G are connected if and only if there is a $\beta \in \mathcal{B}$ so that $u_i = v_i$ if $\beta_i = 0$, and $(u_i, v_i) \in \mathcal{E}(G_i)$ if $\beta_i = 1$, for all $i \in \{1, 2, \ldots, p\}$.

The Kronecker sum is equivalent to the NEPS with basis $\{(1, 0), (0, 1)\}$, the Kronecker product to the NEPS with basis $\{(1, 1)\}$.

EXAMPLE 2.10. It is a remarkable fact that the Kronecker product of two connected graphs need not be connected. In fact, if the graphs are both bipartite, their Kronecker product has always two components, see also Theorem 2.21 in Chapter 5.

The even and odd Aztec rectangles $ER_{n,m}$ and $OR_{n,m}$ depicted in Figures 10 and 11 in the introduction, are the two components of the Kronecker product of two paths P_n and P_m. $ER_{n,m}$ is the component of $P_n \otimes P_m$ which has an even number of vertices, $OR_{n,m}$ has an odd number of vertices.

Clearly, the Kronecker sum of connected graphs is always connected. The ladder L_n, see Figure 7 in the introduction, is the Kronecker sum of K_2 and the path P_n. More generally, the grid graph is the Kronecker sum of two paths P_n and P_m, see Figure 5 above.

DEFINITION 2.11. The *lexicographic product*, also called *local join* is defined for a (weighted) (di)graph G and (weighted) (di)graphs $H_1,\ H_2, \ldots, H_p$ and is denoted by $G\,[H_1, H_2, \ldots, H_p]$. lexicographic product local join

If G is a (di)graph without parallel edges, we construct $G\,[H_1, H_2, \ldots, H_p]$ as follows: Replace in a graph G on p vertices vertex v by the graph H_v, where $v \in \{1, 2, \ldots, p\}$, then join all vertices of H_u and H_v, whenever $(u, v) \in \mathcal{E}(G)$. We will denote vertex i in the subgraph H_u by ui. Hence, the vertex set of $G\,[H_1, H_2, \ldots, H_p]$ is $\bigcup_{v=1}^{p} \mathcal{V}(H_v)$ and its edge set is

$$\{(ux, vy) : (u, v) \in \mathcal{E}(G) \text{ or } \big(u = v \text{ and } (x, y) \in \mathcal{E}(H_u)\big)\}.$$

For general (weighted) (di)graphs G and $H_1,\ H_2, \ldots, H_p$, we define

$$\mathbf{A}_{G\,[H_1,H_2,\ldots,H_p]} = \big((\mathbf{A}_G)_{u,v}\mathbf{J}_{p_u\times p_v}\big)_{u,v\in\mathcal{V}(G)} + \begin{pmatrix} \mathbf{A}_{H_1} & & \\ & \ddots & \\ & & \mathbf{A}_{H_p} \end{pmatrix},$$

where p_v is the number of vertices of the graph H_v. The degree of a vertex ui is

$$d_{G\,[H_1,H_2,\ldots,H_p]}(ui) = \sum_{(u,v)\in\mathcal{E}(G)} p_v + d_{H_v}(i).$$

Note that the lexicographic product $K_p\,[H_1, H_2, \ldots, H_p]$ is equivalent to the complete product of the graphs $H_1,\ H_2, \ldots, H_p$.

DEFINITION 2.12. The *line graph* $\mathfrak{L}(G)$ of a graph G has as vertex set the set of edges of G. Two vertices of $\mathfrak{L}(G)$ are joined, if they are incident in G. Hence: line graph

$$\begin{aligned} &\mathcal{V}(\mathfrak{L}(G)) = \mathcal{E}(G) \text{ and} \\ &\mathcal{E}(\mathfrak{L}(G)) = \{(ux, uy) : u, x, y \in \mathcal{V}(G)\}. \end{aligned}$$

The adjacency matrix of the line graph of a graph G can be expressed in terms of the incidence matrix $\mathbf{B}$ of G:

$$\mathbf{A}_{\mathfrak{L}(G)} = \mathbf{B}_G^t\mathbf{B}_G - 2\mathbf{I}.$$

DEFINITION 2.13. A subgraph $H \subseteq G$ is *deleted* from G by deleting all edges and vertices of H and all edges which were incident to some vertex in H. Hence subgraph deletion

$$\begin{aligned} &\mathcal{V}(G \setminus H) = \mathcal{V}(G) \setminus \mathcal{V}(H) \text{ and} \\ &\mathcal{E}(G \setminus H) = \mathcal{E}(G) \setminus \{(u, v) : u \in \mathcal{V}(H) \text{ or } v \in \mathcal{V}(H)\}. \end{aligned}$$

The *restriction* $G|_E$ ($G|_V$) of a graph G to a set of edges $E \subseteq \mathcal{E}(G)$ (vertices $V \subseteq \mathcal{V}(G)$) is synonymous to the edge (vertex)-induced subgraph of G induced by E (respectively V) and was already explained in Section 1, Definition 1.11. restriction

The *contraction* $G{\cdot}_E$ of a graph G to a set of edges $E \subseteq \mathcal{E}(G)$ has as vertex set all the vertices of G which are not incident to an edge of E^c together with one vertex for each component in $G|_{E^c}$. Two vertices in $G{\cdot}_E$ are connected by as many edges as there are edges connecting the corresponding sets in G. contraction

The contraction $G{\cdot}_V$ of a graph G to a set of vertices $V \subseteq \mathcal{V}(G)$ has as vertex set V plus an additional vertex r, which represents the vertices in V^c. Two vertices in V are connected by as many edges as there are edges connecting them in V, a vertex $v \in V$ is connected to r by as many edges as there are in G connecting v with some vertex in V^c. Finally, the vertex r has as many loops as there are edges in G connecting vertices within V^c.

CHAPTER 3

Equivalent Objects

In this chapter we will show that there are many combinatorial objects which are intimately related to the spanning trees of a graph. Among those there are well known objects as Eulerian tours, spanning forests and perfect matchings. We will also deal with duality in graphs and the so-called Abelian avalanche model, also known as chip-firing game, defined on a graph.

Note that there is even a connection to algebraic topology, although we cannot cover this here: It is possible to associate certain graphs G with 3-manifolds, so that the order of their first homology group is equal to the number of spanning trees. [**30**]

1. Duality

Duality of graphs is a concept which appears in many areas of combinatorial graph theory. It is also of significance to the problem of counting spanning trees of undirected, unweighted graphs. For giving an exact definition, we need some prerequisites:

The following very important concept will accompany us throughout most of the sections:

Definition 1.1. An *embedding* of a (di)graph on a surface S is an injective function i that maps each vertex v onto a point $i(v)$ on the surface and every edge (u, v) onto a path with endpoints $i(u)$ and $i(v)$. It is required that no two paths cross each other, they may only touch in their endpoints. Furthermore, each component of $S \setminus i(G)$ has to be homeomorphic to an open disc. In the following we will identify the paths on the surface with the corresponding edges of the graph and the endpoints of the paths with the corresponding vertices. Note that a graph usually has many embeddings on one surface.

embedding
face
planar graph
dual graph

A *face* of a (di)graph embedded on a surface S by a function $i : G \to S$ is a component of $S \setminus i(G)$. By Euler's formula we have

$$\chi(S) = p - q + f,$$

where $\chi(S)$ is the *Euler characteristic* of S, p is the number of vertices, q is the number of edges and f is the number of faces of G.

A (di)graph is called *planar* if it has an embedding onto the plane or, equivalently, onto the sphere. In planar graphs we have $p - q + f = 2$.

The *dual* graph G^* of an undirected planar graph with respect to some embedding has the same edge set as G and a vertex for each face of G, where an edge connects the faces it bounds.

Remark. Let G be a graph embedded on the plane. When drawing the dual of G, it is often easier to draw the vertex corresponding to the unbounded face of G as a circle, which surrounds the other vertices of G^*. An example is given in Figure 1 of Section 5 on page 32.

Theorem 1.2. *Let G be an undirected, unweighted planar graph G and let G^* be its dual. Then the edges of each cotree of G form a spanning tree of G^* and vice versa. Hence, $\boldsymbol{t}(G) = \boldsymbol{t}(G^*)$.*

Proof. A cotree of G has $q - p + 1$ edges, by Euler's formula, this equals $f - 1$. Hence we only have to show that the edges of a cotree of G do not contain a cycle in G^*. But this is impossible, because by the Jordan-Curve-Theorem such a cycle would divide the sphere into two components, where both would contain a face of G^*, i.e., a vertex of G. Any path in the spanning tree of G joining these two vertices would have to cross the cycle, i.e., contain an edge of the cotree. This is a contradiction. □

Hence, when trying to determine the number of spanning trees of a given graph, it often makes sense to consider its dual. If the number of vertices of a planar graph is larger than its number of faces, its dual is smaller and therefore more accessible to numerical methods.

2. Chip-Firing Games on Graphs

In this section we present a (solitary) game played on a graph G known as 'chip firing game' or 'avalanche model'. We will define certain 'recurrent' configurations of this game and show, that the number of these configurations coincides with the number of spanning trees of G. Most proofs are taken from [**4**], [**5**] and [**18**].

Definition 2.1. A *chip-firing game* can be defined on a (di)graph G with positive edge weights. It depends on non-negative values s_v, l_v and q_v defined on every vertex v of G. The diagonal matrix $\mathbf{S}$ with $\mathbf{S}_{v,v} = s_v$ is called the *dissipation matrix*, the vector $\mathbf{l} = (l_v)_{v \in \mathcal{V}(G)}$ is called a *load vector*. A vector $\mathbf{q} = (q_v)_{v \in \mathcal{V}(G)}$ is called a *configuration* of the game.

chip-firing game
dissipation and load at a vertex, configuration of the game

Given a configuration $\mathbf{q}$, a vertex v is *stable*, if $q_v < d_G(v) + s_v$ (in digraphs $q_v < d^i_G(v) + s_v$). Otherwise, v is *unstable*.

(un)stable vertex

Accordingly, a configuration $\mathbf{q}$ is *stable*, if all vertices are stable. The set of stable configurations will be denoted by $\mathcal{S}(G)$.

The basic action in this game is the *firing* of an unstable vertex of the graph: Let $\mathbf{q}$ be an unstable configuration. For an unstable vertex v we define the configuration $\mathbf{q}'$ *after the firing* of v as follows: $q'_u = q_u + w(u, v)$ for vertices $u \neq v$, where w is the weight function of G, and $q'_v = q_v - d_G(v) - s_v$ (in digraphs $q'_v = q_v - d^i_G(v) - s_v$).

firing of a vertex

In other words, the vertex v which is fired passes the amount of $w(u, v)$ to every vertex u of its neighbourhood $\mathcal{N}_G(v)$ (in digraphs, to every vertex u of its set of predecessors $\mathcal{N}^i_G(v)$). The firing vertex itself looses the amount it distributes and additionally its dissipation s_v. Stable vertices cannot be fired.

More generally, a sequence of vertices $V = (v_1, v_2, \ldots, v_n)$ is called a *legal sequence* for the configuration $\mathbf{q}$, if, starting at this configuration, the vertices can be

legal sequences
representative vector
transition matrix

fired in this order. The *representative vector* of V is the vector $\mathbf{v}$, where $\mathbf{v}(v)$ is the number of occurrences of the vertex v in V.

The configuration $\mathbf{q}'$ after the firing of a legal sequence V is given by

$$\mathbf{q}' = \mathbf{q} - \Delta\mathbf{v}, \tag{+}$$

where $\Delta = \mathbf{C} + \mathbf{S}$, with $\mathbf{C}$ the Kirchhoff matrix of G. We call Δ the *transition matrix* of the game.

We call a game *weakly dissipative*, if for every *non-dissipative* vertex v, i.e., a vertex v with $s_v = 0$, there is a path from a *dissipative* vertex u, i.e., a vertex u with $s_u > 0$. (non-) dissipative vertex, weakly dissipative game

LEMMA 2.2. *In a weakly dissipative game, given a vector* $\mathbf{q}$ *with non-negative components, there is a vector* $\mathbf{q}'$ *with non-negative components so that*

$$\Delta\mathbf{q}' = \mathbf{q}. \tag{$*$}$$

In particular, the matrix Δ *is nonsingular.*

PROOF. We will construct a sequence of approximate solutions that converges towards a solution of Equation $(*)$.

Let $\mathbf{q}_0 = \mathbf{0}$. Given $\mathbf{q}_n$ for $n \geq 0$, let v_n be a vertex of G, so that $(\mathbf{q} - \Delta\mathbf{q}_n)_{v_n}$ is maximal. Let

$$\delta_n(v) = \begin{cases} \frac{(\mathbf{q}-\Delta\mathbf{q}_n)_{v_n}}{\Delta_{v_n,v_n}} & \text{for } v = v_n \\ 0 & \text{otherwise.} \end{cases}$$

We then define $\mathbf{q}_{n+1} = \mathbf{q}_n + \delta_n$. We now show that the sequence $(\mathbf{q}_n)_{n\geq 0}$ converges towards a solution $\mathbf{q}'$ of Equation $(*)$.

We have defined $\mathbf{q}_{n+1}$ so that $(\mathbf{q} - \Delta\mathbf{q}_{n+1})_{v_n}$ vanishes. Furthermore, for $v \neq v_n$ we have

$$\begin{aligned} (\mathbf{q} - \Delta\mathbf{q}_{n+1})_v &= (\mathbf{q} - \Delta\mathbf{q}_n - \Delta\delta_n)_v \\ &= (\mathbf{q} - \Delta\mathbf{q}_n)_v + w(v, v_n)\frac{(\mathbf{q} - \Delta\mathbf{q}_n)_{v_n}}{\Delta_{v_n,v_n}}. \end{aligned} \tag{$**$}$$

Therefore, we inductively see that all components of $\mathbf{q} - \Delta\mathbf{q}_n$ are non-negative.

For two vectors $\mathbf{q}$ and $\mathbf{q}'$ we define a distance $d(\mathbf{q}, \mathbf{q}') = \sum_{v\in\mathcal{V}(G)} |(\mathbf{q} - \mathbf{q}')_v|$. We have to show that $d(\mathbf{q}, \Delta\mathbf{q}_n)$ tends to zero as n approaches infinity. We can express the distance between $\mathbf{q}$ and $\Delta\mathbf{q}_{n+1}$ as

$$\begin{aligned} d(\mathbf{q}, \Delta\mathbf{q}_{n+1}) &= \sum_{v\in\mathcal{V}(G)} |(\mathbf{q} - \Delta\mathbf{q}_{n+1})_v| \\ &= \sum_{v\in\mathcal{V}(G)} (\mathbf{q} - \Delta\mathbf{q}_n - \Delta\delta_n)_v \\ &= d(\mathbf{q}, \Delta\mathbf{q}_n) - s_{v_n}\frac{(\mathbf{q} - \Delta\mathbf{q}_n)_{v_n}}{\Delta_{v_n,v_n}}. \end{aligned}$$

Because v_n was chosen so that $(\mathbf{q} - \Delta\mathbf{q}_n)_{v_n}$ is maximal, we have

$$d(\mathbf{q}, \Delta\mathbf{q}_n) = \sum_{v\in\mathcal{V}(G)} (\mathbf{q} - \Delta\mathbf{q}_n)_v \leq p\,(\mathbf{q} - \Delta\mathbf{q}_n)_{v_n}, \tag{$***$}$$

where, as always, p is the number of vertices of G. Thus we obtain the estimate

$$d(\mathbf{q},\Delta\mathbf{q}_{n+1}) \leq d(\mathbf{q},\Delta\mathbf{q}_n)\left(1-\frac{s_{v_n}}{p\,\Delta_{v_n,v_n}}\right).$$

If v_n is a non-dissipative vertex, $d(\mathbf{q},\Delta\mathbf{q}_{n+1}) = d(\mathbf{q},\Delta\mathbf{q}_n)$. For dissipative vertices, however, we have

$$0 < \left(1-\frac{s_{v_n}}{p\,\Delta_{v_n,v_n}}\right) < \left(1-\min_{\substack{v\in\mathcal{V}(G)\\ s_v>0}}\frac{s_v}{p\,\Delta_{v,v}}\right) < 1.$$

Note that $\lambda = 1-\min_{\substack{v\in\mathcal{V}(G)\\ s_v>0}}\frac{s_v}{p\,\Delta_{v,v}}$ is independent of n. We now have to distinguish between two possible cases: If there is an infinite subsequence $(v_{m_n})_{n\geq 0}$ of $(v_n)_{n\geq 0}$, so that v_{m_n} is dissipative for all n, we have $d(\mathbf{q},\Delta\mathbf{q}_n) \geq \lambda^n d(\mathbf{q},\Delta\mathbf{q}_{m_0})$. Thus $d(\mathbf{q},\Delta\mathbf{q}_{m_n})$ tends to zero as n approaches infinity, which is what we wanted to show.

Suppose that this is not the case and that there is an m_0 so that all vertices v_n with $n \geq m_0$ are non-dissipative. We show, that in this case $d(\mathbf{q},\Delta\mathbf{q}_{m_0})$ must be zero.

Let u be a non-dissipative vertex. Because the game is weakly dissipative, u has a predecessor. Let u' be a predecessor of u.

For each $n > m_0$ with $v_n = u$, we have because of Equation $**$

$$(\mathbf{q}-\Delta\mathbf{q}_{n+1})_{u'} = (\mathbf{q}-\Delta\mathbf{q}_n)_{u'} + w(u',u)\frac{(\mathbf{q}-\Delta\mathbf{q}_n)_u}{\Delta_{u,u}}$$

and because of Inequality $(***)$

$$(\mathbf{q}-\Delta\mathbf{q}_{n+1})_{u'} \geq (\mathbf{q}-\Delta\mathbf{q}_n)_{u'} + w(u',u)\frac{d(\mathbf{q},\Delta\mathbf{q}_n)}{p\,\Delta_{u,u}}.$$

We assumed that for $n \geq m_0$ all vertices v_n are non-dissipative, therefore

$$(\mathbf{q}-\Delta\mathbf{q}_{n+1})_{u'} \geq (\mathbf{q}-\Delta\mathbf{q}_n)_{u'} + \min_{e\in\mathcal{E}(G)} w(e)\frac{d(\mathbf{q},\Delta\mathbf{q}_{m_0})}{p\,\max_{v\in\mathcal{V}(G)}\Delta_{v,v}}.$$

Recall that all the weights are positive. Hence, if $d(\mathbf{q},\Delta\mathbf{q}_{m_0}) > 0$, the sequence $\big((\mathbf{q}-\Delta\mathbf{q}_n)_{u'}\big)_{n\geq m_0}$ tends to infinity. Because of

$$(\mathbf{q}-\Delta\mathbf{q}_n)_{u'} = d(\mathbf{q},\Delta\mathbf{q}_n) - \sum_{v\neq u'}(\mathbf{q}-\Delta\mathbf{q}_n)_v \leq d(\mathbf{q},\Delta\mathbf{q}_n)$$

this is absurd and $d(\mathbf{q},\Delta\mathbf{q}_{m_0})$ must be equal to zero. □

LEMMA 2.3. *If G is weakly dissipative, then every configuration passes over into a stable configuration after a finite number of firings.*

PROOF. Throughout the firings, the total amount held at the vertices cannot exceed its initial value. In particular, there is an upper bound on the amount held at any vertex.

Suppose there is an infinitely long sequence of firings. Then, since the number of vertices is finite, there is a vertex v which is fired infinitely often. This cannot be a dissipative vertex, because then the game would loose the amount $s_v > 0$ at every firing of v, but the total amount of the game is finite.

Suppose that v is a non-dissipative vertex. Since G is weakly dissipative, there is a path from a dissipative vertex r to v. Since v is fired infinitely often, its predecessor v' on the path to r receives an infinite amount. Therefore, as the amount held at any vertex is bounded, v' must be fired infinitely often as well.

Inductively, we see that each of the vertices of the path, including r must be fired infinitely often. This is a contradiction. □

Definition 2.4. A sequence of firings $V = (v_1, v_2, \ldots, v_k)$, $v_i \in \mathcal{V}(G)$ for $i \in \{1, 2, \ldots, k\}$, that transforms an unstable configuration $\mathbf{q}$ into a stable configuration is called an *avalanche starting at* $\mathbf{q}$. avalanche

We will show, that all avalanches starting at a given configuration $\mathbf{q}$, terminate at the same stable configuration $\mathbf{q}'$. This is the so-called Abelian property of the avalanche model. We will follow the argument of Biggs [**4**], who uses a mixing technique:

Let U be a legal sequence for the configuration $\mathbf{q}$ and let $\mathbf{v}$ be a vector with $\mathbf{v}(v) \geq 0$ for all vertices v. Then $U^{\mathbf{v}}$ is the sequence obtained from U by deleting the first $\mathbf{v}(v)$ occurrences of every vertex v from U. If $\mathbf{v}(v)$ is greater than the number of occurrences of v in U, then all occurrences are deleted.

Lemma 2.5. *Let U and V be legal sequences for a configuration $\mathbf{q}$, with representative vectors $\mathbf{u}$ and $\mathbf{v}$. Then the sequence $Z = (V, U^{\mathbf{v}})$ is also legal for $\mathbf{q}$.*

Proof. Let $\mathbf{q}' = \mathbf{q} - \Delta\mathbf{u}$. Suppose that $U^{\mathbf{v}}$ is legal for $\mathbf{q}'$ up to the point where the vertex v is about to be fired for the i^{th} time. Denote the configuration at this point with $\mathbf{p}^{\mathbf{v}}$. Let $\mathbf{p}$ be the configuration which occurs just before the corresponding firing of v in U, which is the $(\mathbf{v}(v) + i)^{\text{th}}$. (Note that $\mathbf{u}(v) > \mathbf{v}(v)$, if v occurs in $U^{\mathbf{v}}$.)

Let $\mathbf{u}_0$ and $\mathbf{u}_0^{\mathbf{v}}$ be the representative vectors of the initial segments of U and $U^{\mathbf{v}}$ up to these points, so that $\mathbf{p} = \mathbf{q} - \Delta\mathbf{u}_0$ and $\mathbf{p}^{\mathbf{v}} = \mathbf{q} - \Delta(\mathbf{v} + \mathbf{u}_0^{\mathbf{v}})$.

We will show that $\mathbf{p}^{\mathbf{v}}(v) \geq \mathbf{p}(v)$. Given that the firing of v is legal at $\mathbf{p}$, this implies that the firing of v at $\mathbf{q}^{\mathbf{v}}$ is legal, too.

Evaluating $\mathbf{p}^{\mathbf{v}}$ and $\mathbf{p}$ at v we obtain

$$\mathbf{p}(v) = \mathbf{q}(v) - \mathbf{u}_0(v)\,(d_G(v) + s_v) + \sum_{u \neq v} \mathbf{u}_0(u) w(u, v)$$

$$\mathbf{p}^{\mathbf{v}}(v) = \mathbf{q}(v) - (\mathbf{v} + \mathbf{u}_0^{\mathbf{v}})(v)\,(d_G(v) + s_v) + \sum_{u \neq v} (\mathbf{v} + \mathbf{u}_0^{\mathbf{v}})(u) w(u, v).$$

Since v is about to be fired for the i^{th} time in $U^{\mathbf{v}}$, we have

$$(\mathbf{v} + \mathbf{u}_0^{\mathbf{v}})(v) = \mathbf{v}(v) + i - 1 = \mathbf{u}_0(v).$$

If $u \neq v$ does occur in $U^{\mathbf{v}}$, suppose it has been fired j times. Then $(\mathbf{v} + \mathbf{u}_0^{\mathbf{v}})(u) = \mathbf{v}(u) + j$. If $j = 0$, then

$$\mathbf{u}_0(u) \leq \mathbf{v}(u) = (\mathbf{v} + \mathbf{u}_0^{\mathbf{v}})(u).$$

This follows, because if $x_0(u)$ were greater than $\mathbf{v}(u)$, then u would have been already fired in $U^{\mathbf{v}}$, too. If $j > 0$, then

$$\mathbf{u}_0(u) = \mathbf{v}(u) + j = (\mathbf{v} + \mathbf{u}_0^{\mathbf{v}})(u).$$

Finally, if u does not occur in $\mathbf{U}^{\mathbf{v}}$, then

$$(\mathbf{v} + \mathbf{u}_0^{\mathbf{v}})(u) = \mathbf{v}(u) \geq \mathbf{u}(u) \geq \mathbf{u}_0(u).$$

Hence $(\mathbf{v} + \mathbf{u}_0^{\mathbf{v}})(u) \geq \mathbf{u}_0(u)$ for all vertices u, and v is a vertex which can be fired. □

LEMMA 2.6. *For any unstable configuration* $\mathbf{q}$*, all avalanches starting at* $\mathbf{q}$ *terminate at the same stable configuration and have the same length.*

PROOF. Let U and V be avalanches starting at $\mathbf{q}$ with representative vectors $\mathbf{u}$ and $\mathbf{v}$. Then, by the preceding lemma, $(V, U^{\mathbf{v}})$ is also a legal sequence for $\mathbf{q}$. We are given that the sequence V transforms $\mathbf{q}$ into a stable configuration, so that no vertex can be fired after V. Therefore, $U^{\mathbf{v}}$ must be empty, which can only be the case if $\mathbf{v}(v) \geq \mathbf{u}(v)$, for all vertices $v \in \mathcal{V}(G)$. Similarly, $V^{\mathbf{u}}$ must be empty, too, and therefore $\mathbf{u}(v) \geq \mathbf{v}(v)$ for $v \in \mathcal{V}(G)$. We conclude, that the sequences U and V have the same representative vectors and thus transform $\mathbf{q}$ into the same stable configuration and have the same length. □

Hence, given a configuration $\mathbf{q}$, we can define an operator $\mathfrak{A}$, which transforms $\mathbf{q}$ into the unique stable configuration reached after an avalanche. Furthermore, for any load vector $\mathbf{l}$ we define a loading operator $\mathfrak{L}_{\mathbf{l}} : \mathbb{R}_+^{\mathcal{V}(G)} \to \mathcal{S}(G), \mathfrak{L}_{\mathbf{l}}\,\mathbf{h} = \mathbf{h} + \mathbf{l}$.

LEMMA 2.7. *Every pair of operators* $\mathfrak{A} \circ \mathfrak{L}_{\mathbf{l}}$ *and* $\mathfrak{A} \circ \mathfrak{L}_{\mathbf{k}}$ *commutes:*

$$\mathfrak{A} \circ \mathfrak{L}_{\mathbf{l}} \circ \mathfrak{A} \circ \mathfrak{L}_{\mathbf{k}} = \mathfrak{A} \circ \mathfrak{L}_{\mathbf{l}+\mathbf{k}},$$

where $\mathbf{l}$ *and* $\mathbf{k}$ *are arbitrary load vectors.*

PROOF. We only need to show that $\mathfrak{A} \circ \mathfrak{L}_{\mathbf{l}} \circ \mathfrak{A} = \mathfrak{A} \circ \mathfrak{L}_{\mathbf{l}}$, or, more explicitly $\mathfrak{A}(\mathfrak{A}\,\mathbf{q} + \mathbf{l}) = \mathfrak{A}(\mathbf{q} + \mathbf{l})$ for any configuration $\mathbf{q}$.

Let u be an avalanche starting at $\mathbf{q}$ and let u' be an avalanche starting at $\mathfrak{A}\,\mathbf{q}+\mathbf{l}$. Then (u, u') is an avalanche starting at $\mathbf{q}+\mathbf{l}$, by the preceding lemma leading to the same configuration as u' starting at $\mathfrak{A}\,\mathbf{q} + \mathbf{l}$. □

DEFINITION 2.8. We now additionally require that the game is *properly loaded*, that is, when $\mathbf{l}$ is the load vector of the game, from every vertex v with $l_v = 0$, there should be a path to a *loaded* vertex u, that is a vertex u with $l_u > 0$. properly loaded

We call a configuration *recurrent*, if it is stable and can be reached after arbitrary long time intervals, i.e., after arbitrary many applications of $\mathfrak{A} \circ \mathfrak{L}_{\mathbf{l}}$. Formally, we define the set of recurrent configurations $\mathfrak{R}(G)$ as recurrent configuration

$$\mathfrak{R}(G) = \bigcap_{t \in \mathbb{N}} \mathfrak{A} \circ \mathfrak{L}_{t\,\mathbf{l}} \left(\mathbb{R}_+^{\mathcal{V}(G)}\right),$$

where $\mathbf{l}$ is the load vector and $\mathbb{R}_+^{\mathcal{V}(G)}$ is the set of all configurations of the game.

THEOREM 2.9. *For a weakly dissipative, properly loaded game, the set* $\mathcal{R}(G)$ *of recurrent configurations does not depend on the loading vector* $\mathbf{l}$ *and has volume* $\det \Delta$.

PROOF. We call two configurations $\mathbf{q}$ and $\mathbf{q}'$ equivalent, when their difference is in the lattice generated by integer combinations of the columns of the transition matrix Δ:

$$\mathbf{q} \sim \mathbf{q}' \Leftrightarrow \mathbf{q} - \mathbf{q}' = \Delta\lambda,$$

where the components of λ are integers.

A set of configurations that contains for every configuration $\mathbf{q}$ exactly one configuration equivalent to $\mathbf{q}$ is called a *fundamental domain.* The determinant of the transition matrix Δ expresses the volume of the parallelepiped spanned by the columns of the matrix, i.e. the volume of

$$\left\{ \sum_{i=1}^{p} \lambda_i \Delta_i : \lambda_i \in (0,1) \right\},$$

where Δ_i is the i^{th} column of Δ. This is precisely the volume of configurations in a fundamental domain. We will show, that the set of recurrent configurations is a fundamental domain.

The rule (+) for firings implies, that $\mathfrak{A}\,\mathbf{q}$ is equivalent to $\mathbf{q}$ for every configuration $\mathbf{q} \in \mathbb{R}_+^{\mathcal{V}(G)}$. Furthermore, by definition of $\mathfrak{A}$, $\mathfrak{A}\,\mathbf{q}$ belongs to the set of stable configurations $\mathcal{S}(G)$. Hence, $\mathcal{S}(G)$ contains a fundamental domain. As this property is translation-invariant, $\mathfrak{L}_{\mathbf{l}}\,\mathcal{S}(G)$ also contains a fundamental domain. Clearly, so does $\mathfrak{A} \circ \mathfrak{L}_{\mathbf{l}}\,\mathcal{S}$.

Next we show, that any two equivalent recurrent configurations $\mathbf{q}$ and $\mathbf{q}'$ are identical, if the load vector $\mathbf{l}$ satisfies

$$\mathbf{l}_v \geq \Delta_{v,v} \text{ for all } v \in \mathcal{V}(G).$$

Let $\mathbf{h}'$ be a stable configuration so that $\mathbf{q}' = \mathfrak{A} \circ \mathfrak{L}_{\mathbf{l}}\,\mathbf{h}'$. Because $\mathfrak{L}_{\mathbf{l}}\,\mathbf{h}'$ and $\mathbf{q}$ are equivalent configurations, there is a vector λ with $\mathfrak{L}_{\mathbf{l}}\,\mathbf{h}' - \mathbf{q} = \Delta\lambda$, where all components of λ are integers. Furthermore, by the condition above we have $\mathfrak{L}_{\mathbf{l}}\,\mathbf{h}'_v \geq \mathbf{q}_v$. Thus, Lemma 2.2 implies that there is a vector μ with $\mathfrak{L}_{\mathbf{l}}\,\mathbf{h}' - \mathbf{q} = \Delta\mu$, where all components of μ are non-negative. Because, again by Lemma 2.2, Δ is nonsingular, λ and μ must be identical, so the components of λ are non-negative integers.

Now we choose t so large, that there is a legal sequence with representative vector λ starting at

$$\overline{\mathbf{h}} = \mathfrak{L}_{t\,\mathbf{l}}\,\mathbf{h} + \Delta\lambda = \mathfrak{L}_{t\,\mathbf{l}}\,\mathbf{h} + \mathfrak{L}_{\mathbf{l}}\,\mathbf{h}' - \mathbf{q},$$

where $\mathfrak{A} \circ \mathfrak{L}_{t\,\mathbf{l}}\,\mathbf{h} = \mathbf{q}$. As all components of $\mathbf{l}$ are positive, this can be done. Then there exists an avalanche starting at $\overline{\mathbf{h}}$ passing through $\mathfrak{L}_{t\,\mathbf{l}}\,\mathbf{h}$. Thus we obtain $\mathfrak{A}\,\overline{\mathbf{h}} = \mathfrak{A} \circ \mathfrak{L}_{t\,\mathbf{l}}\,\mathbf{h} = \mathbf{q}$. Furthermore, an avalanche from $\mathfrak{L}_{t\,\mathbf{l}}\,\mathbf{h}$ to $\mathbf{q}$ started at $\overline{\mathbf{h}}$ terminates at $\mathfrak{L}_{\mathbf{l}}\,\mathbf{h}'$. We obtain $\mathfrak{A}\,\overline{\mathbf{h}} = \mathbf{q}'$, and therefore, by the Abelian property proved in Lemma 2.6, $\mathbf{q} = \mathbf{q}'$.

This proves, that the set of recurrent configurations is a fundamental domain, if the load vector satisfies $\mathbf{l}_v \geq \Delta_{v,v}$, for all vertices v. it remains to show that the set of recurrent configurations is independent of the load vector, provided that the game is properly loaded.

Let $\mathbf{l}$ and $\mathbf{l}'$ be proper load vectors. The proper loading condition guarantees that, for any $t \in \mathbb{N}$, there is a $t' \in \mathbb{N}$ so that there exists a legal sequence starting at $t'\,\mathbf{l}'$ passing through a configuration $\bar{\mathbf{l}}$ with components greater than $t\,\mathbf{l}$. Then, for every configuration $\mathbf{h}$, we have

$$\begin{aligned}\mathfrak{A} \circ \mathfrak{L}_{t\,\mathbf{l}}(\mathcal{S}) &\supseteq \mathfrak{A} \circ \mathfrak{L}_{t\,\mathbf{l}}\left(\mathfrak{A} \circ \mathfrak{L}_{\bar{\mathbf{l}}-t\,\mathbf{l}}(\mathcal{S})\right)\\ &= \mathfrak{A} \circ \mathfrak{L}_{\bar{\mathbf{l}}}(\mathcal{S}).\end{aligned}$$

The last equation follows from Lemma 2.7. Furthermore, we have

$$\begin{aligned}\mathfrak{A} \circ \mathfrak{L}_{\bar{\mathbf{l}}}\mathbf{h} &= \mathfrak{A}(\bar{\mathbf{l}} + \mathbf{h})\\ &= \mathfrak{A}(t'\,\mathbf{l}' + \mathbf{h})\\ &= \mathfrak{A} \circ \mathfrak{L}_{t'\,\mathbf{l}'}\mathbf{h}.\end{aligned}$$

Therefore, $\mathfrak{A}\circ\mathfrak{L}_{t\,\mathbf{l}}(\mathcal{S}) \supseteq \mathfrak{A}\circ\mathfrak{L}_{t'\,\mathbf{l}'}(\mathcal{S})$. This implies, that $\bigcap_{t\in\mathbb{N}} \mathfrak{A}\circ\mathfrak{L}_{t\,\mathbf{l}} \supseteq \bigcap_{t\in\mathbb{N}} \mathfrak{A}\circ\mathfrak{L}_{t\,\mathbf{l}'}$, for any two load vectors $\mathbf{l}$ and $\mathbf{l}'$. We conclude, that the set of recurrent configurations $\mathcal{R}(G) = \bigcap_{t\in\mathbb{N}} \mathfrak{A} \circ \mathfrak{L}_{t\,\mathbf{l}}$ cannot depend on the load vector $\mathbf{l}$. □

The following corollary reveals the connection of the recurrent configurations with the number of spanning trees of G.

COROLLARY 2.10. *Let G be a (di)graph with positive edge weights and let r be one of its vertices. Then the volume of all recurrent configurations of a weakly dissipative, properly loaded game with $s_r = 1$ and $s_v = 0$ for $v \neq r$ is the same as the number of spanning trees of G rooted at r.*

PROOF. By the preceding theorem, the volume of recurrent configurations of the game is $\det \Delta$. Adding all rows of Δ to the row corresponding to r, we obtain a matrix $\mathbf{M}$ with $(\mathbf{M})_{r,r} = 1$ and $(\mathbf{M})_{r,i} = 0$ for $i \neq r$. Hence $\det \Delta = \det \mathbf{C}_{r,r^c}$. By the Matrix-Tree-Theorem, see Chapter 5, Section 1 on page 53, this determinant evaluates to the number of spanning trees of G rooted at r. □

Now consider the vector $\Delta^{-1}\,\mathbf{l}$. If there is a $t \in \mathbb{Z}$ and a vector $\mathbf{n}$ with integer components, so that $t\,\Delta^{-1}\,\mathbf{l} = \mathbf{n}$, then $t\,\mathbf{l} = \Delta\,\mathbf{n}$. This implies that every sequence of load and avalanche operators starting at any given configuration is periodic with a period t, and vertex v fires $\mathbf{n}_v$ times during a period.

This is the case for graphs and Eulerian digraphs G with positive edge weights, when the dissipation is defined as in the corollary above and the load at each vertex is equal to its dissipation: Then we have $\Delta^{-1}\,\mathbf{l} = (1, 1, \ldots, 1)^t$. Note that in this case every vertex fires exactly once during a period. This fact enables us to find a bijection between recurrent configurations and the spanning trees of a graph.

THEOREM 2.11. *Let G be a graph or Eulerian digraph with positive edge weights. Given an arbitrary ordering of the edges of G, let $\prec$ be the lexicographic ordering of all paths from some vertex to a vertex r. Furthermore, let the load and the dissipation at r be equal to one and at all other vertices equal to zero. Then the following two constructions map every spanning in-tree $T \in \mathcal{T}_r(G)$ with weight $w(T)$ onto a set of recurrent configurations of G with volume $w(T)$ and vice versa:*

$\Rightarrow$ Let $\mathbf{q}$ be a recurrent configuration of G. We construct an avalanche $v = (v_1, v_2, \ldots, v_p)$ starting at $\mathfrak{L}_1\mathbf{q}$ as follows: Let $v_1(\mathbf{q}) = r$.

We say that a vertex u is *primed* by the firing of a vertex in the avalanche v, if u has been stable before the firing, but is unstable afterwards.

Clearly, any unstable vertex must have been primed by some other vertex, which in turn must itself have been primed, and so on. Because G is a graph or Eulerian digraph, in every legal sequence each vertex is fired at most once and hence is primed at most once.

It follows that there is a unique path $P_u = (u = u_1, u_2, \ldots, u_k = r)$ from every unstable vertex u to the root r, so that u_i has been primed by u_{i+1} for $i \in \{1, 2, \ldots, k-1\}$. Let $v_{k+1}(\mathbf{q})$ be the vertex which is unstable after the firings of $v_1(\mathbf{q}), v_2(\mathbf{q}), \ldots, v_k(\mathbf{q})$ and whose path P_u is the first in the given lexicographic order.

Let $T_1 = r$ and T_{k+1} be the union of T_k and the vertices which have been primed by $v_k(\mathbf{q})$, together with the edges joining these vertices to $v_k(\mathbf{q})$. Then $T(\mathbf{q}) = T_p$ is a spanning in-tree of G rooted at r. Equivalently, $T(\mathbf{q})$ contains the edges of G so that one end primes the other.

$\Leftarrow$ Let T be a spanning in-tree of G, rooted at r. For any vertex $v \neq r$ of G, let v' be the vertex succeeding v on the path from v to r in T.

For an arbitrary vector λ with $\lambda_v \in [0, 1)$, we define the configuration $\mathbf{q}(T, \lambda)$ by

$$\mathbf{q}_v(T, \lambda)) = \begin{cases} d^i_G(v) - \sum_{u \preceq_T v'} w(v, u) + \lambda_v w(v, v') & \text{for } v \neq r \\ d^i_G(v) + \lambda_v & \text{otherwise,} \end{cases}$$

where $u \prec_T v'$, if the unique path in T from u to r comes before the unique path from v' to r in the given lexicographic order, and $u \preceq_T v'$ if $u = v'$ or $u \prec_T v'$.

Proof. We have to show that each of the configurations defined in the second construction is recurrent and that the two constructions are inverse to each other. Suppose that a spanning in-tree T is given. Let $\mathbf{q} = \mathbf{q}(T, \lambda)$ be a configuration defined by the second construction. We inductively show that the first construction can be applied to $\mathbf{q}$ and the tree produced is the same as T.

We make the induction hypothesis that T_k is a subtree of T, and the sequence $\big(v_1(\mathbf{q}), v_2(\mathbf{q}), \ldots, v_k(\mathbf{q})\big)$ produced by the first construction is an initial segment of the order $\prec_T$. Clearly, this holds for $k = 1$.

Let v be a vertex primed by $v_k(\mathbf{q})$. We show that $v_k(\mathbf{q}) = v'$, hence $v_k(\mathbf{q})$ is adjacent to v in T and thus T_{k+1} is also a subtree of T:

Because v has been primed by $v_k(\mathbf{q})$, and $v_1(\mathbf{q}), v_2(\mathbf{q}), \ldots, v_k(\mathbf{q})$ is an initial segment of the order $\prec_T$, we have

$$0 \leq \mathbf{q}_v + \sum_{u \preceq_T v_k(\mathbf{q})} w(v, u) - d^i_G(v) < w\big(v, v_k(\mathbf{q})\big).$$

By the definition of $\mathbf{q}$ we obtain

$$0 \leq \sum_{u \preceq_T v_k(\mathbf{q})} w(v,u) - \sum_{u \preceq_T v'} w(v,u) + \lambda_v w(v,v') < w\big(v, v_k(\mathbf{q})\big).$$

It follows from the left inequality that $v_k(\mathbf{q}) \preceq_T v'$, and from the right inequality that $v' \preceq_T v_k(\mathbf{q})$. This implies that $v_k(\mathbf{q}) = v'$, which is what we wanted to show.

To complete the induction step, we have to show that if v is the next vertex after $v_k(\mathbf{q})$ in the order $\prec_T$, we have $v = v_{k+1}(\mathbf{q})$.

All vertices preceding v in $\prec_T$ occur in $\{v_1(\mathbf{q}), v_2(\mathbf{q}), \ldots, v_k(\mathbf{q})\}$. Because of $v' = v_k(\mathbf{q})$ and by the definition of $\mathbf{q}$, v is unstable after the firing of the sequence $\big(v_1(\mathbf{q}), v_2(\mathbf{q}), \ldots, v_k(\mathbf{q})\big)$ and is thus in T_{k+1}. We have already shown that T_{k+1} is a subtree of T, so v is also next after $v_k(\mathbf{q})$ in the order $\prec_{T_{k+1}}$, and therefore, by the first construction, $v_{k+1}(\mathbf{q}) = v$.

This also shows, that $\mathbf{q}$ is recurrent, because the sequence

$$\big(v_1(\mathbf{q}), v_2(\mathbf{q}), \ldots, v_p(\mathbf{q})\big)$$

is legal for $\mathfrak{L}_{\mathfrak{l}}\mathbf{q}$ and every vertex is fired exactly once.

Finally, let $\mathbf{q}$ by a recurrent configuration and let $T = T(\mathbf{q})$ be the spanning in-tree produced be the first construction. It remains to show that $\mathbf{q}$ is in the set of configurations defined by the second construction, given T.

We want to show that, if v' primes v,

$$\lambda_v w(v,v') = \mathbf{q}_v - d^i_G(v) + \sum_{u \preceq_T v'} w(v,u)$$

is in $\big[0, w(v,v')\big)$. Because of

$$0 \leq \mathbf{q}_v - d^i_G(v) + \sum_{u \preceq_T v'} w(v,u) < w(v,v'),$$

this is indeed the case. ☐

Note that the preceding correspondence maps the recurrent configurations of a graph onto its in-trees, not its arborescences, although the volume of recurrent configurations equals the number of arborescences of the graph. For (weighted) Eulerian digraphs the number of in-trees and arborescences coincide. For general digraphs, however, the number of in-trees is different from the number of arborescences.

In Section 3 we give a bijection between the spanning in-trees and arborescences of an unweighted Eulerian digraph, but we do not know a bijective proof for weighted Eulerian graphs.

3. Eulerian tours

In this section we will show that in an unweighted Eulerian digraph the number of Eulerian tours is closely related to its number of spanning trees.

BEST-Theorem (de Bruijn, van Ehrenfest, Smith and Tutte). *Let G be an Eulerian digraph, that is, for each vertex of G the in- and outdegree coincide. Then* *the number of arborescences does not depend on the root chosen and can be related to the number of directed Eulerian tours as follows:*

$$\boldsymbol{e}(G) = \prod_{v \in \mathcal{V}(G)} (d^i_G(v) - 1)! \cdot \boldsymbol{t}(G).$$

The following two maps define a correspondence between Eulerian tours with final arc e leading to a vertex r and arborescences of G rooted at r:

($\Rightarrow$) Let E be an Eulerian tour with final arc e leading to a vertex r. Construct the corresponding arborescence as follows:
Let T be the graph consisting only of the vertex r.
WHILE not all vertices of G are in T
    Select the first arc in E which leads to a vertex which is not yet in T.
    Add this arc and the vertex it is leading to, to T.
END WHILE.

($\Leftarrow$) Let T be an arborescence rooted at r. The following construction yields one of the $\prod_{v \in \mathcal{V}(G)} (d^i_G(v) - 1)!$ corresponding Eulerian tours with final arc e, leading to r:
Let E be the tour (v, e, r), where v is the vertex e is incident from. Set $T' = T \cup e$.
REPEAT
    Let v be the initial vertex of E.
    If there is an arc in G incident to v, which is not in T' and not yet in E, then add the vertex v' it is incident from, and the arc itself, to the beginning of E.
    there are $d^i_G(v) - 1$ arcs leading to v which are not in T'
    Otherwise, select the arc in T' which is incident to v and add the vertex v' it is incident from, and the arc itself, to the beginning of E.
UNTIL all arcs incident to v' are in E.

PROOF. It is clear that the first construction produces an arborescence of G. We have to show that the second construction always produces an Eulerian tour of G:

Let E be the walk constructed by the algorithm given an arborescence T. It is clear, that E is a closed walk and all arcs incident from and to r are in E. Let v be any vertex of G. In T there is a walk $(r, e_1, v_1, e_2, \dots, v_{n-1}, e_n, v)$ leading from r to v. We prove inductively, that all arcs incident to and from v are in E.

The arc e_1 is in E. If, for $0 < i < n$, the arc e_i is in E, then all arcs incident *to* v_i must be in E, because $e_i \in \mathcal{E}(T)$ is the last arc incident to v_i added to E. Therefore, all arcs incident *from* v_i are also in E. This applies in particular to e_{i+1}. Hence, as v was arbitrary, all arcs of G are in E, that is, E is Eulerian.

Finally, let T be an arborescence of G and let E be one of the $\prod_{v \in \mathcal{V}(G)} (d^i_G(v) - 1)!$ Eulerian tours produced by the second construction given T. It remains to show that the arborescence produced by the first construction given E is equal to T.

Clearly, for each vertex $v \in \mathcal{V}(G)$ we can fix an arbitrary order in which the arcs incident to v shall be traversed, provided that the arc contained in T is the first arc in this order. On the other hand, constructing the arborescence given an Eulerian tour E, we also select the arc which leads us to a particular vertex the first time in E. All the other arcs leading to this vertex are ignored. □

Note, that we also obtained the important result, that in an Eulerian digraph the number of spanning trees does not depend on the root chosen. This is also true for weighted Eulerian digraphs, i.e., graphs where the sum of the weights of the arcs incident to a vertex is the same as the sum of the weights of the arcs incident from this vertex. This can be shown with the Matrix-Tree-Theorem, see Chapter 5, Section 1, Corollary 1.1.

Furthermore, it is clear that similar constructions define a correspondence between Eulerian tours with first arc e starting from a vertex r and spanning in-trees of G rooted at r. This enables us to construct a bijection between the arborescences and spanning in-trees of G. Again, using the Matrix-Tree-Theorem it can be shown that also in weighted Eulerian digraphs the number of arborescences and the number of spanning in-trees is the same.

4. Forests

As we can count the spanning trees of a graph, we can also try to determine its number of spanning forests. It can be a very difficult task to count the spanning forests of a graph with arbitrary roots. In Chapter 4, Section 3 we will solve this problem for some very simple families of graphs.

However, given a graph G and a subset R of the vertices of G it is very easy to obtain a relationship between the number of spanning forests of G with roots in R and the spanning trees of a related graph:

THEOREM 4.1. *Let G be a (weighted) (di)graph and let R be a subset of the vertices of G. Let $G\cdot_{R^c}$ be the contraction of G to the vertices in $\mathcal{V}(G)\setminus R$ defined in Definition 2.13, denoting the new vertex by r. Then there is a (weight preserving) bijection between the spanning forests of G with roots in R and the spanning trees of $G\cdot_{R^c}$, rooted at r.*

PROOF. This is obvious. □

REMARK. Of course, sometimes it is still easier to count the spanning forests of a graph directly, rather than making this detour, see for example Corollary 3.2 in Chapter 4. But in many cases we can derive the corresponding formula for spanning forests easily from the expression for spanning trees. A particularly nice example is the encoding given in Theorem 3.3 in the same chapter and the following theorems and propositions.

5. Matchings

Apart from spanning trees, the most important type of subgraph of graphs are *perfect matchings*:

DEFINITION 5.1. A *perfect matching* M of a graph G is a one-regular spanning subgraph of G. perfect matching

Remark. If a graph has a perfect matching, its number of vertices must be even.

In 1974, Temperley [**38**] found a bijection between the spanning trees of the $m \times n$ grid and perfect matchings of the $(2m+1) \times (2n+1)$ grid with a corner removed. Kenyon, Propp, and Wilson [**23**] generalized this bijection to spanning trees of weighted planar (di)graphs.

Let $\vec{G}$ be a weighted planar digraph. We define a weighted planar graph G' as follows: Let the vertex set of G' be $\mathcal{V}(\vec{G}) \cup \mathcal{E}(\vec{G}) \cup \mathcal{F}(\vec{G})$. We will denote the vertices in G' by v', e' and f', depending on the corresponding structure in $\vec{G}$. For an example, see Figure 1. Note, that we draw the vertex corresponding to the unbounded face of $\vec{G}$ as remarked in Section 1.

see Definition 1.1

We connect two vertices of G' by an edge if their corresponding structures in $\vec{G}$ are either an edge and its head, or an edge and one of the faces it bounds.

Let the weight of an edge in G' between two vertices v' and e', corresponding to a vertex v and an edge e of $\vec{G}$, be the weight of the edge e in $\vec{G}$. The weight of an edge joining e' and f', corresponding to an edge e and a face f of $\vec{G}$, is always 1.

Theorem 5.2. *Let $\vec{G}$ be a planar digraph and let G be its underlying graph. Construct G' as above. Let $\mathring{v}$ be a vertex and $\mathring{f}$ a face of $\vec{G}$. If $\mathring{v}'$ is a vertex on the border of $\mathring{f}'$, then the following correspondence is a weight-preserving bijection between the spanning trees of $\vec{G}$ rooted at $\mathring{v}$ and the perfect matchings of $G' \setminus \{\mathring{v}', \mathring{f}'\}$:*

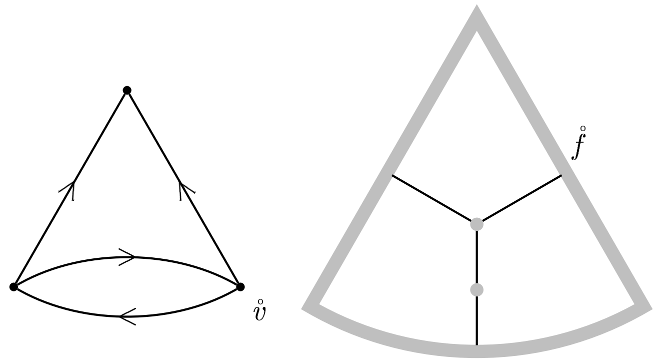
- ($\Rightarrow$) Let $\vec{T}$ be a spanning tree of $\vec{G}$ rooted at $\mathring{v}$, and let T be the underlying spanning tree in G. Let T^* be the spanning tree of G^* corresponding to the edges not in T. Let $\vec{T}^*$ be the digraph obtained by orienting the edges of T^* away from $\mathring{f}$. Then we can construct a perfect matching M of G' as follows: For each vertex $v \in \mathcal{V}(\vec{G})$, pair v' with the unique vertex e', $e \in \mathcal{E}(\vec{G})$, such that v is the head of e in $\vec{T}$, and for each face $f \in \mathcal{F}(\vec{G})$, pair f' with the unique e', such that f is the head of e in $\vec{T}^*$.
- ($\Leftarrow$) Let M be a perfect matching of G'. Then the edge set of the corresponding spanning tree of $\vec{G}$ consists of the edges e, such that e' is paired with a vertex of G' corresponding to a vertex of $\vec{G}$.

see the proof of Theorem 1.2 in Chapter 2

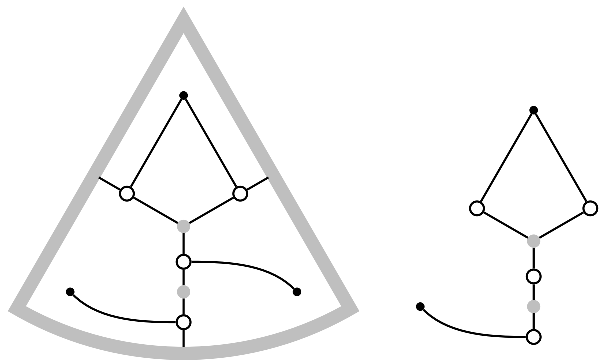
Remark. We can extend this theorem to undirected weighted graphs by thinking of each undirected edge as two arcs, one in each direction. (See also Definition 1.1 in Chapter 2.)

Proof. As we can recover the spanning tree from the matching it is mapped to, the mapping is injective. Let $\vec{T}$ be the spanning subgraph of $\vec{G}$ formed by the set of edges e, such that e' is paired with a vertex of G' corresponding to a vertex of $\vec{G}$. We have to show, that $\vec{T}$ indeed is a spanning tree of $\vec{G}$, rooted at $\mathring{v}$. First we show that T, the underlying graph of T, is acyclic:

Suppose T contained a cycle. Because G and G' are planar, by the Jordan-Curve-Theorem the cycle divides the plane (and hence G and G') into two regions, one of which contains both $\mathring{v}$ and $\mathring{f}$ and the other of which contains neither. We claim that each part contains an odd number of vertices of G':

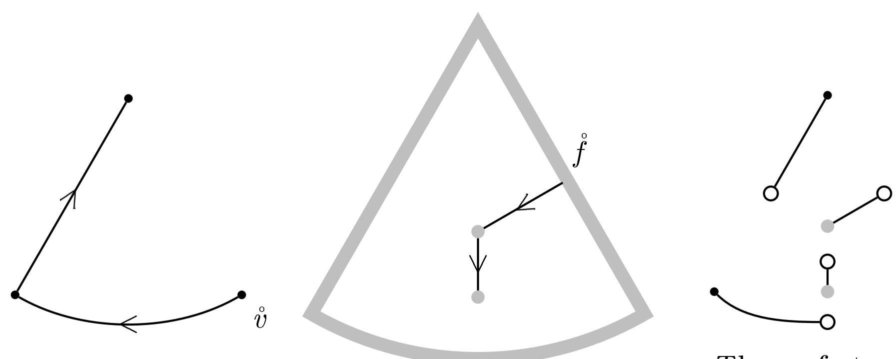


A digraph $\vec{G}$ The dual of G



The graph G' constructed from $\vec{G}$ $G' \setminus \{\mathring{v}', \mathring{f}'\}$

FIGURE 1



A spanning tree $\vec{T}$ of $\vec{G}$, rooted at $\mathring{v}$ $\vec{T}^*$ The perfect matching corresponding to $\vec{T}$

FIGURE 2

Modify G by replacing either of the two regions by a single face. By Euler's formula, the number of vertices, edges and faces in the resulting graph must be even. Since there are an even number of these elements on the cycle (as many vertices as edges) and an odd number in the modified region (1 face), the unmodified region must contain an odd number of elements, corresponding to the vertices of G', as well.

Since the edges of the cycle disconnect G' into parts lying in the two regions, the matching must pair elements within one region. This is impossible, since each region has been shown to contain an odd number of vertices of G'.

As $\vec{T}$ has $\left|\mathcal{V}(\vec{G})\right| - 1$ edges, it remains to show that all edges are directed away from $\mathring{v}$. This is the case, as two edges in $\vec{T}$ pointing towards the same vertex are adjacent in G', and therefore cannot be contained in one matching. □

REMARK. Ciucu [**14**] used this correspondence in conjunction with a theorem which expresses the number of perfect matchings in a graph in terms of the number of perfect matchings of a related graph to show that the even Aztec rectangle has exactly four times as many spanning trees as the corresponding odd Aztec rectangle.

CHAPTER 4

Combinatorial Proofs

In this chapter we will be concerned with some combinatorial methods that enable us to determine the number of spanning trees of a graph. Although these methods apply only to rather restricted classes of graphs, sometimes strikingly simple calculations reveal the number of spanning trees of seemingly complex graphs.

1. Reduction Procedures

It is obvious that any spanning tree of a graph either does or does not contain a given edge e. Furthermore, the number of spanning trees which contain a specified edge e is the same as the total number of spanning trees of the graph with e and its endnodes contracted to a single node. Hence, as already Feussner [**16, 17**] noted,

$$\boldsymbol{t}(G) = \boldsymbol{t}(\,G|_{e^c}) + \boldsymbol{t}(G\boldsymbol{\cdot}_{e^c}).$$

More generally, let $E \subseteq \mathcal{E}(G)$ be a set of edges of a connected graph G. Suppose that the subgraphs $G|_E$ and $G|_{E^c}$ have exactly two vertices in common. Then

$$\boldsymbol{t}(G) = \boldsymbol{t}(\,G|_{E^c}) \cdot \boldsymbol{t}(G\boldsymbol{\cdot}_E) + \boldsymbol{t}(\,G|_E) \cdot \boldsymbol{t}(G\boldsymbol{\cdot}_{E^c}).$$

For example, take E to be a set of k parallel edges, then

$$\boldsymbol{t}(G) = \boldsymbol{t}(\,G|_{E^c}) + k \cdot \boldsymbol{t}(G\boldsymbol{\cdot}_{E^c}).$$

Similarly, if E is the edge set of k parallel paths with lengths $l_1,\ l_2,\ \ldots\ ,\ l_k$ joining two vertices, then

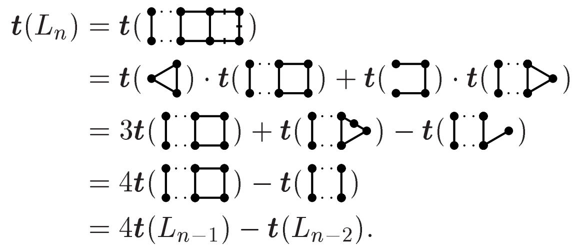
$$\boldsymbol{t}(G) = l_1 \cdots l_k \cdot \boldsymbol{t}(\,G|_{E^c}) + \sum_{i=1}^{k} l_1 \cdots \check{l}_i \cdots l_k \cdot \boldsymbol{t}(G\boldsymbol{\cdot}_{E^c}).$$

Subdividing an edge e of G, denoting the resulting graph with G_s, we get

$$\boldsymbol{t}(G_s) = \boldsymbol{t}(\,G|_{e^c}) + \boldsymbol{t}(G).$$

Using these relations we get recursions for the number of spanning trees for some families of graphs:

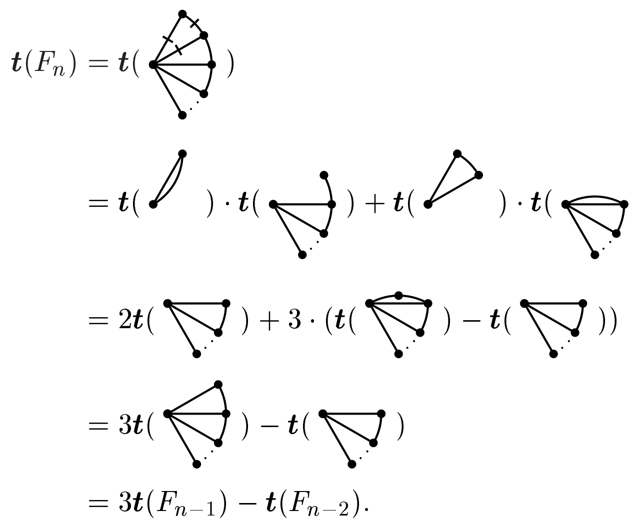
EXAMPLE 1.1. The *ladder* is defined as $L_n = K_2 \oplus P_n$. For $n \geq 3$ we have



Furthermore, we have $\boldsymbol{t}(L_1) = 1$ and $\boldsymbol{t}(L_2) = 4$. By standard methods for solving linear recursions we obtain

$$\boldsymbol{t}(L_n) = \frac{1}{2\sqrt{3}}\left[\left(2+\sqrt{3}\right)^n - \left(2-\sqrt{3}\right)^n\right].$$

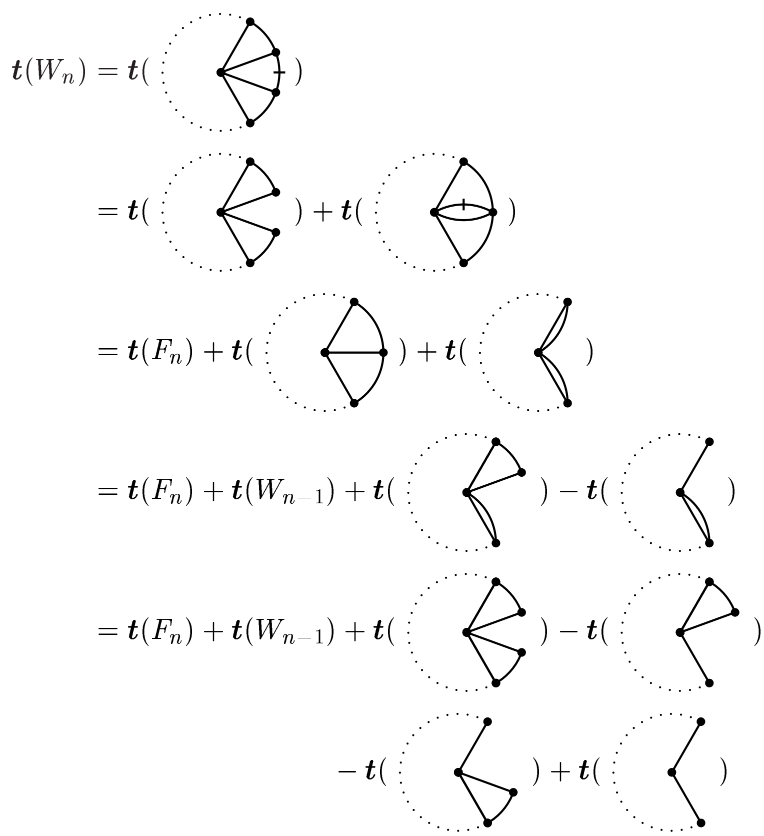
EXAMPLE 1.2. The *fan* is defined as $F_n = K_1 \bigtriangledown P_n$. For $n \geq 3$ we have



Clearly, $\boldsymbol{t}(F_1) = 1$ and $\boldsymbol{t}(F_2) = 3$. Solving this linear recursion we obtain

$$\boldsymbol{t}(F_n) = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right].$$

EXAMPLE 1.3. The *wheel* is defined by $W_n = K_1 \bigtriangledown C_n$. For $n \geq 4$ we have



$$
\begin{aligned}
&= \boldsymbol{t}(W_{n-1}) + 2\boldsymbol{t}(F_n) - 2\boldsymbol{t}(F_{n-1}) + \boldsymbol{t}(F_{n-2}) \\
&= \boldsymbol{t}(W_{n-1}) + \boldsymbol{t}(F_n) + \boldsymbol{t}(F_{n-1}).
\end{aligned}
$$

Defining C_2 as two parallel edges, this recursion holds also for $n = 2$. Therefore, with $\boldsymbol{t}(W_2) = 3$ and $\boldsymbol{t}(W_4) = 16$, we have

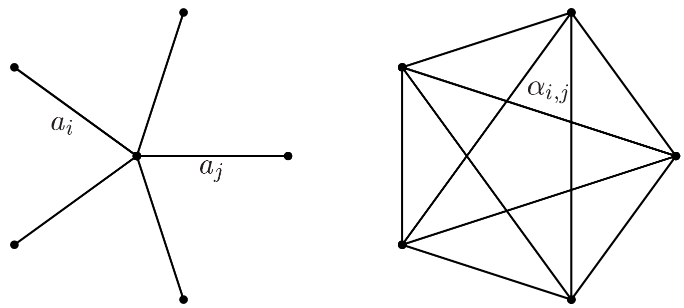
$$\boldsymbol{t}(W_n) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2.$$

For weighted graphs G there is another reduction process, which consists of replacing a star $S_n \subseteq G$ by a complete graph K_n with appropriately chosen weights, thus reducing G by one vertex.

We number the vertices of the star except the one in the middle clockwise from 1 to n, and label the vertex in the middle with 0. Suppose that the edge connecting vertex i with the vertex in the center of the star has weight a_i, $i \in \{1, 2, \ldots, n\}$. Let $\delta = \sum_{i=1}^{n} a_i$ and let $\alpha_{i,j} = \frac{a_i a_j}{\delta}$.

Let G' be the graph obtained from G by replacing the star by the complete graph, where the edge connecting vertices i and j has weight $\alpha_{i,j}$. Then

$$\boldsymbol{t}(G) = \delta \boldsymbol{t}(G').$$



Unfortunately, we do not have a combinatorial proof for this, so we have to use the Matrix-Tree-Theorem proved in Chapter 5, Section 1 on page 53. It expresses the number of spanning trees of a graph as the determinant of any principal minor of its Laplacian matrix.

The Laplacian matrix of G can be written as

$$\mathbf{C}_G = \left(\begin{array}{ccccc|c}
\delta & -a_1 & -a_2 & \dots & -a_n & 0 \ \dots \ 0 \\
-a_1 & a_1+d_1 & 0 & \dots & 0 & \\
-a_2 & 0 & a_2+d_2 & & \vdots & \mathbf{A} \\
\vdots & \vdots & & \ddots & 0 & \\
-a_n & 0 & \dots & 0 & a_n+d_n & \\
\hline
0 & & & & & \\
\vdots & & \mathbf{A}^t & & & \mathbf{B} \\
0 & & & & &
\end{array}\right),$$

where $d_i = d_G(i) - a_i$. By the Matrix-Tree-Theorem, $\boldsymbol{t}(G)$ equals the determinant of any principal minor of $\mathbf{C}_G$:

$$\boldsymbol{t}(G) = \det(\mathbf{C}_G)_{r^c}.$$

For our purposes we demand $r > n$. We can then transform $\mathbf{C}_G$ into the Laplacian matrix of G' by adding $\frac{a_i}{\delta}$ times the first row to each row i, $i \in \{1, 2, \dots, n\}$. The result of these operations is the matrix

$$\left(\begin{array}{ccccc|c}
\delta & -a_1 & -a_2 & \dots & -a_n & \\
0 & a_1(1-\frac{a_1}{\delta})+d_1 & \alpha_{1,2} & \dots & \alpha_{1,n} & \\
0 & \alpha_{1,2} & & & \vdots & \\
\vdots & \vdots & & \ddots & \alpha_{n-1,n} & \mathbf{A} \\
0 & \alpha_{1,n} & \dots & \alpha_{n-1,n} & a_n(1-\frac{a_n}{\delta})+d_n & \\
\hline
 & & & & & \\
 & & \mathbf{A}^t & & & \mathbf{B} \\
 & & & & &
\end{array}\right).$$

It remains to check that $a_i(1-\frac{a_i}{\delta})+d_i$ is indeed the degree of vertex i in G':

$$\sum_{\substack{j=1\\j\neq i}}^{n}\alpha_{i,j}+d_i=\sum_{\substack{j=1\\j\neq i}}^{n}\frac{a_ia_j}{\delta}+d_i=a_i(1-\frac{a_i}{\delta})+d_i.$$

The inverse operation is not quite as nice to describe. Furthermore, $\alpha_{i,j}=\frac{a_ia_j}{\delta}$ implies that the weights of the complete subgraph have to obey the following boundary conditions: $\frac{\alpha_{i,j}}{a_i}$ is constant for all i.

If these conditions are satisfied, it can be checked that the following weights of the star are appropriate:

$$a_1=\frac{\alpha_{1,2}\alpha_{1,3}}{\alpha_{2,3}}+\sum_{\substack{l=1\\l\neq 1}}^{n}\alpha_{1,l}$$

and, for $k\neq 1$

$$a_k=\alpha_{1,k}(1+\frac{\alpha_{2,3}}{\alpha_{1,2}\alpha_{1,3}}\sum_{\substack{l=1\\l\neq 1}}^{n}\alpha_{1,l}).$$

The variable δ then evaluates to

$$\delta=\frac{\alpha_{2,3}}{\alpha_{1,2}\alpha_{1,3}}\left(\sum_{\substack{l=1\\l\neq 1}}^{n}\alpha_{1,l}+\frac{\alpha_{1,2}\alpha_{1,3}}{\alpha_{2,3}}\right)^2.$$

An interesting special case occurs when all weights before and after the transformation are integers. In this case, G and G' can be represented without weights by replacing an edge with weight k by k parallel edges.

Example 1.4. A complete subgraph K_n of a graph G – with all edge weights $\alpha_{i,j}$ equal to one – can be replaced by a star S_n, where each edge has weight $a_i=n$. This applies also for the complete graph itself. The number of spanning trees of the corresponding star with all edge weights equal to n is n^n. The variable δ evaluates to n^2. Hence,

$$\boldsymbol{t}(K_n)=\frac{1}{n^2}n^n=n^{n-2}.$$

In fact, these transformations can be generalized to apply to digraphs.

Remark. By summing up over all edges of a graph we obtain another nice identity: Let $T_1,\ T_2,\ \dots,\ T_t$ be the spanning trees of G. Then we have

$$\sum_{i=1}^{q}\chi(e_i\in T_j)=p-1$$
$$\sum_{j=1}^{\boldsymbol{t}}\chi(e_i\in T_j)=\boldsymbol{t}(G\boldsymbol{\cdot}_{e_i{}^c}).$$

Hence

$$\begin{aligned}\sum_{e\in\mathcal{E}(G)} \boldsymbol{t}(G\boldsymbol{\cdot}_{e^c}) &= \sum_{i=1}^{q} \boldsymbol{t}(G\boldsymbol{\cdot}_{e_i{}^c}))\\ &= \sum_{j=1}^{\boldsymbol{t}}\sum_{i=1}^{q} \chi(e_i \in T_j)\\ &= \sum_{j=1}^{\boldsymbol{t}} (p-1)\\ &= (p-1)\cdot \boldsymbol{t}(G).\end{aligned}$$

And

$$\begin{aligned}\sum_{e\in\mathcal{E}(G)} \boldsymbol{t}(\,G|_{e^c}) &= \sum_{i=1}^{q} \boldsymbol{t}(\,G|_{e_i{}^c}))\\ &= \sum_{i=1}^{q} (\boldsymbol{t}(G) - \boldsymbol{t}(G\boldsymbol{\cdot}_{e_i{}^c}))\\ &= q\cdot \boldsymbol{t}(G) - (p-1)\cdot \boldsymbol{t}(G)\\ &= (q-p+1)\cdot \boldsymbol{t}(G).\end{aligned}$$

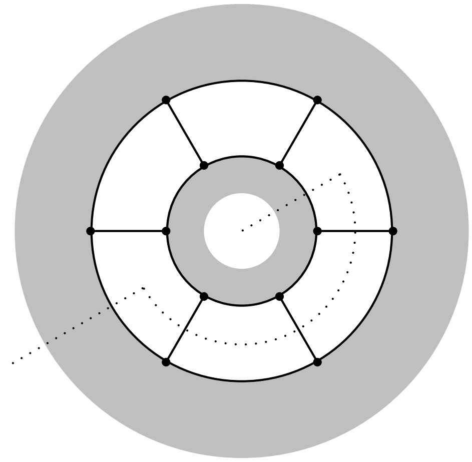
2. Dividing Graphs

Although the method of restriction and contraction works well for families of 'linear' graphs, it is already difficult to count the spanning trees of the wheel and it is not applicable for graphs like the square of the cycle. Following the method presented below we can count spanning trees in some graphs with rotational symmetry like the square of a circle C_n^2, the Möbius ladder M_n or the cyclic ladder $K_2 \oplus C_n$.

First we embed the graph on a suitable surface. This can be the cylinder or the Möbius strip.

Then we cut the surface along some Jordan path to obtain something homeomorphic to a rectangle. In the following we will identify a Jordan path with the set of edges it crosses. Hence we consider two Jordan paths as different only if they cross different sets of edges.

Example 2.1. The following figure depicts the cyclic ladder $C_6 \oplus K_2$ embedded in a gray cylinder and a dotted Jordan path which cuts the cylinder:



By deleting all the edges of the graph the Jordan path crosses we arrive at a spanning subgraph. We now try to find a set $\mathcal{P}(G)$ of Jordan paths, so that the concatenation of any two of them induces a cutset of the graph and every spanning tree is contained in exactly one of the graphs $G|_{P^c}$, where P is the set of edges crossed by a particular Jordan path.

This is easy for planar graphs embedded in a cylinder, as $K_2 \oplus C_n$ or C_{2n}^2: Consider the Jordan paths running from one to the other border of the cylinder. Concatenating any two such Jordan paths we obtain a closed path which separates the surface into two regions. As the paths are supposed to induce different cuts, there must be at least one vertex in each of the regions. Hence, deleting the edges crossed by any of the two paths, we obtain a disconnected graph. Thus, as a spanning tree is connected, it cannot be induced by two different Jordan paths.

Given a spanning tree, it is always possible to find a Jordan path as described above, that does not cross any edge of the tree, as a tree cannot contain a cycle.

For non-planar graphs embedded in a Möbius strip, like the Möbius ladder M_n or C_{2n+1}^2, we may use the family of closed Jordan paths starting at an arbitrary fixed point on the border of the Möbius strip and cutting it into something homeomorphic to a rectangle. Again, two such paths induce a cutset of the graph, as the first path cuts the Möbius strip into a rectangle and the second cuts the rectangle into two regions.

Therefore, every Jordan path in $\mathcal{P}(G)$ corresponds to a set of spanning trees and any two such sets are disjoint. Furthermore, every spanning tree is contained in one of these sets. So all we have to do is to add up the number of spanning trees corresponding to each Jordan path in $\mathcal{P}(G)$:

$$\boldsymbol{t}(G) = \sum_{P \in \mathcal{P}(G)} \boldsymbol{t}(G|_{E^c}).$$

In the various ladders we deal with below, we will use the term 'rungs' for the edges connecting the two paths or circles. The explicit formulas can easily be obtained using the recursions of Section 1.

Example 2.2. The cyclic ladder $K_2 \oplus C_n$ can be embedded in a cylinder. Any Jordan path crossing $k < n$ 'rungs' results in a ladder of length $n - k$ with two

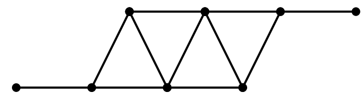


FIGURE 1. A strip graph

pending paths, one on each end. For any $k > 0$ there are $2n$ such paths, since there are n edges to start with and two directions to go: clockwise or counterclockwise. For $k = 0$, that is, for the Jordan path which crosses no 'rungs', there is no counterpart, of course. Hence we have

$$\begin{aligned}\boldsymbol{t}(K_2 \oplus C_n) &= 2n \sum_{k=1}^{n-1} \boldsymbol{t}(L_k) + n\boldsymbol{t}(L_n) \\ &= \frac{n}{2}\left[\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n - 2\right].\end{aligned}$$

EXAMPLE 2.3. The Möbius ladder M_n can be embedded in a Möbius strip. Again, a Jordan path crossing $k < n$ 'rungs' and any two other edges results in a ladder of length $n-k$ with two pending paths, one on each end. For every pair of edges which are not 'rungs' we have two different Jordan paths, except for the case, where the two edges are exactly opposite of each other belonging to the same face of M_n. Furthermore, there are $2n$ Jordan paths crossing all the 'rungs' and only one other edge, inducing a path of length $2n-1$. Hence, we have

$$\begin{aligned}\boldsymbol{t}(M_n) &= 2n \sum_{k=1}^{n-1} \boldsymbol{t}(L_k) + n\boldsymbol{t}(L_n) + 2n \\ &= \frac{n}{2}\left[\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n + 2\right].\end{aligned}$$

EXAMPLE 2.4. Fore even n, the square of the circle C_n^2 is a planar graph. Its spanning trees can be counted exactly the same way as the spanning trees of $K_2 \oplus C_n$, except that the Jordan paths now induce strip graphs as in Figure 1. Note that these strip graphs have the same dual as the fan. Hence

$$\boldsymbol{t}(C_n^2) = 2n \sum_{k=1}^{n-1} \boldsymbol{t}(F_k) + n\boldsymbol{t}(F_n) = nf_n^2,$$

where f_n is the n^{th} Fibonacci number.

For odd n, the square of the circle C_n^2 must be embedded in the Möbius strip. Proceeding similarly to the example of the Möbius ladder, we obtain the same expression as for even n.

3. Codes

In 1918, Prüfer [**34**] constructed a correspondence between the trees of the complete graph K_p for $p > 1$, and words of $p-2$ letters from a p-element set, showing $\boldsymbol{t}(K_p) = p^{p-2}$.

With a minor change, Prüfer's algorithm can be used for encoding any spanning forest of the complete graph:

THEOREM 3.1 (Prüfer 1918). *The following two maps define a correspondence between labelled forests on p vertices with roots in R ($p > |R|$) and words of $p - |R|$ letters from a p-element set, the last letter being an element of R.*

$\Rrightarrow$ Let F_R be a forest on p vertices labelled with numbers from 1 to p with roots in R. Produce the corresponding word as follows:
WHILE there is at least one edge in the forest
  Write down the label of the vertex adjacent to the leaf with the smallest label.
  Remove this leaf and its incident edge.
END WHILE.

in a forest a leaf is not a root and has degree one (outdegree zero)

$\Lleftarrow$ Let $w = (v_1, v_2, \ldots, v_{p-|R|})$ be a word with $p - |R|$ letters, all in $\{1, 2, \ldots, p\}$, the last in R. Let V be a set containing the vertices labelled from 1 to p which are not roots. Produce the corresponding tree as follows:
WHILE the word is not empty
  Let u be the vertex with the smallest label in $V \setminus R$ which does not appear in w, let v be the first letter of w.
  Add (v, u) to the edge set. Drop the first letter of the word w and remove u from the set V.
END WHILE.

in digraphs, an arc (v, u) starts in v and terminates in u

REMARK. In Prüfer's original encoding for trees, the last edge would never be removed as it must always be incident to the root. Hence his encoding produces codes of length $p - 2$. Using the variant of his encoding described above, though, the following corollary is obtained much easier:

COROLLARY 3.2. *For a given set of roots R there are $|R| \cdot p^{p-|R|-1}$ spanning forests of the complete graph K_p.*

Clearly, Prüfer's encoding can be applied in just the same manner when we consider spanning trees of any graph or digraph. In these cases some codes simply will never be produced. When multiple edges are allowed, some codes will be produced more often. It seems though, that it is not any easier to count those codes which may be output of the procedure than to count the spanning trees of the graph in some other way.

Knuth [**26**], later Kelmans [**22**] and finally Pak and Postnikov [**33**] generalized Prüfer's encoding – although the constructions in the latter two papers were incorrect, the idea was right – to deal with the generalized lexicographic product $G[H_1, H_2, \ldots, H_p]$ of graphs G and H_v for $v \in \mathcal{V}(G)$.

Given a linear order on the vertices of G and H_1, H_2, … , H_p, in what follows we will use the lexicographic order on the vertices of the lexicographic product $G[H_1, H_2, \ldots, H_p]$, id est:

$$ui < vj :\Leftrightarrow u < v \text{ or } u = v \text{ and } i < j.$$

The encoding we will present enables us to encode the spanning trees of any (di)graph of the form $G\,[H_1, H_2, \dots, H_p]$. An illustration will be given in Example 3.4 below.

THEOREM 3.3 (Knuth 1968, Kelmans 1989, Pak and Postnikov 1990). *The two maps below define a one to one correspondence between the spanning trees of $G\,[H_1, H_2, \dots, H_p]$ with root rx $(p > 1)$ and structures w as follows:*

$$w = \left(\overline{T}, (F_v)_{v\in\mathcal{V}(G)}, (\overline{w}_v)_{v\in\mathcal{V}(G)}, (w_v)_{v\in\mathcal{V}(G)}\right), \text{ where}$$

- *$\overline{T}$ is a tree in $\mathfrak{T}_r(G)$*
- *F_v is a rooted forest of H_v*
- *$\overline{w}_v$ is a word with $d^o_{\overline{T}}(v)$ letters in H_v, and the last letter of $\overline{w}_r$ belongs to the component of F_r which contains the root*
- *w_v is a word with $c(F_v) - 1$ letters in $\mathcal{N}^i_{G\,[H_1,H_2,\dots,H_p]}(H_v)$*

$\Rightarrow$ Let T be a spanning tree of $G\,[H_1, H_2, \dots, H_p]$ with root rx. Produce w as follows:
WHILE there is at least one edge in the tree T
  Let ui be the leaf with the smallest label, vj the vertex incident to ui, hence $(vj, ui) \in \mathcal{E}(T)$.
  If $v = u$ then let (vj, ui) be a new edge of F_v.
  If $v \neq u$ and ui is not the last vertex of H_u remaining in T then write vj to w_u.
  Otherwise write vj to $\overline{w}_v$ and let (v, u) be a new edge of $\overline{F}$.
  Remove this leaf and its incident edge.
END WHILE.

$\Leftarrow$ Let w be a structure as described above. Let V be a set containing the vertices of $G\,[H_1, H_2, \dots, H_p]$ except of rx. Produce the corresponding tree T as follows:
WHILE not all of the words in w are empty
  Let ui be the vertex with the smallest label in V which has outdegree zero in F_u and does neither occur in w_v, nor in $\overline{w}_u$. Remove ui from V. *ui might be a root in F_u!*
  If u_i has a predecessor in F_u, then let vj be this vertex and remove (vj, ui) from F_u.
  If u_i does not have a predecessor in F_u but $w_u \neq \emptyset$, then let vj be the first letter in w_u.
  Otherwise let vj be the first letter in $\overline{w}_v$, where v is the predecessor of u in $\overline{F}$.
  Remove this occurrence of vj and add (vj, ui) to the edge-set of T.
END WHILE.

REMARK. Using Theorem 4.1 in Chapter 3 we can transform this theorem into a theorem on spanning forests: Let G and $H_1, H_2, \dots, H_p$ be graphs and R be a subset of the set of vertices of G. Then the number of spanning forests $\mathfrak{F}_R(G\,[H_1, H_2, \dots, H_p])$ – forests that have roots in $\bigcup_{v\in R} H_v$ – is equal to the number of spanning trees in $\mathfrak{T}_r(G' \cdot_{R^c} [H_1, H_2, \dots, H_{p-|R|}, H_R])$, where H_R is the single vertex to which the vertices in R are contracted, and G' has the same vertex set as G but every edge in G between a vertex $r \in R$ and some other vertex not in R is replaced by $|\mathcal{V}(H_r)|$ edges, and every edge between two vertices r_1 and r_2 in R is replaced by $|\mathcal{V}(H_{r_1})|\,|\mathcal{V}(H_{r_2})| + |\mathcal{E}(H_{r_1})| + |\mathcal{E}(H_{r_2})|$ edges.

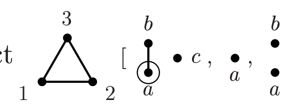
Applying Theorem 3.3 to $G'\bullet_{R^c}[H_1, H_2, \ldots, H_{p-|R|}, H_R]$, we see that the spanning forests of $G[H_1, H_2, \ldots, H_p]$ with roots in $\bigcup_{r \in R} H_r$, where $R \subset \mathcal{V}(G)$ are mapped onto structures w as follows (the details are left to the reader):

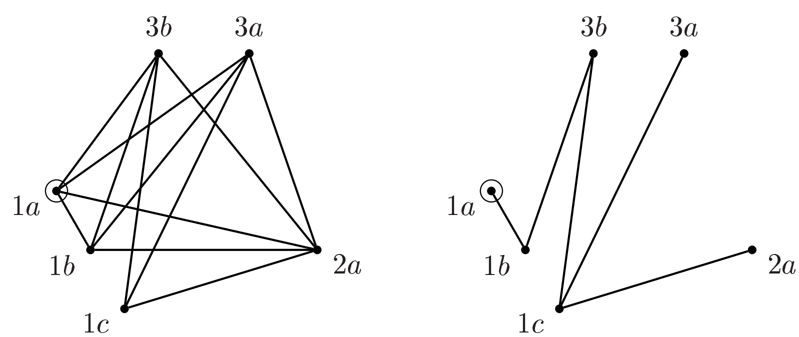
$$w = \big(\overline{F}, (F_v)_{v \in \mathcal{V}(G)\setminus R}, (\overline{w}_v)_{v \in \mathcal{V}(G)}, (w_v)_{v \in \mathcal{V}(G)\setminus R}\big), \text{ where}$$

- $\overline{F}$ is a forest in $\mathfrak{F}_R(G)$,
- F_v is a rooted forest of H_v,
- $\overline{w}_v$ is a word with $d^o_{\overline{F}}(v)$ letters in H_v,
- w_v is a word with $c(F_v) - 1$ letters in $\mathcal{N}^i_{G[H_1, H_2, \ldots, H_p]}(H_v)$.

The maps between these structures and $\mathfrak{F}_R(G[H_1, H_2, \ldots, H_p]$ are the same as above, except that the root rx must be replaced with the set of roots $\bigcup_{r \in R} H_r$.

Before we embark on the proof we give an example:

Example 3.4. Consider the lexicographic product  (The encircled vertex ⊙ denotes the root.) The following figure depicts the resulting graph and one of its spanning trees.

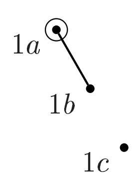


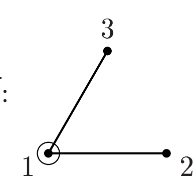
Applying the algorithm, we arrive at the following structure w:

$w_1 : 3b$

$\overline{w}_1 : 1c, 1b$

$w_3 : 1c$

F_1: 

$\overline{F}$: 

Proof of Theorem 3.3. We prove that the two maps are well defined and inverses of each other.

⇒ produces a set w as demanded:

We prove this by induction on the number of vertices of $G[H_1, H_2, \ldots, H_p]$:

Suppose the statement holds for any tree T' of $G'[H'_1, H'_2, \ldots, H'_p]$ with a given number of vertices. Let T be a tree of $G[H_1, H_2, \ldots, H_p]$ with one more vertex. Let ui be the leaf with the smallest label and vj the vertex incident to ui. Now there are three possibilities:

- $v = u$:
  Removing ui and (vj, ui) gives a tree T' of $G[H_1, \ldots, H_u \setminus ui, \ldots, H_p]$. By induction, T' is encoded by a structure w' of the type described above. We have to show that $w = w'$ with F'_v replaced with $F_v = F'_v \cup (vj, ui)$ is an encoding of T as demanded. This is the case, because F_v is still a forest of H_v and $c(F_v) = c(F'_v)$.

- $v \neq u$, but there is another vertex besides ui in H_u:
  Removing ui and (vj, ui) again yields a tree T', just as above. Again $\overline{T}$ and $\overline{w}_v$ for $v \in \mathcal{V}(G)$ remain unchanged, but now $c(F_u) = c(F'_u) + 1$ and $w_u = vj, w'_u$. Again the algorithm does the right thing.

- $v \neq u$, and ui is the only vertex in H_u:
  In this case we get a tree T' of $(G \setminus u)[H_1, \ldots, \check{H}_u, \ldots, H_p]$. Now $\overline{w}_v = vj, \overline{w}'_v$ and $\mathcal{E}(\overline{T}) = \mathcal{E}(\overline{T}') \cup (v, u)$, accordingly $d^o_{\overline{T}}(v) = d^o_{\overline{T}'}(v) + 1$. Hence, for $v \neq r$, all conditions imposed on w are still satisfied. Now suppose $v = r$. We have to show, that the last letter of $\overline{w}_r$ belongs to the component of F_r which contains the root. If $\overline{w}'_r \neq \emptyset$ we are done, as the last letter of $\overline{w}'_r$ belongs to the component of F'_r which contains the root by induction. Otherwise $d^o_{\overline{F}'}(r) = 0$ which means that $H_u = ui$ is the only subgraph connected to H_r, hence it must be connected to a component of F'_r which contains a root.

$\Leftarrow$ produces a tree in $\mathcal{T}_r(G[H_1, H_2, \ldots, H_p])$:

Let W be a multiset containing the letters of $(w_v)_v$ and $(\overline{w}_v)_v$, and all letters ui with $d^o_{F_u}(ui) \neq 0$. Let W' be the set of distinct letters in $W \setminus \{rx\}$. We have to show that there is always a vertex ui in $V \setminus W'$, unless W is empty:

At most $|W| - |W'| - 1$ letters can be removed from W without decreasing the size of W', as the root rx is removed last, which we will show later. Hence, as $|W| \leq |V|$, there are at least $|W'| + 1$ letters remaining in V, whereas W' has not changed.

It remains to show that the last letter removed is the root:

First we have to show that the last letter of a word $\overline{w}_v$ can be removed only, if for $(v, u) \in \overline{T}$ all words $\overline{w}_u$ and w_u are already empty and F_u consists of isolated vertices only. In order to see this, suppose that $\overline{w}_v$ contains only a single letter vj, and, when $(v, u) \in \overline{T}$, ui is the vertex with the smallest label in $V \setminus \{rx\}$. If $\overline{w}_u$ still contains a letter ux (which must be different from ui because $ui \notin W$), not all of the words w_u can be empty or F_u still contains an edge: Either ui and ux belong to different components of F_u, then w_u contains at least one vertex. Or, ui and ux belong to the same component of F_u.

Hence, before the last letter of the word $\overline{w}_r$ can be removed, all words except w_r and all forests except F_r must be empty. The word w_r cannot contain letters of H_r, hence it must be empty, too. (Otherwise, supposing H_r contained a letter ux, not all of the words w_u would be empty, or F_u would still contain an edge.) Now we can distinguish between two possible cases:

Suppose $\overline{w}_r = ry$, which is not the root. Then ry belongs – as we required – to the component of F_r that contains the root. But edges from this component can be removed only after the removal of ry.

Now suppose that $\overline{w}_r$ contains only the root and there is an edge in a component T_0 of F_r which does not contain the root. But T_0 should contain no edges anymore, because all letters ry except those in T_0 are in $V \setminus W'$, but none of the letters of H_u with $u \neq r$, because all the other words are already empty. This is a contradiction.

⇛ and ⇚ are inverse to each other:

We only have to check that each step of ⇛ is the inverse of the corresponding step of ⇚. This is trivial. ☐

This correspondence makes it possible to compute the number of spanning trees and forests of $G[H_1, H_2, \dots, H_p]$:

THEOREM 3.5. *For (di)graphs G and H_v, where $v \in G$, and an arbitrary vertex $r \in G[H_1, H_2, \dots, H_p]$ we have*

$$\begin{aligned}&\boldsymbol{t}_r(G[H_1, H_2, \dots, H_p])\\ &\qquad = \Big(\prod_{v \in \mathcal{V}(G)} \sum_{i=1}^{|H_v|} f_i(H_v) \big(\sum_{\substack{u \in \mathcal{V}(G)\\ (u,v) \in \mathcal{E}(G)}} |H_u| \big)^{i-1} \Big) \cdot \sum_{\overline{T} \in \mathfrak{T}_r(G)} \prod_{v \in \mathcal{V}(G)} |H_v|^{d_{\overline{T}}(v)-1}.\end{aligned}$$

For (di)graphs G and H_v, where $v \in G$, and a set of roots $R = \bigcup_{u \in U} \mathcal{V}(H_u)$, where $U \subset \mathcal{V}(G)$ we have

$$\begin{aligned}&\boldsymbol{f}_R(G[H_1, H_2, \dots, H_p])\\ &\qquad = \Big(\prod_{v \in \mathcal{V}(G) \setminus R} \sum_{i=1}^{|H_v|} f_i(H_v) \big(\sum_{\substack{u \in \mathcal{V}(G)\\ (u,v) \in \mathcal{E}(G)}} |H_u| \big)^{i-1} \Big) \cdot \sum_{\overline{F} \in \mathfrak{F}_R(G)} \prod_{v \in \mathcal{V}(G)} |H_v|^{d^o_{\overline{F}}(v)},\end{aligned}$$

where $f_i(F)$ denotes the number of forests in F with i roots.

PROOF. We show the statement about the number of spanning trees first: Given a set of rooted forests F_v of H_v, where $v \in \mathcal{V}(G)$, there are

$$\prod_{v \in \mathcal{V}(G)} \big(\sum_{\substack{u \in \mathcal{V}(G)\\ (u,v) \in \mathcal{E}(G)}} |H_u| \big)^{c(F_v)-1}$$

sets of words w_v, where $v \in \mathcal{V}(G)$, given a tree $\overline{F}$ of G, rooted in r, there are

$$\prod_{v \in \mathcal{V}(G)\setminus\{r\}} |H_v|^{d^o_{\overline{F}}(v)} \cdot |H_r|^{d^o_{\overline{F}}(r)-1}$$

sets of words $\overline{w}_v$, where $v \in \mathcal{V}(G)$. Each combination of those sets corresponds to a tree of $G[H_1, H_2, \ldots, H_p]$. Hence

$$\begin{aligned}
&\boldsymbol{t}_r(G[H_1, H_2, \ldots, H_p]) \\
&\qquad = \Big(\sum_{F_v \in \mathfrak{F}(H_v)} \prod_{v \in \mathcal{V}(G)} \big(\sum_{\substack{u \in \mathcal{V}(G) \\ (u,v) \in \mathcal{E}(G)}} |H_u| \big)^{c(F_v)-1} \Big) \cdot \sum_{\overline{F} \in \mathfrak{T}_r(G)} \prod_{v \in \mathcal{V}(G)} |H_v|^{d_{\overline{F}}-1},
\end{aligned}$$

which is what we wanted to show. The statement about the number of spanning forests follows by Theorem 4.1 in Chapter 3. See also the remark just after Theorem 3.3. □

Using this awkward looking, but powerful theorem we can deal with quite a few families of graphs. For multipartite graphs we can simplify the second factor of the formula a little bit. For doing so we need a simple lemma:

Lemma 3.6. *The number of occurrences of a letter v in the Prüfer code – even in its generalized form as in Theorem 3.3 – of some forest F equals $d_F(v)$ if v is a root, $d_F(v) - 1$ otherwise.*

Remark. We consider the elements of the set W as described in the proof of Theorem 3.5 as elements of the generalized Prüfer code.

Proof. Consider F as a directed forest with every edge directed away from the corresponding root. Each time a successor of a vertex v is removed, the letter v is added to the code. Now the result follows, as every vertex v in a forest F has $d^o_F(v)$ successors. □

Proposition 3.7. *Let G be any bipartite graph with parts G_1 and G_2 and*

$$|H_v| = \begin{cases} h_1 & \text{for } v \in G_1 \\ h_2 & \text{for } v \in G_2. \end{cases}$$

Furthermore, let r be a vertex of $G[H_1, H_2, \ldots, H_p]$. Then we have

$$\begin{aligned}
&\boldsymbol{t}_r(G[H_1, H_2, \ldots, H_p]) \\
&\qquad = \Big(\prod_{v \in G_1} \sum_{i=1}^{h_1} f_i(H_v)(h_2 d_G(v))^{i-1} \Big) \\
&\qquad\quad \Big(\prod_{v \in G_2} \sum_{i=1}^{h_2} f_i(H_v)(h_1 d_G(v))^{i-1} \Big) \cdot {h_1}^{p_2-1} {h_2}^{p_1-1} \cdot \boldsymbol{t}_r(G),
\end{aligned}$$

where $p_1 = |\mathcal{V}(G_1)|$ and $p_2 = |\mathcal{V}(G_2)|$.

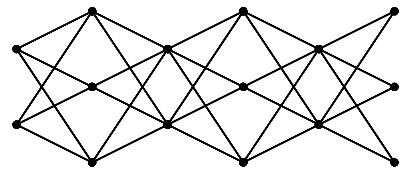


FIGURE 2. The graph $P_6\left[O_2, O_3, O_2, O_3, O_2, O_3\right]$

PROOF. Observe that for $v \in G_1$ we have

$$\sum_{\substack{u \in \mathcal{V}(G) \\ (u,v) \in \mathcal{E}(G)}} |H_u| = h_2 d_G(v),$$

and similarly for $v \in G_2$. Hence, the first factor in the formula given by Theorem 3.5 can be split up into two factors, as above.

We denote the parts of any tree $\overline{T}$ of G by $\overline{T}_1$ and $\overline{T}_2$. Then we can rewrite the second factor in the formula given by Theorem 3.5 as follows:

$$\begin{aligned}
\sum_{\overline{T} \in \mathfrak{T}_r(G)} \prod_{v \in \mathcal{V}(G)} |H_v|^{d_{\overline{T}}(v)-1} &= \sum_{\overline{T} \in \mathfrak{T}_r(G)} \prod_{v \in \mathcal{V}(G_1)} {h_1}^{d_{\overline{T}_1}(v)} \prod_{v \in \mathcal{V}(G_2)} {h_2}^{d_{\overline{T}_2}(v)} \\
&= \sum_{\overline{T} \in \mathfrak{T}_r(G)} {h_1}^{\sum_{v \in \mathcal{V}(G_1)} d_{\overline{T}_1}(v)} {h_2}^{\sum_{v \in \mathcal{V}(G_2)} d_{\overline{T}_2}(v)} \\
&= \sum_{\overline{T} \in \mathfrak{T}_r(G)} {h_1}^{p_2-1} {h_2}^{p_1-1} \\
&= {h_1}^{p_2-1} {h_2}^{p_1-1} \cdot \boldsymbol{t}_r(G),
\end{aligned}$$

which is what we wanted to show. □

EXAMPLE 3.8. As an example we will compute the number of spanning trees of the graphs $P_p\left[O_l, O_m, O_l, O_m, \ldots\right]$. An example is depicted in Figure 2. Both P_{2p} and P_{2p+1} are bipartite graphs, therefore we can use Proposition 3.7:

$$\begin{aligned}
&\boldsymbol{t}(P_{2p}\left[O_l, O_m, O_l, O_m, \ldots\right]) \\
&\qquad = \left(\sum_{i=1}^{l} f_i(O_l)(2m)^{i-1}\right)^{p-1} \left(\sum_{i=1}^{l} f_i(O_l) m^{i-1}\right) \\
&\qquad \cdot \left(\sum_{i=1}^{m} f_i(O_m)(2l)^{i-1}\right)^{p-1} \left(\sum_{i=1}^{m} f_i(O_m) l^{i-1}\right) \cdot l^{p-1} m^{p-1}.
\end{aligned}$$

The number of rooted forests of O_m with i roots $f_i(O_m)$ is nonzero only for $i = m$, in this case it is equal to 1. Therefore,

$$\boldsymbol{t}(P_{2p}\left[O_l, O_m, O_l, O_m, \ldots\right]) = 2^{(l+m-2)(p-1)} m^{lp-1} l^{mp-1}.$$

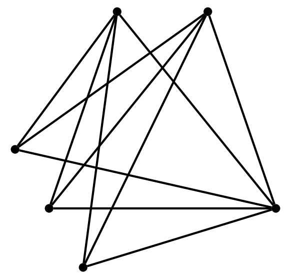


FIGURE 3. The complete multipartite graph $K_{3,2,1}$

Similarly, for $P_{2p+1}\left[O_l, O_m, O_l, O_m, \dots\right]$ we get

$$\begin{aligned}
&\boldsymbol{t}(P_{2p+1}\left[O_l, O_m, O_l, O_m, \dots\right]) \\
&\qquad = \left(\sum_{i=1}^{l} f_i(O_l)(2m)^{i-1}\right)^{p-1} \left(\sum_{i=1}^{l} f_i(O_l)m^{i-1}\right)^2 \\
&\qquad \cdot \left(\sum_{i=1}^{m} f_i(O_m)(2l)^{i-1}\right)^{p} \cdot l^{p-1}m^p \\
&\qquad = 2^{(l+m-2)(p-1)+m-1}m^{lp-1+l}l^{mp-1} \\
&\qquad = 2^{m-1}m^l \cdot \boldsymbol{t}(P_{2p}\left[O_l, O_m, O_l, O_m, \dots\right]).
\end{aligned}$$

PROPOSITION 3.9. *Let G be the complete multipartite graph $K_{n_1,n_2,\dots,n_p}$. This graph can be expressed as $K_p\left[O_{n_1}, O_{n_2}, \dots O_{n_p}\right]$, where O_n denotes the graph consisting of n isolated vertices. Let $N_u = \sum_{v\in\mathcal{V}(O_{n_u})} |H_v|$ and $N = \sum_{v\in\mathcal{V}(K_{n_1,n_2,\dots,n_p})} |H_v|$. Then we have*

$$\begin{aligned}
&\boldsymbol{t}(G\left[H_1, H_2, \dots, H_{\sum n_i}\right]) \\
&\qquad = \Big(\prod_{v\in\mathcal{V}(G)} \sum_{i=1}^{|H_v|} f_i(H_v)\big(\sum_{\substack{u\in\mathcal{V}(G)\\(u,v)\in\mathcal{E}(G)}} |H_u|\big)^{i-1}\Big) \cdot N^{p-2} \cdot \prod_{u\in\mathcal{V}(K_p)} \big(N - N_u\big)^{n_u-1}.
\end{aligned}$$

PROOF. Any spanning tree T' of G corresponds by the bijection to a tree $\overline{T} \in \mathfrak{T}(K_p)$, a set of words $\overline{w}_1,\ \overline{w}_2,\ \dots,\ \overline{w}_p$, each having $d^o_{\overline{T}}(v)$ letters in $\mathcal{V}(O_{n_v})$, and a set of words $w_1,\ w_2,\ \dots,\ w_p$ with each word w_v having $n_v - 1$ letters in $\bigcup_{u\in\mathcal{V}(G),u\neq v} \mathcal{V}(O_{n_u})$. By Prüfer's (original) encoding, finally, $\overline{T}$ corresponds to a $p-2$ letter word w. The forests F_v consist of isolated vertices only.

Hence we can rewrite the second factor of the formula:

$$
\begin{aligned}
&\sum_{T' \in \mathfrak{T}(G)} \prod_{v \in \mathcal{V}(G)} |H_v|^{d_{T'}(v)} \\
&= \sum_{\substack{\overline{T} \in \mathfrak{T}(K_p) \\ \overline{w}_1, \overline{w}_2, \ldots, \overline{w}_p}} \sum_{w_1, w_2, \ldots, w_p} \prod_{v \in \mathcal{V}(G)} |H_v|^{\substack{\#v \text{ in } \overline{w}_1, \overline{w}_2, \ldots, \overline{w}_p \\ \text{and } w_1, w_2, \ldots, w_p}} \\
&= \Big(\sum_{\substack{\overline{T} \in \mathfrak{T}(K_p) \\ \overline{w}_1, \overline{w}_2, \ldots, \overline{w}_p}} \prod_{v \in \mathcal{V}(G)} |H_v|^{\#v \text{ in } \overline{w}_1, \overline{w}_2, \ldots, \overline{w}_p} \Big) \\
&\qquad\qquad\qquad\qquad \cdot \Big(\sum_{w_1, w_2, \ldots, w_p} \prod_{v \in \mathcal{V}(G)} |H_v|^{\#v \text{ in } w_1, w_2, \ldots, w_p} \Big) \\
&= \Big(\sum_{T \in \mathfrak{T}(K_p)} \prod_{u \in \mathcal{V}(K_p)} \sum_{\overline{w}_u} \prod_{v \in \mathcal{V}(O_{n_u})} |H_v|^{\#v \text{ in } \overline{w}_u} \Big) \\
&\qquad\qquad\qquad\qquad \cdot \Big(\prod_{u \in \mathcal{V}(K_p)} \sum_{w_u} \prod_{v \in \bigcup_{k \neq u} \mathcal{V}(O_{n_k})} |H_v|^{\#v \text{ in } w_u} \Big) \\
&= \Big(\sum_{w} \prod_{u \in \mathcal{V}(K_p)} \big(\sum_{v \in \mathcal{V}(O_{n_u})} |H_v| \big)^{\#u \text{ in } w} \Big) \\
&\qquad\qquad\qquad\qquad \cdot \Big(\prod_{u \in \mathcal{V}(K_p)} \big(\sum_{v \in \bigcup_{k \neq u} \mathcal{V}(O_{n_k})} |H_v| \big)^{n_u - 1} \Big) \\
&= \Big(\sum_{u \in \mathcal{V}(K_p)} \big(\sum_{v \in \mathcal{V}(O_{n_u})} |H_v| \big)^{p-2} \Big) \\
&\qquad\qquad\qquad\qquad \cdot \Big(\prod_{u \in \mathcal{V}(K_p)} \big(\sum_{v \in \bigcup_{k \neq u} \mathcal{V}(O_{n_k})} |H_v| \big)^{n_u - 1} \Big) \\
&= \Big(\sum_{v \in \mathcal{V}(K_{n_1, n_2, \ldots, n_p})} |H_v| \Big)^{p-2} \cdot \Big(\prod_{u \in \mathcal{V}(K_p)} \big(\sum_{v \in \bigcup_{k \neq u} \mathcal{V}(O_{n_k})} |H_v| \big)^{n_u - 1} \Big) \\
&= N^{p-2} \cdot \prod_{u \in \mathcal{V}(K_p)} \big(N - N_u \big)^{n_u - 1}.
\end{aligned}
$$

□

EXAMPLE 3.10. The complete multipartite graph $K_{p_1, p_2, \ldots, p_n}$ has

$$
\boldsymbol{t}(K_n [O_{p_1}, O_{p_2}, \ldots O_{p_n}]) = p^{n-2} \prod_{i=1}^{n} (p - p_i)^{p_i - 1}
$$

spanning trees, where $p = \sum_{i=1}^{n} p_i$.

Clearly, the major difficulty we encounter when using Theorem 3.5 lies in the calculation of the numbers $f_i(H)$. Still, for some simple families of graphs it is possible to obtain nice formulas:

PROPOSITION 3.11. *The number of rooted forests with i components of the path P_n is*

$$f_i(P_n) = \binom{n+i-1}{2i-1}.$$

PROOF 1.

$$\begin{aligned} f_i(P_n) &= \sum_{\substack{k_1,k_2,\ldots,k_i \\ k_1+k_2+\cdots+k_i=n \\ k_j\geq 1}} k_1k_2\cdots k_i \\ &= \langle x^n\rangle \left(\frac{x}{(1-x)^2}\right)^i \\ &= \langle x^{n-i}\rangle \frac{1}{(1-x)^{2i}} \\ &= \binom{n+i-1}{n-i}. \end{aligned}$$

□

PROOF 2. Draw $n+i-1$ dots and choose $n-i$ of them to be edges. The $2i-1$ dots which were not selected then alternately represent a root (i items) and separation of two components ($i-1$ items). Hence a rooted forest of P_n with i components can be represented by a selection of $n-i$ in $n+i-1$ items. □

PROPOSITION 3.12. *The number of rooted forests with i components of the circle C_n is*

$$f_i(C_n) = \frac{n}{i}\binom{n+i-1}{2i-1}.$$

PROOF. Suppose the vertices of the path are numbered from 0 to $n-1$, so that $(i, i+1)$ is an edge of P_n for $i \in \{0, 1, \ldots, n-2\}$. Similarly, let the vertices of the circle C_n be numbered from 1 to n, so that $(i, i+1 \mod n)$ is an edge of C_n for $i \in \{0, 1, \ldots, n-1\}$.

Let F_P be a rooted forest of the path P_n with i roots, and select a vertex u. We can then construct a rooted forest F_C of the circle C_n with one component selected as follows: Let $(i, i+1 \mod n)$ be an edge of F_C if and only if $(u+i \mod n, u+i+1 \mod n)$ is an edge of F_P. Finally, select the component of F_C which contains vertex $n-u$.

Conversely, let F_C be a rooted forest of the circle C_n with i roots, and select one component. Let v be the smallest vertex in this component. Let $(i, i+1)$ be an edge of F_P if and only if $(v+i \mod n, v+i+1 \mod n)$ is an edge of F_C. Finally, select vertex $n-v$ in F_P. □

PROPOSITION 3.13. *The number of rooted forest with i components of the star S_n is*

$$f_i(S_n) = (n-i+2)\binom{n}{i-1}.$$

PROOF. Select $i-1$ vertices from $\{1,2,\ldots,n\}$ which shall be isolated vertices. The other $n-i+2$ vertices give the i^{th} component, in which a root has to be selected. $\square$

PROPOSITION 3.14. *The number of rooted forests with i components of the complete graph K_n is*

$$f_i(K_n) = n^{n-i}\binom{n-1}{i-1}.$$

PROOF. This is Corollary 3.2:

$$f_i(K_n) = \binom{n}{i} i n^{n-i-1}.$$

$\square$

EXAMPLE 3.15. The number of spanning trees of the fan F_n can be calculated as follows:

$$\begin{aligned}\boldsymbol{t}(K_2[K_1, P_n]) &= \sum_{i=1}^{n} f_i(P_n)\\ &= \sum_{i=1}^{n}\binom{n+i-1}{2i-1}.\end{aligned}$$

Replacing i with $n-i$ we get

$$\begin{aligned}\boldsymbol{t}(F_n) &= \sum_{i=0}^{n-1}\binom{2n-1-i}{2n-1-2i}\\ &= \sum_{i=0}^{n-1}\binom{2n-1-i}{i}\\ &= f_{2n-1},\end{aligned}$$

where f_n is the n^{th} Fibonacci number.

EXAMPLE 3.16. The wheel W_n has

$$\begin{aligned}\boldsymbol{t}(K_2[K_1, C_n]) &= \sum_{i=1}^{n} f_i(C_n)\\ &= \sum_{i=1}^{n}\frac{n}{i}\binom{n+i-1}{2i-1} \text{ spanning trees.}\end{aligned}$$

CHAPTER 5

Algebraic Proofs

This chapter covers the most powerful methods for determining the number of spanning trees. The famous Matrix-Tree-Theorem and the theory of graph spectra enables us to obtain very general theorems.

1. The Matrix-Tree-Theorem

The following theorem is probably the most important theorem when counting spanning trees. Among the first who proved it were Kirchhoff [**24**] in 1847, Sylvester [**37**] in 1855 and Borchardt [**7**] in 1860.

Matrix-Tree-Theorem. *Given a (weighted) (di)graph G, its number of spanning forests can be computed by the formula*

$$f_R(G) = \det(\mathbf{C}_{R^c}).$$

Here $\mathbf{C}$ denotes the Kirchhoff matrix or Laplacian of G and $\mathbf{C}_{R^c}$ is the principal minor of $\mathbf{C}$ that we obtain by deleting the rows and columns indexed by R.

Proof. Consider G as a weighted complete digraph K_p so that for any pair of vertices u and v there is exactly one arc (u,v) from u to v which has weight $a_{u,v}$. (See Definition 1.3)

By the definition of the determinant we have

$$\det(\mathbf{C}_{R^c}) = \sum_{\substack{\pi \text{ permutation} \\ \text{of the vertices in } \mathcal{V}(K_p)\setminus R}} \operatorname{sgn} \pi \prod_{v \in \mathcal{V}(G)\setminus R} c_{\pi(v),v}.$$

Substituting $\delta_{u,v} d^i(v) - a_{u,v}$ for $c_{u,v}$ we arrive at

$$\begin{aligned}\det(\mathbf{C}_{R^c}) &= \sum_{\pi} \operatorname{sgn} \pi \prod_{v:\pi(v)=v} \sum_{u} a_{u,v} \prod_{v:\pi(v)\neq v} (-a_{\pi(v),v}) \\ &= \sum_{(\pi,f)} \operatorname{sgn} \pi \prod_{v:\pi(v)=v} a_{f(v),v} \prod_{v:\pi(v)\neq v} (-a_{\pi(v),v}),\end{aligned}$$

where f is any function from the set of fixed points of π into $\mathcal{V}(K_p)$.

It is well known that $\operatorname{sgn} \pi = (-1)^{p-|R|+\#(\text{cycles in } \pi)}$, furthermore we have $\#(\pi(v) \neq v) = p - |R| - \#(\text{trivial cycles in } \pi)$, hence

$$\det(\mathbf{C}_{R^c}) = \sum_{(\pi,f)} (-1)^{\#(\text{nontrivial cycles in } \pi)} \prod_{v:\pi(v)=v} a_{f(v),v} \prod_{v:\pi(v)\neq v} a_{\pi(v),v}.$$

Now, (π, f) defines a spanning subdigraph H of K_p containing arcs from $\pi(v)$ to v for $\pi(v) \neq v$ and from $f(v)$ to v otherwise, for any $v \in \mathcal{V}(K_p) \setminus R$.

Because either f or π, but not both, apply to a vertex $v \in H$ we have

$$d^i_H(v) = \begin{cases} 1 & \text{for } v \in \mathcal{V}(K_p) \setminus R \\ 0 & \text{otherwise.} \end{cases}$$

Hence any circuit in H must be a cycle and all cycles in H are disjoint. Furthermore, any cycle in H must either belong completely to π or to f. Thus we can define the following involution on the set of all pairs (π, f):

If there is at least one cycle in (π, f), take the cycle containing the smallest vertex and put it to π if it belonged to f, and vice versa.

Clearly, this involution preserves weight but alternates sign, hence all terms in the sum which contain cycles cancel. Therefore only terms with $\pi = \textit{identity}$ actually count. In all the other terms, f contains no cycles, consists of $p - |R|$ arcs, and exactly the vertices in R have indegree zero, which implies that it must be a forest with roots in R. This proves the theorem. □

Remark. We remarked after Proposition 1.6 in Chapter 2, Section 1, that in [**13**] a Laplacian matrix for vertex-weighted graphs is defined. The principal minor obtained from this matrix by deleting the row and column corresponding to vertex r counts the vertex-weighted spanning trees rooted at r of the graph, where the weight of a tree T is

$$w(T) = \prod_{(u,v) \in \mathcal{E}(T)} w(u) w(u,v).$$

The following important corollary for Eulerian digraphs is easily deduced from the Matrix-Tree-Theorem. Note that we proved a special case already in Chapter 3, Section 3.

Corollary 1.1. *In Eulerian digraphs, the number of spanning trees does not depend on the root chosen.*

Proof. Consider the Laplacian matrix of an Eulerian digraph G on p vertices. We have to show that there are as many spanning trees rooted at vertex 1, as there are spanning trees rooted at vertex 2. By the Matrix-Tree-Theorem we have

$$\boldsymbol{t}_1(G) = \det \begin{pmatrix} \sum_{v=1}^p a_{2,v} & -a_{2,3} & \dots & -a_{2,p} \\ -a_{3,2} & \sum_{v=1}^p a_{3,v} & & \\ \vdots & & \ddots & \\ -a_{p,2} & & & \sum_{v=1}^p a_{p,v} \end{pmatrix}.$$

Adding all columns but the first to the first column we obtain

$$\boldsymbol{t}_1(G) = \det \begin{pmatrix} a_{2,1} & -a_{2,3} & \dots & -a_{2,p} \\ a_{3,1} & \sum_{v=1}^p a_{3,v} & & \\ \vdots & & \ddots & \\ a_{p,1} & & & \sum_{v=1}^p a_{p,v} \end{pmatrix}.$$

Finally, we add all rows but the first to the first row. Because G is Eulerian we have $\sum_{v=1}^{p} a_{u,v} = \sum_{v=1}^{p} a_{v,u}$. Therefore, we obtain

$$\boldsymbol{t}_1(G) = \det\begin{pmatrix} \sum_{v=1}^{p} a_{1,v} & a_{1,3} & \cdots & a_{1,p} \\ a_{3,1} & \sum_{v=1}^{p} a_{3,v} & & \\ \vdots & & \ddots & \\ a_{p,1} & & & \sum_{v=1}^{p} a_{p,v} \end{pmatrix}$$
$$= \det\begin{pmatrix} \sum_{v=1}^{p} a_{1,v} & -a_{1,3} & \cdots & -a_{1,p} \\ -a_{3,1} & \sum_{v=1}^{p} a_{3,v} & & \\ \vdots & & \ddots & \\ -a_{p,1} & & & \sum_{v=1}^{p} a_{p,v} \end{pmatrix} = \boldsymbol{t}_2(G).$$

□

There is an alternative approach to the Matrix-Tree-Theorem – relying on the cycle and cocycle spaces of a graph – which shall lead us towards a more general result for weighted graphs. (For more details on these matters see [**3**]).

Recall that the Kirchhoff matrix $\mathbf{C}$ of an undirected weighted graph G without loops can be expressed as $\mathbf{BXB}^t$, where $\mathbf{B}$ is the incidence matrix of G and $\mathbf{X}$ is its weight matrix. Thus, we have

$$\boldsymbol{t}(G) = \det(\mathbf{BXB}^t)_{r^c},$$

where r is an arbitrary vertex of G. In the following we will show, that this equation holds for any matrix $\mathbf{B}$, whose rows form an *integral basis* of the *lattice of integer cocycles* of an arbitrary orientation of G. We will first give the necessary definitions:

Definition 1.2. Let G be a graph and let $\vec{G}$ be an arbitrary orientation of G. Consider the linear space $\mathcal{C}^1(\vec{G};\mathbb{R})$ of real valued functions on the edges of $\vec{G}$. The standard inner product of two elements x and y of $\mathcal{C}^1(\vec{G};\mathbb{R})$ is $\langle x, y\rangle = \sum_{e\in\mathcal{E}(G)} x(e)y(e)$.

Let E be a subgraph of G and let $\vec{E}$ be an orientation of E. Then we can represent $\vec{E}$ as an element $\delta_{\vec{E}}$ of $\mathcal{C}^1(\vec{G};\mathbb{R})$ as follows:

$$\delta_{\vec{E}}(e) = \begin{cases} +1 & \text{if the orientation of } e \text{ is the same in } \vec{E} \text{ as in } \vec{G} \\ -1 & \text{if the orientation of } e \text{ is different in } \vec{E} \text{ and in } \vec{G} \\ 0 & \text{if } e \text{ is not an element of } E. \end{cases}$$

We will call two edge sets E_1 and E_2 linearly independent, if the corresponding elements $\delta_{\vec{E}_1}$ and $\delta_{\vec{E}_2}$ are linearly independent.

Recall that a cycle is a closed path, i.e., a walk with all vertices distinct, and contains at least one edge. A cocycle is a minimal cutset, i.e., a minimal set of edges of G, whose removal from G increases the number of components of G.

For a cycle C of G let $\vec{C}$ be an orientation of C, so that all vertices have indegree and outdegree equal to 1. Then the *cycle space* $\mathcal{C}$ of a digraph $\vec{G}$, is the linear subspace of $\mathcal{C}^1(\vec{G};\mathbb{R})$ generated by the functions $\delta_{\vec{C}}$, where C is a cycle of G. cycle space

Similarly, let C^* be a cocycle of G and let V_1 and V_2 be the vertex sets the edges of C^* are incident to. Let $\vec{C}^*$ be the orientation of C^*, so that all edges of C^* have cocycle space their tail in V_1 and their head in V_2. The *cocycle space* $\mathcal{C}^*$ of a digraph $\vec{G}$, is the linear subspace of $\mathcal{C}^1(\vec{G};\mathbb{R})$ generated by the functions $\delta_{\vec{C}^*}$, where C^* is a cocycle of G.

Note, that the rows of the incidence matrix of a graph are equal to $\pm\delta_{\vec{v}^*}$, where v^* is the cocycle defined by all edges incident to G.

Clearly, for any two orientations of a graph, the corresponding cycle and cocycle spaces are isomorphic. Hence, we can informally speak of the cycle space of a graph G, instead of the cycle space of an orientation of G.

The following lemma reveals the connection between the cycle and the cocycle spaces of a graph and its incidence matrix:

Lemma 1.3. *Let* $\mathbf{B}$ *be the incidence matrix of an arbitrary orientation* $\vec{G}$ *of a graph* G*. Then the cycle space* $\mathcal{C}$ *is the kernel of* $\mathbf{B}$*, and the cocycle space* $\mathcal{C}^*$ *is its orthogonal complement with respect to the standard inner product of* $\mathcal{C}^1(\vec{G};\mathbb{R})$*. The dimension of* $\mathcal{C}$ *is* $q-p+c$*, where* q *is the number of edges,* p *is the number of vertices and* c *is the number of components of* G*. The dimension of* $\mathcal{C}^*$ *equals* $p-c$*.*

Remark. The number $q-p+c$ is often called the *cyclomatic number* of G. Similarly, $p-c$ is called the *cocyclomatic number* of G. (co)cyclomatic number

Proof. First, we show that there is a linearly independent set of $q-p+c$ cycles and a linearly independent set of $p-c$ cocycles of G. In fact, given any spanning forest F of G, the so called 'fundamental' cycles (respectively cocycles) associated with F are linearly independent:

Let e be an edge of the forest F, and let T be the component of F which contains e. Then the removal of the edge e from F separates the set of vertices of T into two parts, one containing the head of e, the other its tail. The edges incident to both sets form a cocycle C^* of G, so that for any edge $f \in \mathcal{E}(F)$, $\delta_{\vec{C}^*}(f)$ is equal to ± 1 if $e=f$ and zero otherwise. Thus we obtain a linearly independent set of $p-c$ cocycles, one for each edge e that is not in F. These cocycles are called the fundamental cocycles associated with F.

Similarly, for any edge $e \in \mathcal{E}(G)\setminus\mathcal{E}(F)$, the subgraph of G induced by the edges of F and the edge e contains exactly one cycle C, a so called fundamental cycle of G. For any other edge $f \in \mathcal{E}(G)\setminus\mathcal{E}(F)$, $\delta_{\vec{C}}(f)$ equals ± 1 if $e=f$ and zero otherwise. Again, the set of $q-p+c$ cycles obtained is linearly independent.

Next, we prove that $\mathcal{C} \subset \operatorname{Ker}\mathbf{B}$ and $\mathcal{C}^* \subset (\operatorname{Ker}\mathbf{B})^*$:

Let C be a cycle of G. Then, for any vertex $v \in \mathcal{V}(G)$ we have

$$\langle \delta_{\vec{C}}, \delta_{\vec{v}^*} \rangle = 0,$$

where $\delta_{\vec{v}^*}$ is the cocycle defined by the edges incident to v: We only have to consider the case where v is a vertex traversed by the cycle. Otherwise, the inner product is trivially zero. It is easy to check that in all four remaining cases the inner product evaluates to zero, too.

Hence, as $\delta_{\vec{v}^*}$ equals the row of $\mathbf{B}$ corresponding to vertex v, the product $\mathbf{B}\delta_{\vec{C}}$ is the zero vector, that is, δ_C is in the kernel of $\mathbf{B}$.

Similarly, let C^* be a cocycle of G and let V be one of the sets of vertices the edges of C^* are incident to. Then $\delta_{\vec{C}^*} = \sum_{v \in V} \delta_{\vec{v}^*}$. Furthermore, for $\mathbf{z} \in$

$\operatorname{Ker}\mathbf{B}$ and an arbitrary vertex v of G, the inner product of $\mathbf{z}$ and $\delta_{\vec{v}^*}$ vanishes. Consequently, $\langle \mathbf{z}, \delta_{\vec{C}^*}\rangle$ is equal to zero as well, which implies that $\delta_{\vec{C}^*}$ is in the orthogonal complement of $\operatorname{Ker}\mathbf{B}$.

It remains to show, that $\dim\operatorname{Ker}\mathbf{B} = q-p+c$ and $\dim(\operatorname{Ker}\mathbf{B})^* = p-c$: Let $\mathbf{x}$ be any vector with p components, and for any edge e, let $h(e)$ be its head and $t(e)$ its tail in the (arbitrary) orientation of G. Then we have

$$\mathbf{B}^t\mathbf{x}(e) = \mathbf{x}(h(e)) - x(t(e)).$$

Hence, $\mathbf{x}$ is in the kernel of $\mathbf{B}^t$, if and only if $\mathbf{x}$ is constant on each component of G, which implies that $\dim\operatorname{Ker}\mathbf{B}^t = c$.

$\mathbf{B}^t$ is a function defined on the vertices of G, therefore we have

$$\dim\operatorname{Im}\mathbf{B}^t = p - \dim\operatorname{Ker}\mathbf{B}^t = p-c.$$

By the 'row rank=column rank' theorem, $\dim\operatorname{Im}\mathbf{B} = \dim\operatorname{Im}\mathbf{B}^t = p-c$.

Now consider the orthogonal decomposition

$$\mathcal{C}^1(\vec{G};\mathbb{R}) = \operatorname{Ker}\mathbf{B} \oplus (\operatorname{Ker}\mathbf{B})^*.$$

By standard results of linear algebra we obtain

$$\dim\operatorname{Ker}\mathbf{B} = \dim\mathcal{C}^1(\vec{G};\mathbb{R}) - \dim\operatorname{Im}\mathbf{B} = q-p+c,$$

and

$$\dim(\operatorname{Ker}\mathbf{B})^* = p-c,$$

which concludes our proof. □

Up to this point we have considered linear spaces defined on a graph. We will now turn our attention to the corresponding lattices:

DEFINITION 1.4. Consider the Abelian group $\mathcal{C}^1(\vec{G};\mathbb{Z})$ of integer valued functions defined on the edges of $\vec{G}$. The *lattice of integer cycles* is the Abelian group $\mathcal{C}_I = \mathcal{C} \cap \mathcal{C}^1(\vec{G};\mathbb{Z})$. Similarly, the *lattice of integer cocycles* is the Abelian group $\mathcal{C}_I^* = \mathcal{C}^* \cap \mathcal{C}^1(\vec{G};\mathbb{Z})$. lattice of integer (co)cycles

An *integral basis* $\mathcal{B}$ of a lattice $\mathcal{L}$ is a basis of the lattice so that each element $L \in \mathcal{L}$ can be written as an integral linear combination of the elements of the basis $\mathcal{B}$: integral basis

$$L = \sum_i \lambda_i b_i,$$

where $L \in \mathcal{L}$, $b_i \in \mathcal{B}$ and $\lambda_i \in \mathbb{Z}$.

The following fundamental lemma is the foundation of the generalized form of the Matrix-Tree-Theoremfor graphs:

LEMMA 1.5. *Let G be a graph and let $\vec{G}$ be an arbitrary orientation of G. Let $\mathbf{M}$ (resp. $\mathbf{M}^*$) be a matrix, so that its rows form an integral basis of the lattice of integer cycles (cocycles) of $\vec{G}$. Let $E \subseteq \mathcal{E}(G)$ be a set of $q-p+1$ ($p-1$) edges of G and let $\mathbf{M}_E$ be the restriction of $\mathbf{M}$ to the columns corresponding to E. Then* $\det\mathbf{M}_E = 0$ ($\det\mathbf{M}_E^* = 0$) *if E contains a cocycle (cycle) of G and* $\det\mathbf{M}_E = \pm 1$ ($\det\mathbf{M}_E^* = \pm 1$) *if E is a cotree (spanning tree) of G.*

PROOF. Let $\mathbf{M}$ be an integral basis of the lattice of integer cycles of G. We first show that $\mathbf{M}_E$ is singular, if E contains a cocycle C^* of G: Consider the element $\delta_{\vec{C}^*}$ of the cocycle space as in Definition 1.2 and its restriction $\delta_{\vec{C}^*}\big|_E$ to the edges in E. Then

$$\mathbf{M}_E\ \delta_{\vec{C}^*}\big|_E = \mathbf{M}\ \delta_{\vec{C}^*} = \mathbf{0}.$$

The first equation holds, because $\delta_{\vec{C}^*}$ is nonzero on the edges of E only, and the second equation holds, because the inner product of a cycle – i.e. a row of $\mathbf{M}$ – with a cocycle is zero by Lemma 1.3.

The vector $\delta_{\vec{C}^*}\big|_E$ is not the zero vector, therefore the matrix $\mathbf{M}_E$ must be singular.

On the other hand, if E does not contain a cocycle, the edges of E form a cotree of G. Consider the fundamental cycles associated with the spanning tree E^c, as explained in the proof of Lemma 1.3. Let $\delta_{\vec{C}_e}$ be the function corresponding to the fundamental cycle determined by the edge $e \in E$. These functions form a basis Δ of the cycle space of G. As $\mathbf{M}$ also is a basis of the cycle space, there is a matrix $\mathbf{T}$, so that $\mathbf{M} = \mathbf{T}\Delta$ and therefore

$$\mathbf{M}_E = (\mathbf{T}\ \Delta)_E = \mathbf{T}\ \Delta_E.$$

For any two edges e and f in E, we have

$$\delta_{\vec{C}_e}(f) = \begin{cases} 1 \text{ if } e = f \\ 0 \text{ otherwise.} \end{cases}$$

Thus, the determinant of Δ_E equals ± 1. Because $\mathbf{T}$ must be non-singular, too, so is $\mathbf{M}_E$.

In fact, Δ even is an integral basis of the lattice of integer cycles: Let C be any element of $\mathcal{C}_I$. As Δ is a basis of the cycle space, we have

$$C = \sum_{e \in E} \lambda_e \delta_{\vec{C}_e},$$

where $\lambda_e \in \mathbb{R}$ for $e \in E$. Evaluating at an edge $f \in E$ we obtain $C(f) = \lambda_f$, hence the λ_e, $e \in E$ are integers.

Finally note that every matrix $\mathbf{T}$ transforming an integral basis of the lattice of integer cycles into another must have determinant equal to ± 1. This follows, because $\mathbf{T}$ has only integral entries – and so has its inverse. This in turn implies that $\det \mathbf{M}_E = \pm 1$, which is what we wanted to show.

An analogous argument shows the statement concerning an integral basis of the cocycle space: We only need to replace the word cycle with the word cocycle, spanning tree with cotree and vice versa. □

The proof of the generalized Matrix-Tree-Theorem for graphs requires the following well known lemma from the theory of determinants:

CAUCHY-BINET-THEOREM. *For $(n \times m)$ matrices $\mathbf{A}$ and $\mathbf{B}$ we have*

$$\det(\mathbf{A}\ \mathbf{B}^t) = \sum_K \det \mathbf{A}_K \det \mathbf{B}_K,$$

where $n < m$ and K ranges over all n-element subsets of $\{1, 2, \ldots, m\}$.

Proof. Let $\mathbf{A} = (a_{i,j})_{\substack{i=1\dots n\\ j=1\dots m}}$ and $\mathbf{B} = (b_{i,j})_{\substack{i=1\dots n\\ j=1\dots m}}$. Using the definition of the determinant we get

$$\begin{aligned}\det(\mathbf{A}\,\mathbf{B}^t) &= \sum_{\pi \text{ permutation of } [n]} \operatorname{sgn}\pi \cdot \prod_{i=1}^{n}\sum_{j=1}^{m} a_{i,j}b_{\pi(i),j}\\ &= \sum_{\substack{\pi\\ f:[n]\mapsto[m]}} \operatorname{sgn}\pi \cdot \prod_{i=1}^{n} a_{i,f(i)}b_{\pi(i),f(i)}.\end{aligned}$$

We define $w(\pi, f) = \operatorname{sgn}\pi \cdot \prod_{i=1}^{n} a_{i,f(i)}b_{\pi(i),f(i)}$ and get

$$= \sum_{\substack{\pi\\ f:[n]\mapsto[m]\\ f \text{ injective}}} w(\pi,f) + \sum_{\substack{\pi\\ f:[n]\mapsto[m]\\ f \text{ not injective}}} w(\pi,f).$$

Next we show that the first sum equals $\sum_K \det \mathbf{A}_K \det \mathbf{B}_K$:

$$\begin{aligned}&\sum_{\substack{\pi\\ f:[n]\mapsto[m]\\ f \text{ injective}}} \operatorname{sgn}\pi \cdot \prod_{i=1}^{n} a_{i,f(i)}b_{\pi(i),f(i)}\\ &= \sum_{\substack{K\subseteq[m]\\ |K|=n}} \sum_{\substack{\pi\\ f:[n]\mapsto K\\ f \text{ injective}}} \operatorname{sgn}\pi \cdot \prod_{i} a_{i,f(i)}b_{\pi(i),f(i)}\\ &= \sum_{K} \sum_{\substack{f:[n]\mapsto K\\ f \text{ injective}}} \operatorname{sgn} f \cdot \prod_{i=1}^{n} a_{i,f(i)} \cdot \Big(\sum_{\pi} \operatorname{sgn}\pi \operatorname{sgn} f \cdot \prod_{i=1}^{n} b_{\pi(i),f(i)}\Big)\\ &= \sum_{K} \det \mathbf{A}_K \det \mathbf{B}_K.\end{aligned}$$

Finally we have to show that the second sum vanishes. To achieve this we define an involution on the pairs (π, f) where π is a permutation of $[n]$ and $f : [n] \mapsto [m]$ is not injective: Let α be the smallest number so that $f(k) = f(l) = \alpha$ for distinct k and l. Let (k, l) be minimal in $\{(k, l), f(k) = f(l) = \alpha\}$ with respect to the lexicographic ordering. Define $\varphi(\pi, f) = (\pi \circ (k, l), f)$. Clearly $\varphi^2 = \varepsilon$ and

$$\begin{aligned}w(\varphi(\pi, f)) &= \operatorname{sgn}(\pi \circ (k,l)) \cdot \prod_{i=1}^{n} a_{i,f(i)}b_{(\pi\circ(k,l))(i),f(i)}\\ &= -\operatorname{sgn}\pi \prod_{i=1}^{n} a_{i,f(i)}b_{\pi(i),f(i)}\\ &= -w(\pi, f).\end{aligned}$$

This proves the theorem. □

We are now ready to prove the generalized Matrix-Tree-Theorem for graphs:

THEOREM 1.6 (Maurer 1976). *Let G be a connected graph and let $\mathbf{M}$ (resp. $\mathbf{M}^*$) be a matrix, so that its rows form an integral basis of the lattice of integer cycles (cocycles) of G. Then the number $\boldsymbol{t}(G)$ of spanning trees of G is*

$$\boldsymbol{t}(G) = \det\left(\mathbf{M}^*\ \mathbf{X}\ (\mathbf{M}^*)^t\right) = \pm\det\begin{pmatrix}\mathbf{M}\\ \mathbf{M}^*\ \mathbf{X}.\end{pmatrix}$$

Similarly, the number $\boldsymbol{c}(G)$ of cotrees of G is

$$\boldsymbol{c}(G) = \det\left(\mathbf{M}\ \mathbf{X}\ \mathbf{M}^t\right) = \pm\det\begin{pmatrix}\mathbf{M}^*\\ \mathbf{M}\ \mathbf{X}.\end{pmatrix}$$

REMARK. Note that for unweighted graphs, $\boldsymbol{t}(G) = \boldsymbol{c}(G)$.

PROOF. The first equation follows from the Cauchy-Binet-Theorem and from Lemma 1.5:

$$\begin{aligned}\det\left(\mathbf{M}^*\ \mathbf{X}\ (\mathbf{M}^*)^t\right) &= \sum_K \det(\mathbf{M}^*_K\mathbf{X}_K)\det(\mathbf{M}^*_K)^t\\ &= \sum_K \det\mathbf{X}_K(\det\mathbf{M}^*_K)^2.\end{aligned}$$

By Lemma 1.5 we know that $\det\mathbf{M}^*_K = \pm 1$ whenever $K \subseteq \mathcal{E}(G)$ corresponds to a spanning tree and zero otherwise. Hence we have

$$\det(\mathbf{M}^*\ \mathbf{X}\ (\mathbf{M}^*)^t) = \sum_{K \text{ spanning tree}} \det\mathbf{X}_K.$$

To prove the second equation, consider the square matrix $\mathbf{P} = \left(\begin{smallmatrix}\mathbf{M}\\ \mathbf{M}^*\end{smallmatrix}\right)$. Calculating the determinant of $\mathbf{P}\mathbf{P}^t$ we get

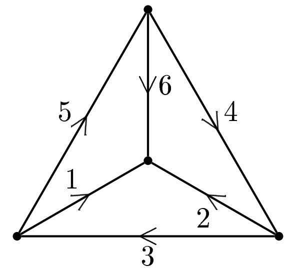
$$\begin{aligned}\det(\mathbf{P}\ \mathbf{P}^t) &= \det\begin{pmatrix}\mathbf{M}\ \mathbf{M}^t & \mathbf{M}\ \mathbf{M}^{*t}\\ \mathbf{M}^*\ \mathbf{M}^t & \mathbf{M}^*\ \mathbf{M}^{*t}\end{pmatrix}\\ &= \det\begin{pmatrix}\mathbf{M}\ \mathbf{M}^t & \mathbf{0}\\ \mathbf{0} & \mathbf{M}^*\ \mathbf{M}^{*t}\end{pmatrix} = \left(\det(\mathbf{M}\ \mathbf{M}^t)\right)^2.\end{aligned}$$

Hence $\det\mathbf{P} = \pm\det(\mathbf{M}\ \mathbf{M}^t)$. Now consider the product

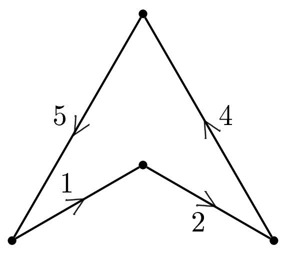
$$\begin{aligned}\det\left(\begin{pmatrix}\mathbf{M}\\ \mathbf{M}^*\ \mathbf{X}\end{pmatrix}\mathbf{P}\right) &= \det\begin{pmatrix}\mathbf{M}\ \mathbf{M}^t & \mathbf{M}\ \mathbf{M}^{*t}\\ \mathbf{M}^*\ \mathbf{X}\ \mathbf{M}^t & \mathbf{M}^*\ \mathbf{X}\ \mathbf{M}^{*t}\end{pmatrix}\\ &= \det\begin{pmatrix}\mathbf{M}\ \mathbf{M}^t & \mathbf{0}\\ \mathbf{M}^*\ \mathbf{X}\ \mathbf{M}^t & \mathbf{M}^*\ \mathbf{X}\ \mathbf{M}^{*t}\end{pmatrix} = \pm\det\mathbf{P}\cdot\boldsymbol{t}(G).\end{aligned}$$

Provided that $\det\mathbf{P} \neq 0$, we get the desired result by dividing both sides by $\det\mathbf{P}$.
The expressions for the number of cotrees of G follow similarly. □

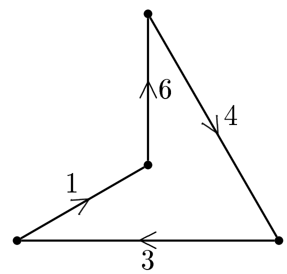
The question remains, how to get an integral basis of the lattice of integer cycles (cocycles). In the proof of Lemma 1.5 we have already seen that – informally spoken – the fundamental cycles associated with some spanning forest form such a basis. One might expect (see [**36**]) that any set of $q - p + 1$ linearly independent cycles is an integral basis. However, as the following example shows, this is not true:



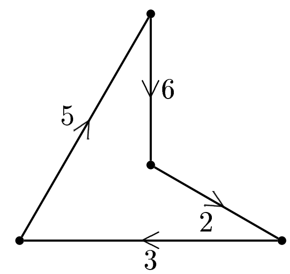
An orientation of the complete graph K_4



A cycle x



A cycle y



A cycle z

FIGURE 1

EXAMPLE 1.7. Consider the orientation of the complete graph on four vertices K_4 depicted in Figure 1. Then the cycles x, y and z indicated in the figure are certainly linearly independent, but they do not form an integral basis of the lattice of integer cycles! Consider the matrix which has rows δ_x, δ_y and δ_z:

$$\begin{array}{cc} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{pmatrix} 1 & -1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

The determinant of any submatrix corresponding to a cotree is equal to ± 2.

However, we can show the following useful fact:

LEMMA 1.8. *Let C_1, C_2, ..., $C_{\mathbf{c}}$ be a set of subgraphs of a graph G, where $\mathbf{c}$ is the (co)cyclomatic number of G. Suppose that the corresponding functions $\delta_{\vec{C}_1}, \delta_{\vec{C}_2}, \dots \delta_{\vec{C}_{\mathbf{c}}}$ form a basis of the (co)cycle space. If for any two subgraphs C_i and C_j, the orientation of edges common to both is always the same, or always different, then the corresponding functions form an integral basis of the lattice of integer (co)cycles of G.*

PROOF. Let C be an element of the lattice of integer (co)cycles of G. Then we can express C as a linear combination of the functions δ_e, $e \in \mathcal{E}(G)$, with $\delta_e(f) = 1$, if $e = f$ and $\delta_e(f) = 0$ otherwise:

$$(*) \qquad C = \sum_{e \in \mathcal{E}(G)} \kappa_e \delta_e,$$

where $\kappa_e \in \mathbb{Z}$ for $e \in \mathcal{E}(G)$.

We show by induction on the number of edges of G, that C is a $\mathbb{Z}$-linear combination of the functions δ_{C_i}, $i \in \{1, 2, \ldots, \mathbf{c}\}$.

If G has no edges, the conclusion is trivial. Suppose that there is an edge $e \in \mathcal{E}(G)$ that occurs only in one subgraph C_0. Then $\delta_{C_0}(e) = \pm 1$ and $\delta_{C_i}(e) = 0$ for $i \neq 0$. Consider the graph $G|_{e^c}$, which is obtained by deleting this edge from G. Because e occurred only in the subgraph C_0, the set of functions δ_{C_i}, $i \neq 0$ is a basis of its (co)cycle space.

Now express C as an $\mathbb{R}$-linear combination of the functions δ_{C_i}:

$$C = \sum_{i=1}^{q-p+1} \lambda_i \delta_{C_i},$$

where $\lambda_i \in \mathbb{R}$ for $i \in \{1, 2, \ldots, \mathbf{c}\}$. There are now two possible cases to distinguish: if $\lambda_0 = 0$, we can consider C as an element of the lattice of integer (co)cycles of the reduced graph $G|_{e^c}$. By the induction hypothesis then, C can be expressed as a $\mathbb{Z}$-linear combination of the δ_{C_i}.

Otherwise, if $\lambda_0 \neq 0$, then $C' = C - \lambda_0 \delta_{C_0}$ is an element of the lattice of integer (co)cycles of $G|_{e^c}$. By Equation $(*)$ above we obtain

$$\lambda_0 = \frac{1}{\delta_{C_0}(e)}(C(e) - C'(e)) = \frac{\kappa_e}{\delta_{C_0}(e)}.$$

Since $\kappa_e \in \mathbb{Z}$ and $\delta_{C_0}(e) = \pm 1$, λ_0 is an integer, too. By the induction hypothesis, C' can be expressed as a $\mathbb{Z}$-linear combination of the δ_{C_i}. Therefore, $C = C' + \lambda_0 \delta_{C_0}$ is also a $\mathbb{Z}$-linear combination of the δ_{C_i}.

In general, there might not be an edge that occurs only in one of the subgraphs C_i. In this case, let C_0 be any of the subgraphs in which e occurs and let $\delta_{C_i'} = \delta_{C_i} - \lambda_0 \lambda_i \delta_{C_0}$ for $i \in \{1, 2, \ldots, \mathbf{c}\}$, $i \neq 0$. Because we required that edges common to C_0 and C_i are always traversed in the same direction or always in the opposite direction, $\delta_{C_i'}$ is again an element of the lattice of integer (co)cycles with entries in $\{0, \pm 1\}$. Since the edge e occurs only in C_0, but not in C_i' for $i \in \{1, 2, \ldots, \mathbf{c}\}$, we can apply the first part of this proof to show that the set of functions $\delta_{C_i'}$ for $i \in \{1, 2, \ldots, \mathbf{c}\}$, $i \neq 0$ together with δ_{C_0} is an integral basis. Because of $\delta_{C_i'} = \delta_{C_i} \pm \delta_{C_0}$, the functions δ_{C_i} with $i \in \{1, 2, \ldots, \mathbf{c}\}$ form an integral basis as well. $\square$

This lemma shows, that not only the fundamental cocycles associated with a spanning tree, but also the cocycles corresponding to the vertices of G form an integral basis of the lattice of integer cocycles. Similarly, for any planar graph G, the cycles corresponding to the faces of G form an integral basis of the lattice of integer cycles of G. This reflects the fact, that a planar graph and its dual have the same number of spanning trees, as the incidence matrix of G^* coincides with this basis.

2. Spectra of Graphs

It is an astonishing fact, that the spectra of the various matrices associated with a graph contain a lot of structural information of the graph. In particular, also the spanning tree number is determined by the spectra of those matrices. This enables us to apply methods known from linear algebra and the theory of spectra to the problem of counting the spanning trees of a graph.

DEFINITION 2.1. The *(ordinary) spectrum* of a (di)graph G consists of the zeros of its characteristic polynomial $P_G(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$, where $\mathbf{A}$ is the adjacency matrix of G. The *C-spectrum* consists of the zeros of $C_G(\lambda) = \det(\lambda\mathbf{I} - \mathbf{C})$, where $\mathbf{C}$ is the Kirchhoff Matrix of G, and the *Q-spectrum* of a graph without isolated vertices consists of the zeros of $Q_G(\lambda) = \det(\lambda\mathbf{I} - \mathbf{D}^{-1}\mathbf{A}) = \det(\lambda\mathbf{I} - \mathbf{A}\mathbf{D}^{-1}) = \det(\lambda\mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2})$, where $\mathbf{D}$ is the degree matrix of G. ordinary, C and Q-spectrum

REMARK. $\mathbf{Q}(\mathbf{A}) = \mathbf{D}^{-1}\mathbf{A}$ is called *stochasticization* of $\mathbf{A}$, as it is the matrix obtained from $\mathbf{A}$ by dividing each entry by the column-sum of the column it is in.

REMARK. Clearly, for r-regular digraphs, these spectra are all equivalent:

$$P_G(\lambda) = (-1)^p C_G(r - \lambda) = r^p Q_G(\lambda/r).$$

Recall, that a graph is semiregular, if it is bipartite and the vertices of each part have the same degree. Surprisingly, for semiregular graphs the ordinary and the Q-spectrum are equivalent as well: Denote the parts of G by X and Y, so that $d_G(x) = r_1$ for $x \in \mathcal{V}(X)$ and $d_G(y) = r_2$ for $y \in \mathcal{V}(Y)$. If we order the vertices of G so that all vertices in X precede those in Y, then the adjacency and the degree matrix of G have the form

$$\mathbf{A}_G = \begin{pmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^t & \mathbf{0} \end{pmatrix} \text{ and } \mathbf{D}_G = \begin{pmatrix} r_1\mathbf{I} & \mathbf{0} \\ \mathbf{0} & r_2\mathbf{I} \end{pmatrix} \text{ respectively.}$$

Hence we have

$$\begin{aligned} Q_G(\lambda) &= \det(\lambda\mathbf{I} - \mathbf{D}_G^{-1/2}\mathbf{A}_G\mathbf{D}_G^{-1/2}) \\ &= \det\left(\lambda\mathbf{I} - \begin{pmatrix} \mathbf{0} & 1/\sqrt{r_1r_2}\,\mathbf{A} \\ 1/\sqrt{r_1r_2}\,\mathbf{A}^t & \mathbf{0} \end{pmatrix}\right) \\ &= \det(\lambda\mathbf{I} - 1/\sqrt{r_1r_2}\,\mathbf{A}_G) \\ &= \frac{1}{\sqrt{(r_1r_2)^p}} P_G(\sqrt{r_1r_2}\,\lambda). \end{aligned}$$

LEMMA 2.2. *For any $(n \times n)$ matrix $\mathbf{M}$ the characteristic polynomial can be expressed in terms of determinants of principal minors of the matrix:*

$$\det(\lambda\mathbf{I} - \mathbf{M}) = \sum_{l=0}^{n} (-1)^{n-l}\lambda^l \sum_{\substack{K \subseteq [n] \\ |K| = l}} \det \mathbf{M}_{K^c}.$$

PROOF. By the definition of the determinant we have

$$\begin{aligned}\det(\lambda\mathbf{I}-\mathbf{M}) &= \sum_{\substack{\pi \text{ permutation} \\ \text{of } [n]}} \operatorname{sgn} \pi \prod_{i\in[n]} (\lambda\mathbf{I}-\mathbf{M})_{i,\pi(i)} \\ &= \sum_{\substack{\pi \text{ permutation} \\ \text{of } [n]}} \operatorname{sgn} \pi \sum_{l=0}^{n} \sum_{\substack{K\subseteq[n] \\ |K|=l}} \prod_{i\in K} (\lambda\mathbf{I})_{i,\pi(i)} \prod_{i\notin K} (-\mathbf{M})_{i,\pi(i)}.\end{aligned}$$

Note that $(\lambda\mathbf{I})_{i,\pi(i)}$ is nonzero only if $\pi|_K = id_K$; in this case $\prod_{i\in K}(\lambda\mathbf{I})_{i,\pi(i)}$ evaluates to λ^l. Hence we get:

$$\det(\lambda\mathbf{I}-\mathbf{M}) = \sum_{l=0}^{n} \sum_{\substack{K\subseteq[n] \\ |K|=l}} \lambda^l \det(-\mathbf{M}_{K^c}),$$

which is what we wanted to show. □

THEOREM 2.3. *The number of spanning trees of a graph or Eulerian digraph can be expressed in terms of its different spectra. When* $\mathbf{A}$ *is the adjacency matrix of* G, $\mathbf{C}$ *the Laplacian matrix of* G *and* $\mathbf{Q}$ *the stochasticization of* $\mathbf{A}$, *we have*

$$\boldsymbol{t}(G) = \frac{1}{p}\, P'_G(\lambda)\big|_{\lambda=r} = \frac{1}{p} \prod_{\substack{\lambda\neq r \\ \textit{eigenvalue of } \mathbf{A}}} (r-\lambda) \tag{1}$$

for r*-regular graphs and Eulerian digraphs, and*

$$\begin{aligned}\boldsymbol{t}(G) &= \frac{\sqrt{r_1 r_2}}{2q} \left(\frac{r_1}{r_2}\right)^{\frac{p_1-p_2}{2}} P'_G(\lambda)\big|_{\lambda=\sqrt{r_1 r_2}} \\ &= \frac{1}{2p_1} \left(\frac{r_1}{r_2}\right)^{\frac{p_1-p_2-1}{2}} \prod_{\substack{\lambda\neq\sqrt{r_1 r_2} \\ \textit{eigenvalue of } \mathbf{A}}} (\sqrt{r_1 r_2}-\lambda)\end{aligned} \tag{2}$$

for semiregular graphs and semiregular Eulerian digraphs with degrees r_1 *and* r_2 *and parts of size* p_1 *and* p_2, *where* q *denotes the sum of the edge-weights of* G. *Furthermore, we have*

$$\boldsymbol{t}(G) = \frac{(-1)^{p-1}}{p}\, C'_G(\lambda)\big|_{\lambda=0} = \frac{1}{p} \prod_{\substack{\lambda\neq 0 \\ \textit{eigenvalue of } \mathbf{C}}} \lambda \tag{3}$$

and

$$\boldsymbol{t}(G) = \frac{\prod_{i=1}^{p} d_i}{2q}\, Q'_G(\lambda)\big|_{\lambda=1} = \frac{\prod_{i=1}^{p} d_i}{2q} \prod_{\substack{\lambda\neq 1 \\ \textit{eigenvalue of } \mathbf{Q}}} (1-\lambda) \tag{4}$$

for graphs and Eulerian digraphs. Again, q denotes the sum of the edge-weights of G.

PROOF. By Lemma 2.2 we have for any $(p \times p)$ matrix $\mathbf{M}$

$$(\det(\lambda \mathbf{I} - \mathbf{M}))'\big|_{\lambda=0} = (-1)^{p-1} \sum_{k=1}^{p} \det \mathbf{M}_{k^c}.$$

Using the BEST-Theorem (see Corollary 1.1) and the Matrix-Tree-Theorem, Equations (1) and (3) follow. For showing Equation (4), observe that

$$\begin{aligned} Q'_G(\lambda)\big|_{\lambda=1} &= \left(\det(\lambda \mathbf{I} - \mathbf{D}^{-1}\mathbf{A})\right)'\Big|_{\lambda=1} \\ &= \left(\det\left((\lambda+1)\mathbf{I} - \mathbf{D}^{-1}\mathbf{A}\right)\right)'\Big|_{\lambda=0} \\ &= \left(\det(\lambda \mathbf{I} - \mathbf{D}^{-1}\mathbf{C})\right)'\Big|_{\lambda=0}. \end{aligned}$$

Equation (2) follows from the equivalence of the ordinary spectrum and the Q-spectrum described in the remark at the beginning of this section. □

REMARK. In some cases the following observation may help, too: Every graph can be 'regularised' by adding $r - d(v)$ loops to vertices v with degree lower than $r = \max_{v \in \mathcal{V}(G)} d_G(v)$. Obviously, the resulting graph G' has the same number of spanning trees as G. Unfortunately though, most operations on graphs produce a different result when G is altered this way. Anyway, we have

$$\begin{aligned} C_G(\lambda) &= (-1)^p P_{G'}(r - \lambda) \\ &= (-r)^p Q_{G'}(1 - \lambda/r). \end{aligned}$$

Note that the C-spectrum remains invariant when loops are added to the graph.

Sometimes graphs occur that are 'nearly k-regular', that is, all vertices except of one – r – have the same degree k. In this case we have

$$\begin{aligned} \boldsymbol{t}(G) &= \det(\mathbf{D}_G - \mathbf{A}_G)_{r^c} = \det(k\mathbf{I} - \mathbf{A}_{r^c}) \\ &= P_{G|_{r^c}}(k). \end{aligned}$$

Note that in general $G|_{r^c}$ is not regular!

The following well known lemmas will be very useful:

LEMMA 2.4. *If $\mathbf{A}$ is a nonsingular, square matrix, we have*

$$\det \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} = \det(\mathbf{A}) \det(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}).$$

LEMMA 2.5. *For $(n \times m)$ matrices $\mathbf{A}$ and $\mathbf{B}$ we have*

$$\det(\lambda \mathbf{I} - \mathbf{A}\mathbf{B}^t) = \lambda^{n-m} \det(\lambda \mathbf{I} - \mathbf{B}^t \mathbf{A}).$$

Proof. Consider the matrices
$$\begin{pmatrix} \mathbf{I} & \mathbf{B}^t \\ \mathbf{A} & \lambda\mathbf{I} \end{pmatrix} \text{ and } \begin{pmatrix} \lambda\mathbf{I} & \mathbf{A} \\ \mathbf{B}^t & \mathbf{I} \end{pmatrix}.$$
Clearly, they have the same determinant. By Lemma 2.4 the first evaluates to $\det(\lambda\mathbf{I} - \mathbf{A}\mathbf{B}^t)$, the second to $\lambda^{n-m}\det(\lambda\mathbf{I} - \mathbf{B}^t\mathbf{A})$. □

Lemma 2.6. *For a bipartite graph G we have*
$$P_G(\lambda) = (-1)^p P_G(-\lambda) \text{ and}$$
$$Q_G(\lambda) = (-1)^p Q_G(-\lambda).$$

Proof. As G is bipartite, its adjacency matrix has the form $\mathbf{A}_G = \left(\begin{smallmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^t & \mathbf{0} \end{smallmatrix}\right)$. Suppose the parts of G have p_1 and p_2 vertices respectively. Then
$$\begin{aligned}\det(\lambda\mathbf{I} - \mathbf{A}_G) &= \det\begin{pmatrix} \lambda\mathbf{I}_{p_1} & -\mathbf{A} \\ -\mathbf{A}^t & \lambda\mathbf{I}_{p_2} \end{pmatrix} \\ &= (-1)^{p_1}\det\begin{pmatrix} -\lambda\mathbf{I}_{p_1} & \mathbf{A} \\ -\mathbf{A}^t & \lambda\mathbf{I}_{p_2} \end{pmatrix} \\ &= (-1)^{p_1+p_2}\det\begin{pmatrix} -\lambda\mathbf{I}_{p_1} & -\mathbf{A} \\ -\mathbf{A}^t & -\lambda\mathbf{I}_{p_2} \end{pmatrix} \\ &= (-1)^p\det(-\lambda\mathbf{I} - \mathbf{A}_G).\end{aligned}$$
The statement about the Q-spectrum follows just as easy, it is only messier to notate. □

Remark. In fact, the converse is true as well. See [**15**].

Lemma 2.7. *If $\mathbf{A}$ is an $(n \times n)$ circulant matrix, i.e., $a_0, \ a_1, \ \dots, \ a_{n-1}$ are arbitrary numbers and $\mathbf{A} = (a_{j-i \mod n})_{i,j\in\{0,1,\dots,n-1\}}$, then*
$$\det(\lambda\mathbf{I} - \mathbf{A}) = \prod_{\omega:\omega^n=1}(\lambda - \sum_{i=0}^{n-1} a_i\omega^i),$$
where ω runs through all n^{th} roots of unity.

Proof. This will be shown in Section 3, using the concept of automorphisms of a graph. □

We are now able to exploit known relations for the adjacency matrix of graphs:

Lemma 2.8. *For the complement $\overline{G}$ of a graph G, the direct sum $\dot{+}_{i=1}^n G_i$ and the complete product $\bigtriangledown_{i=1}^n G_i$ of graphs G_i, $i = 1, 2, \dots, n$, we have the following relations:*
$$C_{\overline{G}}(\lambda) = (-1)^p \frac{\lambda}{\lambda - p} C_G(p - \lambda),$$
$$C_{\dot{+}_{i=1}^n G_i}(\lambda) = \prod_{i=1}^n C_{G_i}(\lambda),$$
$$C_{\bigtriangledown_{i=1}^n G_i}(\lambda) = \lambda(\lambda - p)^{n-1}\prod_{i=1}^n \frac{C_{G_i}(\lambda - p + p_i)}{\lambda - p + p_i}.$$

REMARK. The proofs are taken from [**15**].

PROOF. For the direct sum $G_1 \dotplus G_2$ of two graphs G_1 and G_2 we have

$$\mathbf{C}_{G_1 \dotplus G_2} = \begin{pmatrix} \mathbf{C}_{G_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{G_2} \end{pmatrix}.$$

From the Laplacian development of the determinant, the statement follows.

For the complementary graph $\overline{G}$ of a graph G we have

$$\begin{aligned} C_{\overline{G}}(\lambda) &= \det\big(\lambda\mathbf{I} - (p-1)\mathbf{I} + \mathbf{D} + \mathbf{J} - \mathbf{I} - \mathbf{A}\big) \\ &= \det\big((\lambda - p)\mathbf{I} + \mathbf{J} + \mathbf{D} - \mathbf{A}\big). \end{aligned}$$

Adding all rows except the first to the first row of the determinant, every entry of the first row becomes equal to λ. Taking this factor out and then subtracting the first row from all other rows, we obtain

$$C_{\overline{G}}(\lambda) = \lambda \det \begin{pmatrix} 1 \dots 1 \\ \big((\lambda - p)\mathbf{I} + \mathbf{D} - \mathbf{A}\big)_{1^c} \end{pmatrix},$$

where $(\mathbf{X})_{1^c}$ denotes the submatrix obtained from $\mathbf{X}$ by deleting its first row. On the other hand, we have

$$\det\big((\lambda - p)\mathbf{I} + \mathbf{D} - \mathbf{A}\big) = (-1)^p C_G(p - \lambda).$$

Again, adding all other rows to the first row, every entry of the first row becomes equal to $\lambda - p$. Taking this factor out, we obtain the required result

$$C_{\overline{G}}(\lambda) = (-1)^p \frac{\lambda}{\lambda - p} C_G(p - \lambda).$$

Now the formula for $C_{\bigtriangledown_{i=1}^n G_i}(\lambda)$ follows very easily from the fact that $\bigtriangledown_{i=1}^n G_i = \overline{\dotplus_{i=1}^n \overline{G}_i}$:

$$\begin{aligned} C_{\bigtriangledown_{i=1}^n G_i}(\lambda) &= C_{\overline{\dotplus_{i=1}^n \overline{G}_i}}(\lambda) \\ &= (-1)^p \frac{\lambda}{\lambda - p} C_{\dotplus_{i=1}^n \overline{G}_i}(p - \lambda) \\ &= (-1)^p \frac{\lambda}{\lambda - p} \prod_{i=1}^n C_{\overline{G}_i}(p - \lambda) \\ &= (-1)^p \frac{\lambda}{\lambda - p} \prod_{i=1}^n (-1)^{p_i} \frac{p - \lambda}{p - \lambda - p_i} C_{G_i}(p_i - p + \lambda) \\ &= \lambda(\lambda - p)^{n-1} \prod_{i=1}^n \frac{C_{G_i}(\lambda - p + p_i)}{\lambda - p + p_i}. \end{aligned}$$

□

EXAMPLE 2.9. The complete multipartite graph $K_{p_1,p_2,\dots,p_n}$ can be expressed as the complete product of the graphs $O_{p_1}, O_{p_2}, \dots O_{p_n}$, where O_p denotes the graph consisting of p isolated vertices. O_p again is the direct sum of p copies of a single vertex. Hence, $C_{O_p}(\lambda) = \lambda^p$ and

$$C_{K_{p_1,p_2,\ldots,p_n}}(\lambda) = \lambda(\lambda - p)^{n-1} \prod_{i=1}^{n} (\lambda - p + p_i)^{p_i - 1}.$$

Therefore, by Theorem 2.3 we have

$$\begin{aligned} \boldsymbol{t}(K_{p_1,p_2,\ldots,p_n}) &= \frac{(-1)^{p-1}}{p}\, C'_{K_{p_1,p_2,\ldots,p_n}}(\lambda)\Big|_{\lambda=0} \\ &= \frac{(-1)^{p-1}}{p}\, (\lambda - p)^{n-1} \prod_{i=1}^{n} (\lambda - p + p_i)^{p_i - 1}\Bigg|_{\lambda=0} \\ &= p^{n-2} \prod_{i=1}^{n} (p - p_i)^{p_i - 1}. \end{aligned}$$

Before we give some more examples, we need to calculate the spectra of two very simple graphs, the path P_p and the circle C_p. For doing this, it will be helpful to recall some properties of the well-known Chebyshev polynomials. We will then see the reason for the close connection between certain spanning tree formulas and Chebyshev polynomials.

Remark. The *Chebyshev polynomials of the first kind* are defined as $T_n(x) = \cos(n \arccos x)$. It can be shown that

$$T_n(x) = \frac{1}{2}\Big(\big(x + \sqrt{x^2 - 1}\big)^n + \big(x - \sqrt{x^2 - 1}\big)^n\Big)$$

and

$$T_n(x) = (-1)^n T_n(-x).$$

Similarly, the *Chebyshev polynomials of the second kind* are defined as $U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}$. We can calculate its zeros and obtain

$$U_n(x) = 2^n \prod_{k=1}^{n} \left(x - \cos\frac{k\pi}{n+1}\right).$$

It can be shown, that

$$U_n(x) = \begin{cases} \frac{1}{2\sqrt{x^2-1}}\Big(\big(x + \sqrt{x^2 - 1}\big)^{n+1} - \big(x - \sqrt{x^2 - 1}\big)^{n+1}\Big) \\ \qquad\qquad \text{for } |x| \neq 1 \\ (\operatorname{sgn} x)^n (n+1) \text{ for } x = \pm 1 \end{cases}$$

and

$$U_n(x) = (-1)^n U_n(-x).$$

Furthermore, they satisfy the recursion

$$U_n(x) = 2x U_{n-1}(x) - U_{n-2}(x).$$

LEMMA 2.10. *The ordinary, the C-characteristic and the Q-characteristic polynomials of the path P_p are*

$$P_{P_p}(\lambda) = U_p\left(\frac{\lambda}{2}\right), \qquad C_{P_p}(\lambda) = \lambda\, U_{p-1}\left(\frac{\lambda-2}{2}\right)$$

and

$$Q_{P_p}(\lambda) = \frac{\lambda^2-1}{2^{p-2}} U_{p-2}(\lambda).$$

The ordinary, the C-characteristic and the Q-characteristic polynomials of the circle C_p are

$$P_{C_p}(\lambda) = 2\left(T_p\left(\frac{\lambda}{2}\right) - 1\right), \qquad C_{C_p}(\lambda) = 2\left(T_p\left(\frac{\lambda-2}{2}\right) - (-1)^p\right)$$

and

$$Q_{C_p}(\lambda) = \frac{1}{2^{p-1}}\left(T_p(\lambda) - 1\right).$$

respectively.

PROOF. Let $\mathbf{A}_p$ denote the $(p \times p)$ matrix

$$\begin{pmatrix} \lambda-2 & 1 & 0 & & \cdots\cdots & 0 \\ 1 & \lambda-2 & 1 & 0 & & \\ 0 & & & & & \vdots \\ & & & \ddots & & \\ \vdots & & & & & 0 \\ & & & 0 & 1 \quad \lambda-2 & 1 \\ 0 & \cdots\cdots & & & 0 \quad 1 & \lambda-1 \end{pmatrix}.$$

We adopt the convention $\mathbf{A}_1 = (\lambda - 1)$. Then, for $p > 1$, by Lemma 2.4 $C_{P_p}(\lambda)$ equals

$$\det \begin{pmatrix} \lambda-1 & 1 & 0 & \dots & 0 \\ 1 & & & & \\ 0 & & \mathbf{A}_{p-1} & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix} = (\lambda-1)\det \mathbf{A}_{p-1} - \det \mathbf{A}_{p-2}.$$

By the Laplacian development we get

$$\det \mathbf{A}_p = (\lambda-2)\det \mathbf{A}_{p-1} - \det \mathbf{A}_{p-2}.$$

Clearly, $C_{P_p}(\lambda)$ satisfies the same recursion, with initial conditions $C_{P_2}(\lambda) = \lambda(\lambda-2)$ and $C_{P_3}(\lambda) = \lambda(\lambda-1)(\lambda-3)$. By the remark above we have

$$C_{P_p}(\lambda) = \lambda U_{p-1}(\frac{\lambda-2}{2}).$$

The statements concerning the ordinary and the Q-characteristic polynomials are deduced similarly.

The circle C_p has a so called 'circulant' matrix as its adjacency matrix. Employing Lemma 2.7 we can calculate its eigenvalues: The first row of the adjacency matrix $\mathbf{A}_{C_p}$ of the circle is $(0, 1, 0, \ldots, 0, 1)$. Hence, applying the lemma we obtain

$$\begin{aligned} P_{C_p}(\lambda) &= \prod_{k=1}^{p} (\lambda - \omega - \omega^{-1}) \\ &= \prod_{k=1}^{p} (\lambda - 2\cos\frac{2k\pi}{p}), \end{aligned}$$

where ω is a p^{th} root of unity. As the circle is a regular graph, using Remark 2 we immediately deduce the expressions for the C- and Q-characteristic polynomial. $\square$

Now we can easily compute the number of spanning trees of the fan and the wheel:

Example 2.11. The fan F_n is the complete product of a single vertex and the path P_n. Thus, by Lemma 2.8 and Lemma 2.10

$$\begin{aligned} C_{K_1 \triangledown P_n}(\lambda) &= \lambda(\lambda - n - 1)\frac{C_{K_1}(\lambda - n)}{\lambda - n}\frac{C_{P_n}(\lambda - 1)}{\lambda - 1} \\ &= \frac{\lambda(\lambda - n - 1)}{\lambda - 1} C_{P_n}(\lambda - 1) \\ &= \frac{\lambda(\lambda - n - 1)}{\lambda - 1}(\lambda - 1)\, U_n\left(\frac{\lambda - 3}{2}\right) \\ &= \lambda(\lambda - n - 1)\, U_n\left(\frac{\lambda - 3}{2}\right). \end{aligned}$$

Using Theorem 2.3 we finally get

$$\begin{aligned} \boldsymbol{t}(F_n) &= U_n(\frac{3}{2}) \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]. \end{aligned}$$

Example 2.12. The wheel W_n is the complete product of a single vertex and the circle C_n. Thus, as before, we have

$$\begin{aligned} C_{K_1 \triangledown C_n}(\lambda) &= \lambda(\lambda - n - 1)\frac{C_{K_1}(\lambda - n)}{\lambda - n}\frac{C_{C_n}(\lambda - 1)}{\lambda - 1} \\ &= \frac{\lambda(\lambda - n - 1)}{\lambda - 1} C_{C_n}(\lambda - 1) \\ &= \frac{\lambda(\lambda - n - 1)}{\lambda - 1} 2\left(T_p\left(\frac{\lambda - 3}{2}\right) - (-1)^n\right). \end{aligned}$$

Finally we get

$$\begin{aligned} \boldsymbol{t}(W_n) &= 2\left(T_n\left(\frac{3}{2}\right) - 1\right) \\ &= \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2. \end{aligned}$$

Using the remark after the proof of Theorem 2.3 we can give even easier proofs for more general results:

EXAMPLE 2.13. Let F_n^k be the graph obtained from F_n by replacing each edge on the rim by a path consisting of k edges. Its dual $(F_n^k)^*$ is nearly $(2+k)$-regular and, if r denotes the exceptional vertex, $(F_n^k)^*\big|_{r^c}$ is the path on $n-1$ vertices. Hence

$$\boldsymbol{t}(F_n^k) = P_{P_{n-1}}(2+k) = U_{n-1}\left(\frac{k+2}{2}\right).$$

EXAMPLE 2.14. Similarly, let W_n^k be the graph obtained from W_n by replacing each edge on the rim by a path consisting of k edges. Its dual $(W_n^k)^*$ is nearly $(2+k)$-regular and, if r denotes the exceptional vertex, $(W_n^k)^*\big|_{r^c}$ is the circle on n vertices. Hence

$$\boldsymbol{t}(W_n^k) = P_{C_n}(2+k) = 2\left(T_n\left(\frac{k+2}{2}\right) - 1\right).$$

LEMMA 2.15. *Let G be an r-regular graph. Then its line graph $\mathfrak{L}(G)$ is $2(r-1)$-regular and its characteristic polynomial is*

$$P_{\mathfrak{L}(G)}(\lambda) = (\lambda+2)^{q-p} P_G(\lambda + 2 - r).$$

If G is semiregular, then its line graph is regular of degree $r_1 + r_2 - 2$ and its characteristic polynomial is

$$P_{\mathfrak{L}(G)}(\lambda) = (\lambda+2)^{q-p}\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{p_1-p_2}{2}} P_G(\sqrt{\alpha_1\alpha_2}),$$

for $\alpha_i = \lambda + 2 - r_i$ for $i = 1,\, 2$.

REMARK. Note, that the line graph of a graph G is regular, if and only if G is regular or semiregular!

PROOF. Let $\mathbf{B}$ be the incidence matrix of G with all entries made positive. Then we can express the adjacency matrix of a graph G and its line graph $\mathfrak{L}(G)$ in terms of $\mathbf{B}$:

$$\mathbf{A}_G = \mathbf{B}\mathbf{B}^t - \mathbf{D} \text{ and } \mathbf{A}_{\mathfrak{L}(G)} = \mathbf{B}^t\mathbf{B} - 2\mathbf{I}.$$

Now we can calculate the P-spectrum of $\mathfrak{L}(G)$:

$$\begin{aligned} P_{\mathfrak{L}(G)}(\lambda) &= \det\big((\lambda+2)\mathbf{I}_q - \mathbf{B}^t\mathbf{B}\big) \\ &= (\lambda+2)^{q-p}\det\big((\lambda+2)\mathbf{I}_p - \mathbf{B}\mathbf{B}^t\big) \qquad (*) \\ &= (\lambda+2)^{q-p}\det\left((\lambda+2)\mathbf{I}_p - \mathbf{A}_G - \mathbf{D}_G\right). \end{aligned}$$

If G is r-regular, $\mathbf{D}_G = r\mathbf{I}$, therefore

$$P_{\mathfrak{L}(G)}(\lambda) = (\lambda+2)^{q-p} P_G(\lambda + 2 - r).$$

For semiregular graphs we have

$$\begin{aligned}
\det\left((\lambda+2)\mathbf{I}_p - \mathbf{A}_G - \mathbf{D}_G\right) &= \det\begin{pmatrix} (\lambda+2-r_1)\mathbf{I}_{p_1} & -\mathbf{A} \\ -\mathbf{A}^t & (\lambda+2-r_2)\mathbf{I}_{p_2} \end{pmatrix} \\
&= \det\begin{pmatrix} \alpha_1 \mathbf{I}_{p_1} & -\mathbf{A} \\ -\mathbf{A}^t & \alpha_2 \mathbf{I}_{p_2} \end{pmatrix} \\
&= {\alpha_1}^{p_1} \det\left(\alpha_2 \mathbf{I}_{p_2} - \frac{1}{\alpha_1}\mathbf{A}^t\mathbf{A}\right) \\
&= {\alpha_1}^{p_1 - p_2} \det(\alpha_1\alpha_2 \mathbf{I}_{p_2} - \mathbf{A}^t\mathbf{A}), \text{ but also} \\
&= {\alpha_2}^{p_2 - p_1} \det(\alpha_1\alpha_2 \mathbf{I}_{p_1} - \mathbf{A}\mathbf{A}^t).
\end{aligned}$$

Multiplying the last two lines and taking the square root we get

$$\begin{aligned}
&\det\big((\lambda+2)\mathbf{I}_p - \mathbf{A}_G - \mathbf{D}_G\big) \\
&\qquad = \sqrt{\left(\frac{\alpha_1}{\alpha_2}\right)^{p_1-p_2} \det\left(\alpha_1\alpha_2\mathbf{I} - \begin{pmatrix} \mathbf{A}\mathbf{A}^t & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^t\mathbf{A} \end{pmatrix}\right)},
\end{aligned}$$

and because of $\left(\begin{smallmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^t & \mathbf{0} \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} \mathbf{A}\mathbf{A}^t & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^t\mathbf{A} \end{smallmatrix}\right)$ we obtain

$$\begin{aligned}
\det\big((\lambda+2)\mathbf{I}_p - \mathbf{A}_G - \mathbf{D}_G\big) &= \sqrt{\left(\frac{\alpha_1}{\alpha_2}\right)^{p_1-p_2} P_{G^2}(\alpha_1\alpha_2)} \\
&= \sqrt{\left(-\frac{\alpha_1}{\alpha_2}\right)^{p_1-p_2} P_G(\sqrt{\alpha_1\alpha_2}) P_G(-\sqrt{\alpha_1\alpha_2})}.
\end{aligned}$$

Because the spectrum of a bipartite graph is symmetric by Lemma 2.6, we have

$$\begin{aligned}
\det\left((\lambda+2)\mathbf{I}_p - \mathbf{A}_G - \mathbf{D}_G\right) &= \sqrt{\left(-\frac{\alpha_1}{\alpha_2}\right)^{p_1-p_2} (-1)^p P_G(\sqrt{\alpha_1\alpha_2})^2} \\
&= \left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{p_1-p_2}{2}} P_G(\sqrt{\alpha_1\alpha_2}).
\end{aligned}$$

Combining this with Equation $(*)$ we get the desired result. $\square$

Now we can see how the number of spanning trees of a (semi)regular graph and its line graph are related to each other:

Proposition 2.16. *Let G be an r-regular graph. Then*

$$\boldsymbol{t}(\mathfrak{L}(G)) = \frac{(2r)^{q-p+1}}{r^2}\boldsymbol{t}(G).$$

If G is semiregular of degrees r_1 and r_2 we have

$$\boldsymbol{t}(\mathfrak{L}(G)) = \frac{(r_1+r_2)^{q-p+1}}{r_1 r_2}\left(\frac{r_1}{r_2}\right)^{p_2-p_1}\boldsymbol{t}(G)$$

Proof. For a r-regular graph G we have

$$\begin{aligned} P'_{\mathfrak{L}(G)}(\lambda)\Big|_{\lambda=2(r-1)} &= (\lambda+2)^{q-p}P'_G(\lambda)\big|_{\lambda=2(r-1)} \\ &= (2r)^{q-p}P'_G(r). \end{aligned}$$

By Theorem 2.3 the proposition follows. If G is semiregular, we have

$$\begin{aligned} & P'_{\mathfrak{L}(G)}(\lambda)\Big|_{\lambda=r_1+r_2-2} \\ &= (\lambda+2)^{q-p}\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{p_1-p_2}{2}}\frac{\alpha_1+\alpha_2}{2\sqrt{\alpha_1\alpha_2}}P'_G(\sqrt{\alpha_1\alpha_2})\Bigg|_{\lambda=r_1+r_2-2} \\ &= \frac{(r_1+r_2)^{q-p+1}}{2\sqrt{r_1r_2}}\left(\frac{r_2}{r_1}\right)^{\frac{p_1-p_2}{2}}P'_G(\sqrt{r_1r_2}). \end{aligned}$$

□

This proposition leads to a strange proof of Cayley's theorem:

Example 2.17. The complete graph K_p is the line graph of the star S_p, which is semiregular of degrees p and one. Hence

$$\boldsymbol{t}(K_p) = \frac{(p+1)^0}{p}\left(\frac{p}{1}\right)^{p-1} = p^{p-2}.$$

We now turn to the so-called NEPS of graphs. For regular graphs, everything is very easy. In this case, we can use the ordinary spectrum to calculate the number of spanning trees. Although the following theorem is true for graphs which are not regular as well, it is of no use for our purposes, as it is only valid for the ordinary spectrum!

Theorem 2.18. *The NEPS G with basis $\mathcal{B}$ of the graphs $G_1,\ G_2,\ \ldots,\ G_n$, whose adjacency matrices are $\mathbf{A}_1,\ \mathbf{A}_2,\ \ldots,\ \mathbf{A}_n$, has adjacency matrix*

$$\mathbf{A} = \sum_{\beta\in\mathcal{B}} \mathbf{A}_1^{\beta_1}\otimes\cdots\otimes\mathbf{A}_n^{\beta_n}.$$

Suppose that the graph G_i has p_i vertices and its (ordinary) spectrum is $\lambda_{i1},\ldots,\lambda_{ip_i}$ for $i=1,2,\ldots,n$. Then the spectrum of the NEPS G consists of all possible values of $\Lambda_{i_1,i_2,\ldots,i_n}$, where

$$\Lambda_{i_1,i_2,\ldots,i_n} = \sum_{\beta\in\mathcal{B}} \lambda_{1i_1}^{\beta_1}\ldots\lambda_{ni_n}^{\beta_n}$$

for $i_k=1,2,\ldots,p_k$ and $k=1,2,\ldots,n$.

PROOF. The entries of $\mathbf{A}$ are

$$\mathbf{A}_{(u_1,u_2,\ldots,u_n)(v_1,v_2,\ldots,v_n)} = \sum_{\beta\in\mathcal{B}} (\mathbf{A}_1^{\beta_1})_{u_1v_1}\cdots(\mathbf{A}_n^{\beta_n})_{u_nv_n}.$$

Hence, the vertices $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_n)$ of G are connected if and only if there is a $\beta \in \mathcal{B}$ so that $(\mathbf{A}_i^{\beta_i})_{u_iv_i} \neq 0$ for all $i \in \{1, 2, \ldots, n\}$. This is precisely the definition of the NEPS.

We can now prove the statement about the eigenvalues of the NEPS G: Since $\mathbf{A}_i$, the adjacency matrix of G_i, is normal, there are linearly independent vectors $\mathbf{x}_{ij}$ such that $\mathbf{A}_i\mathbf{x}_{ij} = \lambda_{ij}\mathbf{x}_{ij}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, p_i$. Consider the vector $\mathbf{x} = \mathbf{x}_{1i_1} \otimes \cdots \otimes \mathbf{x}_{ni_n}$. Using the fact that $(\mathbf{A}\otimes\mathbf{B})\cdot(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{AC})\otimes(\mathbf{BD})$ we get

$$\begin{aligned}\mathbf{Ax} &= \sum_{\beta\in\mathcal{B}} (\mathbf{A}_1^{\beta_1}\otimes\cdots\otimes\mathbf{A}_n^{\beta_n})\mathbf{x} \\ &= \sum_{\beta\in\mathcal{B}} (\mathbf{A}_1^{\beta_1}\mathbf{x}_{1i_1})\otimes\cdots\otimes(\mathbf{A}_n^{\beta_n}\mathbf{x}_{ni_n}) \\ &= \sum_{\beta\in\mathcal{B}} (\lambda_{1i_1}^{\beta_1}\mathbf{x}_{1i_1})\otimes\cdots\otimes(\lambda_{ni_n}^{\beta_n}\mathbf{x}_{ni_n}) \\ &= \Lambda_{i_1,i_2,\ldots,i_n}\mathbf{x}.\end{aligned}$$

This yields $p_1p_2\ldots p_n$ eigenvectors. As $\mathbf{A}$ has dimension $p_1p_2\ldots p_n$, a basis of eigenvectors has been determined. □

Unfortunately, for NEPS of graphs which are not regular, we can not apply the preceding theorem to find their eigenvalues. Only for two special cases we can find workarounds:

The Kronecker sum $G \oplus H$ of two graphs G and H with p_G and p_H vertices, respectively, is represented by the NEPS with basis $\{(1,0),(0,1)\}$. Hence, its adjacency matrix and its degree matrix are

$$\begin{aligned}\mathbf{A}_{G\oplus H} &= \mathbf{A}_G\otimes\mathbf{I}_{p_H} + \mathbf{I}_{p_G}\otimes\mathbf{A}_H, \\ \mathbf{D}_{G\oplus H} &= \mathbf{D}_G\otimes\mathbf{I}_{p_H} + \mathbf{I}_{p_G}\otimes\mathbf{D}_H.\end{aligned}$$

Therefore,

$$\begin{aligned}C_{G\oplus H}(\lambda) &= \det(\lambda\mathbf{I} - \mathbf{D}_{G\oplus H} + \mathbf{A}_{G\oplus H}) \\ &= \det(\lambda\mathbf{I} - \mathbf{D}_G\otimes\mathbf{I}_{p_H} - \mathbf{I}_{p_G}\otimes\mathbf{D}_H + \mathbf{A}_G\otimes\mathbf{I}_{p_H} + \mathbf{I}_{p_G}\otimes\mathbf{A}_H) \\ &= \det(\lambda\mathbf{I} - (\mathbf{D}_G - \mathbf{A}_G)\otimes\mathbf{I}_{p_H} - \mathbf{I}_{p_G}\otimes\mathbf{D}_H - \mathbf{A}_H).\end{aligned}$$

Hence, by the theorem above, the C-eigenvalues of $G \oplus H$ are all possible sums of a C-eigenvalue of G with a C-eigenvalue of H. In fact, the Kronecker sum of two graphs is essentially unchanged when loops are added to some to some of the vertices: Loops in G or H are transformed into loops of $G \oplus H$. Hence, we may 'regularise' G and H as described after Theorem 2.3, by adding an appropriate number of loops. Then we can use Theorem 2.18 directly.

EXAMPLE 2.19. The $(l \times m)$ square lattice is the Kronecker sum of two paths P_l and P_m. Therefore we have

$$\boldsymbol{t}(P_l \oplus P_m) = \frac{1}{lm} \prod_{\substack{j\in\{1,2,\dots,l\} \\ k\in\{1,2,\dots,m\} \\ (j,k)\neq(l,m)}} \left(4 + 2\left(\cos\frac{\pi j}{l} + \cos\frac{\pi k}{m}\right)\right).$$

This does not look very nice, but at least we get a nice formula for the ladder $L_n = P_2 \oplus P_n$:

$$\begin{aligned}
\boldsymbol{t}(P_2 \oplus P_n) &= \frac{1}{2n} \prod_{k\in\{1,2,\dots,n-1\}} \left(2 + 2\cos\frac{\pi k}{n}\right) \prod_{k\in\{1,2,\dots,n\}} \left(4 + 2\cos\frac{\pi k}{n}\right) \\
&= \frac{2}{2n} U_{n-1}(-1) U_{n-1}(-2) \\
&= U_{n-1}(2) \\
&= \frac{1}{2\sqrt{3}} \left[\left(2+\sqrt{3}\right)^n - \left(2-\sqrt{3}\right)^n\right].
\end{aligned}$$

By very similar calculations it can be shown that the number of spanning trees of the complete prism $K_n \oplus P_m$ equals

$$\boldsymbol{t}(K_n \oplus P_m) = n^{n-2}\left(U_{m-1}\left(\frac{n+2}{2}\right)\right)^{n-1},$$

and the number of spanning trees of the complete cyclic prism $K_n \oplus C_m$ equals

$$\boldsymbol{t}(K_n \oplus C_m) = \frac{m2^{n-1}}{n}\left(T_m\left(\frac{n+2}{2}\right) - 1\right)^{n-1}.$$

The Kronecker product $G \otimes H$ of graphs G and H is represented by the NEPS with basis $\{(1,1)\}$. Its adjacency matrix and its degree matrix are

$$\begin{aligned}
\mathbf{A}_{G\otimes H} &= \mathbf{A}_G \otimes \mathbf{A}_H, \\
\mathbf{D}_{G\otimes H} &= \mathbf{D}_G \otimes \mathbf{D}_H.
\end{aligned}$$

It turns out that, this time, the Q-spectrum is the right choice, as we have

$$\begin{aligned}
Q_{G\otimes H}(\lambda) &= \det\left(\lambda\mathbf{I} - \mathbf{D}_{G\otimes H}^{-1}\mathbf{A}_{G\otimes H}\right) \\
&= \det\left(\lambda\mathbf{I} - (\mathbf{D}_G^{-1} \otimes \mathbf{D}_H^{-1})(\mathbf{A}_G \otimes \mathbf{A}_H)\right) \\
&= \det\left(\lambda\mathbf{I} - (\mathbf{D}_G^{-1}\mathbf{A}_G) \otimes (\mathbf{D}_H^{-1}\mathbf{A}_H)\right).
\end{aligned}$$

Hence, by Theorem 2.18, the Q-eigenvalues of $G \otimes H$ are all possible products of a Q-eigenvalue of G with a Q-eigenvalue of H. Note, that loops in G or H do affect the Kronecker product $G \otimes H$!

Chow [**12**] used this fact to prove an interesting theorem about the Kronecker product of bipartite graphs. Before proving his result, we need the following lemma:

LEMMA 2.20. *Let $\mathbf{A}_1,\ \mathbf{A}_2, \ldots, \mathbf{A}_n$ be matrices whose column sums are nonzero. Then, for the stochasticization of the tensor product of these matrices we have*

$$\mathbf{Q}\left(\bigotimes_{i=1}^{n}\mathbf{A_i}\right) = \bigotimes_{i=1}^{n}\mathbf{Q}\left(\mathbf{A_i}\right).$$

PROOF. We have

$$\mathbf{Q}\left(\bigotimes_{i=1}^{n}\mathbf{A_i}\right) = \left(\bigotimes_{i=1}^{n}\mathbf{D}_i^{-1}\right)\left(\bigotimes_{i=1}^{n}\mathbf{A}_i\right) = \bigotimes_{i=1}^{n}\left(\mathbf{D}_i^{-1}\mathbf{A}_i\right) = \bigotimes_{i=1}^{n}\mathbf{Q}\left(\mathbf{A}_i\right),$$

where $\mathbf{D}_i$ is the degree matrix of the graph represented by $\mathbf{A}_i$. ☐

THEOREM 2.21. *Let $G_1,\ G_2, \ldots, G_n$ be connected bipartite weighted (di)graphs. Then $G = \bigotimes_{i=1}^{n} G_i$ has 2^{n-1} connected components, each of which is also a connected bipartite weighted (di)graph. The Q-spectra of the connected components are all equal, up to the multiplicity of the eigenvalue zero.*

PROOF. For $i \in \{1, 2, \ldots, n\}$, the adjacency matrix of G_i can be written as $\mathbf{A}_{G_i} = \left(\begin{smallmatrix} \mathbf{0} & \mathbf{A}_i^{(0)} \\ \mathbf{A}_i^{(1)} & \mathbf{0} \end{smallmatrix}\right)$. (Note that the adjacency matrix of a digraph need not be symmetric.) By reordering the vertices of the Kronecker product G we find that its adjacency matrix can be represented by

$$\mathbf{A}_G = \begin{pmatrix} & & \cdot^{\cdot^{\cdot}} \\ & \bigotimes_{i=1}^{n}\mathbf{A}_i^{(\beta_i)} & \\ \cdot^{\cdot^{\cdot}} & & \end{pmatrix},$$

where β_i can be either zero or one, and β runs through all 2^n possible combinations. For example, for $n = 2$ we get

$$\begin{pmatrix} & & & & \mathbf{0} & \mathbf{A}_1^{(0)} \otimes \mathbf{A}_2^{(0)} \\ & & & & \mathbf{A}_1^{(0)} \otimes \mathbf{A}_2^{(1)} & \mathbf{0} \\ & & \mathbf{0} & \mathbf{A}_1^{(1)} \otimes \mathbf{A}_2^{(0)} & & \\ \mathbf{0} & & & & & \\ \mathbf{A}_1^{(1)} \otimes \mathbf{A}_2^{(1)} & & \mathbf{0} & & & \end{pmatrix}.$$

Hence, for the component of G corresponding to β we have

$$\mathbf{A}_{H_\beta} = \begin{pmatrix} \mathbf{0} & \bigotimes_{i=1}^{n}\mathbf{A}_i^{(\beta_i)} \\ \bigotimes_{i=1}^{n}\mathbf{A}_i^{(1-\beta_i)} & \mathbf{0} \end{pmatrix}$$

Now we can compute the Q-characteristic polynomials of the components of G:

$$\begin{aligned}
Q_{H_\beta}(\lambda) &= \det\left(\lambda\mathbf{I} - \mathbf{Q}(\mathbf{A}_{H_\beta})\right) \\
&= \det\left(\lambda\mathbf{I} - \begin{pmatrix} \mathbf{0} & \mathbf{Q}\left(\bigotimes_{i=1}^n \mathbf{A}_i^{(\beta_i)}\right) \\ \mathbf{Q}\left(\bigotimes_{i=1}^n \mathbf{A}_i^{(1-\beta_i)}\right) & \mathbf{0} \end{pmatrix}\right) \\
&= \lambda^k \det\left(\lambda\mathbf{I} - \frac{1}{\lambda}\mathbf{Q}\left(\bigotimes_{i=1}^n \mathbf{A}_i^{(\beta_i)}\right)\mathbf{Q}\left(\bigotimes_{i=1}^n \mathbf{A}_i^{(1-\beta_i)}\right)\right) \\
&= \lambda^{k-l} \det\left(\lambda\mathbf{I} - \bigotimes_{i=1}^n \mathbf{Q}\left(\mathbf{A}_i^{(\beta_i)}\right)\bigotimes_{i=1}^n \mathbf{Q}\left(\mathbf{A}_i^{(1-\beta_i)}\right)\right) \\
&= \lambda^{k-l} \det\left(\lambda\mathbf{I} - \bigotimes_{i=1}^n \mathbf{Q}\left(\mathbf{A}_i^{(\beta_i)}\right)\mathbf{Q}\left(\mathbf{A}_i^{(1-\beta_i)}\right)\right).
\end{aligned}$$

By Lemma 2.5, the spectra of the matrices

$$\mathbf{Q}\left(\mathbf{A}_i^{(\beta_i)}\right)\mathbf{Q}\left(\mathbf{A}_i^{(1-\beta_i)}\right) \text{ and } \mathbf{Q}\left(\mathbf{A}_i^{(1-\beta_i)}\right)\mathbf{Q}\left(\mathbf{A}_i^{(\beta_i)}\right)$$

are equal up to the multiplicity of the eigenvalue zero. To conclude the proof, note that by Theorem 2.18 the spectrum of a Kronecker product consists of all possible products of eigenvalues of its factors counting multiplicities. □

For $n = 2$, this theorem provides a simple relation between the spanning trees of the two connected components of $G_1 \otimes G_2$:

Corollary 2.22. *Let $G_1 = (X_1, Y_1)$ and $G_2 = (X_2, Y_2)$ be weighted bipartite graphs. (In fact, one of them can even be a digraph.) Let $H_1 = (X_1 \times X_2, Y_1 \times Y_2)$ and $H_2 = (X_1 \times Y_2, Y_1 \times X_2)$ be the two connected components of $G_1 \otimes G_2$. Then*

$$\frac{\boldsymbol{t}(H_1)}{\boldsymbol{t}(H_2)} = \left(\frac{\prod_{v\in X_1} d(v)}{\prod_{v\in Y_1} d(v)}\right)^{|X_2|-|Y_2|}\left(\frac{\prod_{v\in X_2} d(v)}{\prod_{v\in Y_2} d(v)}\right)^{|X_1|-|Y_1|}$$

Proof. Suppose that G_2 has symmetric adjacency matrix. Then any edge of $G = G_1 \otimes G_2$ has weight

$$\begin{aligned}
(\mathbf{A}_G)_{(ux,vy)} &= (\mathbf{A}_{G_1})_{(u,v)}(\mathbf{A}_{G_2})_{(x,y)} \\
&= (\mathbf{A}_{G_1})_{(u,v)}(\mathbf{A}_{G_2})_{(y,x)} \\
&= (\mathbf{A}_G)_{(uy,vx)},
\end{aligned}$$

hence the bijection between the edges of H_1 and H_2 that sends (ux, vy) to (uy, vx) is weight-preserving. Therefore, the sums of the edge-weights are the same in H_1 and H_2. By Theorem 2.3, (4) we get

$$\boldsymbol{t}(H_1)\Big(\prod_{v\in\mathcal{V}(H_2)} d_v\Big)\Big(\prod_{\substack{\lambda\neq 1\\ \text{Q-eigenvalue of } H_2}} (1-\lambda)\Big) = \boldsymbol{t}(H_2)\Big(\prod_{v\in\mathcal{V}(H_1)} d_v\Big)\Big(\prod_{\substack{\lambda\neq 1\\ \text{Q-eigenvalue of } H_1}} (1-\lambda)\Big).$$

By Theorem 2.21, the Q-eigenvalues of H_1 and H_2 are the same up to the multiplicity of the eigenvalue zero. Hence we are left with

$$\boldsymbol{t}(H_1)\Big(\prod_{v\in\mathcal{V}(H_2)} d_v\Big)=\boldsymbol{t}(H_2)\Big(\prod_{v\in\mathcal{V}(H_1)} d_v\Big).$$

By the definition of the Kronecker product we find that the product of the degrees of the vertices in H_1 is

$$\Big(\prod_{v\in X_1} d_v\Big)^{|X_2|}\Big(\prod_{v\in X_2} d_v\Big)^{|X_1|}\Big(\prod_{v\in Y_1} d_v\Big)^{|Y_2|}\Big(\prod_{v\in Y_2} d_v\Big)^{|Y_1|},$$

and the product of the degrees of the vertices in H_2 is

$$\Big(\prod_{v\in X_1} d_v\Big)^{|Y_2|}\Big(\prod_{v\in Y_2} d_v\Big)^{|X_1|}\Big(\prod_{v\in Y_1} d_v\Big)^{|X_2|}\Big(\prod_{v\in X_2} d_v\Big)^{|Y_1|}.$$

This concludes the proof. □

This corollary yields some rather nice proportions. For instance:

Example 2.23. The even and odd Aztec rectangles are the components of the graph $P_{2n+1}\otimes P_{2m+1}$. Therefore, the even Aztec rectangle has exactly four times as many spanning trees as the odd Aztec rectangle.

3. Automorphisms on Graphs

Often we want to determine the number of spanning trees of graphs with a high degree of symmetry. For such graphs, the method presented in this section is appropriate.

Definition 3.1. An *automorphism* $\mathbf{T}$ on a (di)graph G is a permutation of the vertices of G which leaves their incidence relation invariant: $(u,v)\in\mathcal{E}(G)\Leftrightarrow(\mathbf{T}u,\mathbf{T}v)\in\mathcal{E}(G)$. Equivalently, the matrix representing the automorphism and the adjacency matrix of the graph commute. In the following, we will use the notation $\mathbf{T}$ both for the permutation and its matrix. automorphism

The following well known lemma leads us directly towards a method for counting the spanning trees of graphs which possess an automorphism with large orbits:

Lemma 3.2. *Let $\mathbf{A}$ and $\mathbf{B}$ be real matrices which can be diagonalised. Then $\mathbf{A}$ and $\mathbf{B}$ have a common basis of eigenvectors if and only if they commute.*

Proof. Let λ be an eigenvalue of $\mathbf{A}$ with multiplicity k. Consider a basis of eigenvectors $\{\mathbf{x}_1,\mathbf{x}_2,\dots,\mathbf{x}_k\}$ associated with λ. Let $\mathbf{x}$ be an eigenvector of $\mathbf{A}$. Then

$$\mathbf{A}(\mathbf{B}\mathbf{x})=\mathbf{B}\mathbf{A}\mathbf{x}=\mathbf{B}\lambda\mathbf{x}=\lambda(\mathbf{B}\mathbf{x}).$$

Thus, for every eigenvector $\mathbf{x}$ of $\mathbf{A}$, the vector $\mathbf{B}\mathbf{x}$ is also an eigenvector of $\mathbf{A}$. Hence, $\mathbf{B}\mathbf{x}$ must be a linear combination of the $\mathbf{x}_i$'s, $i\in\{1,2,\dots,k\}$. Therefore

$$\mathbf{B}\mathbf{x}_i=\sum_{j=1}^{k}c_{ij}\mathbf{x}_j.$$

Now consider the effect of $\mathbf{B}$ on a linear combination $\sum_{j=1}^{k} a_i \mathbf{x}_i$:

$$\mathbf{B}\big(\sum_{j=1}^{k} a_i \mathbf{x}_i\big) = \sum_{i=1}^{k} a_i \sum_{j=1}^{k} c_{ij}\mathbf{x}_j = \sum_{j=1}^{k}\Big(\sum_{i=1}^{k} a_i c_{ij}\Big)\mathbf{x}_j$$

This implies, that $\sum_{j=1}^{k} a_i \mathbf{x}_i$ is an eigenvector of $\mathbf{B}$ if and only if

$$\sum_{i=1}^{k} a_i c_{ij} = \mu a_j \text{ for } j \in \{1, 2, \dots, k\}$$

or equivalently, if

$$\mathbf{C}\mathbf{a} = \mu \mathbf{a}.$$

Two linear combinations $\sum_{j=1}^{k} a_i \mathbf{x}_i$ and $\sum_{j=1}^{k} a_i' \mathbf{x}_i$ are linearly independent if and only if $\mathbf{a}$ and $\mathbf{a}'$ are linearly independent. Therefore we need k linearly independent eigenvectors of $\mathbf{C}$. These exist if and only if $\mathbf{C}$ can be diagonalised. Because of

$$\mathbf{B}\mathbf{T} = \mathbf{T}\begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}' \end{pmatrix},$$

it follows that $\mathbf{C}$ can be diagonalised if and only if $\mathbf{B}$ can be diagonalised. □

Let G be a graph with vertices $1,\ 2,\ \dots,\ p$. Consider an automorphism $\mathbf{T}$ of G and let $\mathbf{x} = (a_1, a_2, \dots, a_p)$ be an eigenvector $\mathbf{T}$ with corresponding eigenvalue ω. Clearly,

$$\omega\mathbf{x} = \mathbf{T}\mathbf{x} = (a_{\mathbf{T}1}, a_{\mathbf{T}2}, \dots, a_{\mathbf{T}p})^t.$$

Hence, $a_i = \frac{1}{\omega} a_{\mathbf{T}i}$, $a_{\mathbf{T}i} = \frac{1}{\omega} a_{\mathbf{T}^2 i}$, $\dots$, $a_{\mathbf{T}^{k_1-1}i} = \frac{1}{\omega} a_i$ for all vertices i, where k_i is the length of the orbit the vertex belongs to. It follows that $a_i = \frac{1}{\omega^{k_i}} a_i$. Therefore, whenever $a_i \neq 0$, we have $\omega^{k_i} = 1$.

Summarizing, after rearranging the vertices of G, we see that the eigenvectors of $\mathbf{T}$ can be chosen to be of the form

$$\mathbf{x} = (a_1, a_1\omega, \dots, a_1\omega^{k_1-1}, \dots, a_n, a_n\omega, \dots, a_n\omega^{k_n-1})^t,$$

where $\omega^{k_i} = 1$ for $a_i \neq 0$, and n is the number of orbits of $\mathbf{T}$. Because of the lemma above, for all of the matrices $\mathbf{A}$, $(\mathbf{D} - \mathbf{A})$ and $\mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$, there exists a basis of eigenvectors which is also a basis of eigenvectors of $\mathbf{T}$ and whose elements are of the form displayed above.

Now we can easily derive the characteristic polynomial of the graph by solving its characteristic equation for $a_1,\ a_2,\ \dots,\ a_n$, i.e.

$$\lambda\mathbf{x} = \begin{cases} \mathbf{A}\mathbf{x}, \text{ for obtaining the ordinary spectrum} \\ (\mathbf{D} - \mathbf{A})\mathbf{x}, \text{ for obtaining the C-spectrum} \\ \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}\mathbf{x}, \text{ for obtaining the Q-spectrum} \end{cases}$$

and so on. Note that, if $\mathbf{T}$ is an automorphism of G, the matrices $\mathbf{D}$ and $\mathbf{T}$ commute as well.

Now we see that Lemma 2.7 stated in Section 2 is an almost trivial consequence of the preceding paragraphs:

Lemma 3.3. *If* $\mathbf{A}$ *is an* $(n \times n)$ *circulant matrix, i.e.,* $a_0, \ a_1, \ \dots, \ a_{n-1}$ *are arbitrary numbers and* $\mathbf{A} = (a_{j-i \mod n})_{i,j \in \{0,1,\dots,n-1\}}$*, then*

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \prod_{\omega^n = 1} (\lambda - \sum_{i=0}^{n-1} a_i \omega^i),$$

where ω *runs through all* n^{th} *roots of unity.*

Proof. Just note that $\mathbf{T} = (1\ 2\ \dots\ p)$ is an automorphism of $\mathbf{A}$. □

Remark. The distance of two vertices u and v is the length of the shortest path joining u and v. A graph is called distance-regular, if there are integers b_i and c_i $(i \geq 0)$, such that for any two vertices u and v at distance i, there are precisely c_i neighbours of u at distance $i-1$ to v and b_i neighbours of u at distance $i+1$ to v.

Distance-regular graphs have 'large' automorphism-groups. For some special families of such graphs the eigenvalues can be computed using this properties. See, for example, [**8**], Theorem 8.3.1.

4. Restriction of Infinite Graphs

In this section we will exploit the fact that the eigenfunctions of the infinite grid-graph are known and can be used to guess the eigenvectors of finite subgraphs of the infinite grid-graph. This is an idea of Kenyon, Propp and Wilson, see [**23**].

Consider the infinite grid-graph $\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$. We can identify a vertex with a pair of integers, with $(0,0)$ as origin. Sometimes though it will be convenient to specify a different origin, usually $(\frac{1}{2}, \frac{1}{2})$. In any case, two vertices (u,v) and (x,y) are connected by an edge if and only if $|u-x| = 1$ and $|v-y| = 0$ or $|u-x| = 0$ and $|v-y| = 1$. Clearly, the Laplacian operator, i.e., the operator which maps (x,x) to the degree of vertex x and (x,y) to the weight of the edge (x,y), is

$$\begin{aligned} &\mathbf{C}_{\mathbb{Z}^2} : \mathcal{C}(\mathbb{Z}^2) \to \mathcal{C}(\mathbb{Z}^2) \\ &f(x,y) \mapsto 4f(x,y) + f(x-1,y) + f(x+1,y) + f(x,y-1) + f(x,y+1) \end{aligned}$$

An eigenfunction of $\mathbf{C}_{\mathbb{Z}^2}$ must satisfy

$$\mathbf{C}_{\mathbb{Z}^2}(f) = \lambda f.$$

Such an eigenfunction can be constructed for each pair of complex numbers ζ and η by putting $f(x,y) = \zeta^x \eta^y$ and $\lambda = 4 - \zeta - \zeta^{-1} - \eta - \eta^{-1}$.

Now consider a finite subgraph G of $\mathbb{Z}^2$. Select eigenfunctions of $\mathbb{Z}^2$ which satisfy the following additional boundary conditions: For all edges (u,v) of $\mathbb{Z}^2$, where u is a vertex of G but v is not, we require $f(u) = f(v)$. By checking the equation $\mathbf{C}_G(f|_{\mathcal{V}(G)}) = \lambda\, f|_{\mathcal{V}(G)}$, where $\mathbf{C}$ is the Laplacian matrix of G, we see that the restriction to the vertices of G of an eigenfunction of the graph $\mathbb{Z}^2$, which satisfies the condition above, is an eigenfunction of $\mathbf{C}_G$, too.

We can allow one exceptional vertex r, so that $G \setminus r$ is a restriction of $\mathbb{Z}^2$. In this case we require that $f(x,y) = 0$ for all $(x,y) \in \mathbb{Z}^2$ that have distance one to some vertex of $G \setminus r$ embedded in $\mathbb{Z}^2$. It is probably best to imagine r drawn as remarked in Section 1 of Chapter 3, an example is given in Figure 2 in this section.

As an algorithmic procedure for finding the eigenvectors is not available, we have to rely on our intuition: Depending on the shape of G we may set $f(x,y) = g(x)h(y)$ for rectangular regions or $f(x,y) = g(x+y)h(x-y)$ for diamond-shaped regions, where g and h are any of $x \mapsto \sin(\alpha x)$ or $x \mapsto \cos(\alpha x)$. For more complicated regions, we need to try sums of these functions.

Example 4.1. Consider the $l \times m$ grid $P_l \oplus P_m$. An example of this graph is depicted in Figure 5 in Chapter 2, Section 2 on page 16. We set $f(x,y) = g(x)h(y)$. Let (x_{min}, y_{min}) be the lower left corner of the graph, similarly, let (x_{max}, y_{max}) be its upper right corner. The boundary conditions force g to satisfy $g(x_{min}-1) = g(x_{min})$ and $h(y_{min}-1) = h(y_{min})$, hence it will be best to set (x_{min}, y_{min}) to $(\frac{1}{2}, \frac{1}{2})$ and set $g(x) = \cos(\alpha x)$ and $h(y) = \cos(\beta y)$. Furthermore, we have $g(x_{max}) = g(x_{max}+1)$ and $h(y_{max}) = h(y_{max}+1)$, where $x_{max} = l - \frac{1}{2}$ and $y_{max} = m - \frac{1}{2}$. We arrive at the equations

$$\cos\left(\alpha\left(l-\frac{1}{2}\right)\right) - \cos\left(\alpha\left(l+\frac{1}{2}\right)\right) = 0 \text{ and}$$
$$\cos\left(\beta\left(m-\frac{1}{2}\right)\right) - \cos\left(\beta\left(m+\frac{1}{2}\right)\right) = 0.$$

From this we obtain $\alpha = \frac{j}{l}\pi$ and $\beta = \frac{k}{m}\pi$ for integers j and k. Summarizing, we get eigenfunctions

$$f_{j,k}(x,y) = \cos\frac{\pi j x}{l}\cos\frac{\pi k y}{m},$$

where x runs from $\frac{1}{2}$ to $l-\frac{1}{2}$ by integer steps and y from $\frac{1}{2}$ to $m-\frac{1}{2}$ by integer steps, $j \in \{0,1,\ldots,l-1\}$ and $k \in \{0,1,\ldots,m-1\}$. The eigenvalue corresponding to $f_{j,k}$ is $4 - 2\cos\frac{\pi j}{l} - 2\cos\frac{\pi k}{m}$, which is zero for $j = k = 0$. Hence, the number of spanning trees is

$$\boldsymbol{t}(P_l \oplus P_m) = \frac{1}{lm}\prod_{\substack{j\in\{0,1,\ldots,l-1\}\\ k\in\{0,1,\ldots,m-1\}\\ (j,k)\neq(0,0)}}\left(4 - 2\left(\cos\frac{\pi j}{l} + \cos\frac{\pi k}{m}\right)\right).$$

Using the dual graph we can compute the number of spanning trees in a different way: For the dual graph, the boundary conditions demand that $g(x_{min}) = h(y_{min}) = 0$. Hence, it makes sense to put $(x_{min}, y_{min}) = (0,0)$ and $g(x) = \sin(\alpha x)$, $h(y) = \sin(\beta y)$. Furthermore, it is required that $g(x_{max}) = h(y_{max}) = 0$, which results in $\alpha = \frac{j}{l}\pi$ and $\beta = \frac{k}{m}\pi$. Therefore, the eigenvectors of $(P_l \oplus P_m)^*$ are

$$f_{j,k}(x,y) = \sin\frac{\pi j x}{l}\sin\frac{\pi k y}{m},$$

where x runs from 0 to l by integer steps and y from 0 to m by integer steps, $j \in \{1,2,\ldots,l-1\}$ and $k \in \{1,2,\ldots,m-1\}$. The eigenvalues are the same as above. Hence, the number of spanning trees is

$$\boldsymbol{t}(P_l \oplus P_m) = \prod_{\substack{j\in\{1,2,\ldots,l-1\}\\ k\in\{1,2,\ldots,m-1\}}}\left(4 - 2\left(\cos\frac{\pi j}{l} + \cos\frac{\pi k}{m}\right)\right).$$

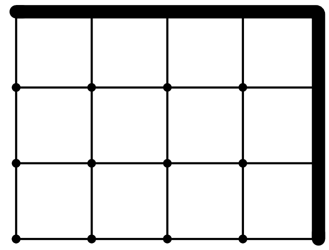


FIGURE 2. A grid with an extra vertex

The equivalence of these two formulas follows from the well-known identity for $l = \prod_{j=1}^{l-1}\left(2 - 2\cos\frac{\pi j}{l}\right)$.

EXAMPLE 4.2. Consider the graph $(P_l \oplus P_m)'$ resulting from $P_l \oplus P_m$ by attaching an extra vertex r which is joined to all vertices (x_{max}, y) and (x, y_{max}), when (x_{max}, y_{max}) denotes the upper right vertex of $P_l \oplus P_m$. Note that there are two edges joining r and (x_{max}, y_{max}). An example is depicted in Figure 2. Again we put $f(x,y) = g(x)h(y)$. Let (x_{min}, y_{min}) denote the lower left corner of the graph. Then the boundary conditions require that $g(x_{min}) = g(x_{min}+1)$ and $h(y_{min}) = h(y_{min}+1)$, hence we set (x_{min}, y_{min}) to $(\frac{1}{2}, \frac{1}{2})$ and put $g(x) = \cos(\alpha x)$ and $h(y) = \cos(\beta y)$. Furthermore, we want $g(x_{max}+1) = 0$ and $h(y_{max}+1) = 0$, where $x_{max} = l - \frac{1}{2}$ and $y_{max} = m - \frac{1}{2}$. Therefore,

$$\cos\left(\alpha\left(l + \frac{1}{2}\right)\right) = 0 \text{ and}$$
$$\cos\left(\beta\left(m + \frac{1}{2}\right)\right) = 0,$$

which results in $\alpha = \frac{2j+1}{2l+1}\pi$ and $\beta = \frac{2j+1}{2m+1}\pi$. Putting the pieces together we get

$$f_{j,k}(x,y) = \cos\frac{\pi(2j+1)x}{2l+1}\cos\frac{\pi(2k+1)y}{2m+1},$$

where x runs from $\frac{1}{2}$ to $l + \frac{1}{2}$ by integer steps and y from $\frac{1}{2}$ to $m + \frac{1}{2}$ by integer steps, $j \in \{0, 1, \ldots, l-1\}$ and $k \in \{0, 1, \ldots, m-1\}$. The eigenvalue corresponding to $f_{j,k}$ is $4 - 2\cos\frac{\pi(2j+1)}{2l+1} - 2\cos\frac{\pi(2k+1)}{2m+1}$, which is zero for $j = l$ or $k = m$. Hence, the number of spanning trees is

$$\boldsymbol{t}((P_l \oplus P_m)') = \prod_{\substack{j\in\{0,1,\ldots,l-1\}\\ k\in\{0,1,\ldots,m-1\}}} \left(4 - 2\left(\cos\frac{\pi(2j+1)}{2l+1} + \cos\frac{\pi(2k+1)}{2m+1}\right)\right).$$

Again it is also possible to compute the number of spanning trees using the dual graph of $(P_l \oplus P_m)'$.

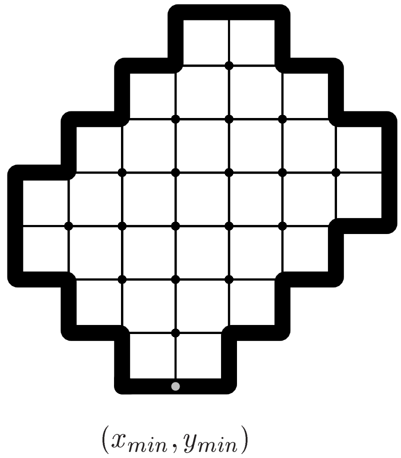


FIGURE 3. The dual of $ER_{4,3}$

EXAMPLE 4.3. The graph $P_l \otimes P_m$ consists of two components, known as the even and odd Aztec rectangles. See Section 2 in Chapter 2 for an exact definition. In Figures 10 and 11 on page 6 the two components of $P_5 \otimes P_3$ are depicted. It seems to be difficult to compute their eigenvalues directly, but it is easy to find the eigenvalues of their duals. As the regions are diamond-shaped, we put $f(x, y) = g(x+y)h(x-y)$. First, consider the dual of the even Aztec diamond. We set (x_{min}, y_{min}) as indicated in Figure 3 by a gray dot.

The boundary conditions require that the following equations are satisfied:

$$\begin{aligned}
f(x_{min}+k, y_{min}+k) =& g(x_{min}+y_{min}+2k)h(x_{min}-y_{min}) = 0\\
f(x_{min}-k, y_{min}+k) =& g(x_{min}+y_{min})h(x_{min}-y_{min}-2k) = 0\\
f(x_{min}-m+k, y_{min}+m+k) =& g(x_{min}+y_{min}+2k)\\
& h(x_{min}-y_{min}-2m) = 0\\
f(x_{min}+l-k, y_{min}+l+k) =& g(x_{min}+y_{min}-2l)\\
& h(x_{min}-y_{min}-2k) = 0,
\end{aligned}$$

where k is an appropriate integer.

When we set $g(z) = \sin(\alpha z)$ and $h(z) = \sin(\beta z)$, and (x_{min}, y_{min}) to $(0, 0)$, the first two equations are satisfied. Furthermore, we want $g(x_{min}+y_{min}-2l) = 0$ and $h(x_{min}-y_{min}+2m) = 0$. Therefore,

$$\begin{aligned}
&\sin(\alpha(2l)) = 0 \text{ and}\\
&\sin(\beta(-2m)) = 0,
\end{aligned}$$

which results in $\alpha = \frac{j}{2l}\pi$ and $\beta = \frac{k}{2m}\pi$. Putting the pieces together we get

$$f_{j,k}(x, y) = \sin\frac{\pi j(x+y)}{2l}\sin\frac{\pi k(x-y)}{2m},$$

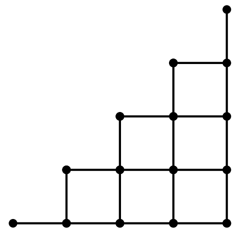


FIGURE 4
The triangle T_5

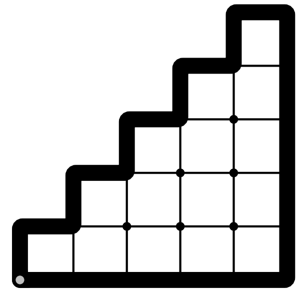


FIGURE 5
The dual of the triangle T_5

where $x \in [-m, l]$ and $y \in [0, l+m]$, with $0 \leq x+y \leq 2l$ and $0 \leq y-x \leq 2m$. Note that $f_{j,k}(x,y) = f_{2l-j,2m-k}(x,y)$. Hence, j runs from 1 to $2l-1$ and k runs from 1 to m when $j \leq l$, from 1 to $m-1$ otherwise. The eigenvalue corresponding to $f_{j,k}$ is $4 - 4\cos\frac{\pi j}{2l}\cos\frac{\pi k}{2m}$. Hence, the number of spanning trees is

$$\boldsymbol{t}(ER_{l,m}) = \prod_{(j,k)\in[1,2l-1]\times[1,m-1]\cup[1,l]\times\{m\}} \left(4 - 4\cos\frac{\pi j}{2l}\cos\frac{\pi k}{2m}\right).$$

For the odd Aztec rectangle, all of the above goes through unchanged, except that $x \in [-m-\frac{1}{2}, l+\frac{1}{2}]$ and $y \in [-\frac{1}{2}, l+m+\frac{1}{2}]$. Furthermore, $f_{l,m}$ is no longer an eigenvector, as it is the zero vector. Thus, for arbitrary l and m, the even Aztec rectangle has exactly 4 times as many spanning trees as the odd Aztec rectangle. This can be also shown using the Q-spectrum of the graph, as in Section 2, Example 2.23.

EXAMPLE 4.4. Consider the triangular graph T_m, which is depicted in Figure 4 for $m = 5$. Again it seems to be difficult to compute its eigenvalues directly, but it is possible to find the eigenvalues of its dual. Let (x_{min}, y_{min}) the lower left corner of T_m^* as indicated in Figure 5, and set $(x_{max}, y_{max}) = (x_{min}+m, y_{min}+m)$.

The boundary conditions force f to satisfy $f(x, y_{min}) = 0$, $f(x_{max}, y) = 0$, and $f(x_{min}+k, y_{min}+k) = 0$. We set (x_{min}, y_{min}) to $(0,0)$ and put $f(x,y) = g_1(x)h_1(y) + g_2(x)h_2(y)$. The first condition suggests that $h_1(y) = \sin(\alpha y)$ and $h_2(y) = \sin(\beta y)$, the second suggests $g_1(x) = \sin\frac{\pi j}{m}$ and $g_2(x) = \sin\frac{\pi k}{m}$. Now $f(x,x) = 0$ requires that $\alpha = \frac{\pi k}{m}$ and $\beta = -\frac{\pi j}{m}$. We therefore obtain

$$f_{j,k}(x,y) = \sin\frac{\pi j x}{m}\sin\frac{\pi k y}{m} - \sin\frac{\pi k x}{m}\sin\frac{\pi j y}{m},$$

where $0 \leq y \leq x \leq m$ and $0 < j < k < m$. The eigenvalue corresponding to $f_{j,k}$ is $4 - 2\cos\frac{\pi j}{m} - 2\cos\frac{\pi k}{m}$. Hence, the number of spanning trees is

$$\boldsymbol{t}(T_m) = \prod_{0<j<k<m} \left(4 - 2\cos\frac{\pi j}{m} - 2\cos\frac{\pi k}{m}\right).$$

Bibliography

[1] W. Ahrens, *Über das Gleichungssystem einer Kirchhoffschen Stromverzweigung*, Mathematische Annalen **49** (1897), 311–324.

[2] Gerd Baron, Helmut Prodinger, R. F. Tichy, F. T. Boesch, and J. F. Wang, *The number of spanning trees in the square of a cycle*, The Fibonacci Quaterly **23** (1985), no. 3, 258–264.

[3] Norman L. Biggs, *Algebraic potential theory on graphs*, Bulletin of the London Mathematical Society **29** (1997), 641–682.

[4] ———, *Chip-firing and the critical group of a graph*, Journal of Algebraic Combinatorics **1** (1999), 25–45.

[5] Norman L. Biggs and Peter Winkler, *Chip-firing and the chromatic polynomial*, CDAM Research Report Series **3** (1997).

[6] F. T. Boesch and Helmut Prodinger, *Spanning Tree Formulas and Chebyshev Polynomials*, Graphs and Combinatorics **2** (1986), 191–200.

[7] C. W. Borchardt, *Über eine der Interpolation entsprechende Darstellung der Eliminations-Resultante*, Journal für die Reine und Angewandte Mathematik **57** (1860), 111–121.

[8] Andries E. Brouwer, Arjeh M. Cohen, and Arnold Neumaier, *Distance-Regular Graphs*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 18, A Series of Modern Surveys in Mathematics, Springer-Verlag, 1989.

[9] T. J. N. Brown, R. B. Mallion, P. Pollak, and A. Roth, *Some methods for counting the spanning trees in labelled molecular graphs, examined in relation to certain fullerenes*, Discrete Applied Mathematics **67** (1996), no. 1–3, 51–66.

[10] A. Cayley, *A theorem on trees*, Quarterly Journal on Pure and Applied Mathematics **23** (1889), 376–378.

[11] Seth Chaiken, *A combinatorial proof of the all minors matrix tree theorem*, SIAM Journal on Algebraic and Discrete Methods **3** (1982), no. 3, 319–329.

[12] Timothy Y. Chow, *The Q-spectrum and spanning trees of tensor products of bipartite graphs*, Proceedings of the American Mathematical Society **125** (1997), no. 11, 3155–3161.

[13] Fan R. K. Chung and Robert P. Langlands, *A combinatorial Laplacian with vertex weights*, Journal of Combinatorial Theory, Series A **75** (1996), no. 2, 316–327.

[14] Mihai Adrian Ciucu, *A complementation theorem for perfect matchings of graphs having a cellular completion*, Journal of Combinatorial Theory, Series A **81** (1998), 34–68.

[15] Dragoš M. Cvetković, Michael Doob, and Horst Sachs, *Spectra of graphs*, Academic Press, 1980.

[16] W. Feussner, *Über Stromverzweigung in Netzförmigen Leitern*, Annalen der Physik **9** (1902), 1304–1329.

[17] ———, *Zur Berechnung der Stromstärke in Netzförmigen Leitern*, Annalen der Physik **15** (1904), 385–394.

[18] Andrei Gabrielov, *Abelian avalanches and Tutte polynomials*, Physica A **195** (1993), 253–274.

[19] I. Gutman, R. B. Mallion, and J. W. Essam, *Counting the spanning trees of a labelled molecular graph*, Molecular Physics **50** (1983), no. 4, 859–877.

[20] A. J. W. Hilton, *Spanning trees and Fibonacci and Lucas numbers*, The Fibonacci Quaterly **12** (1974), 259–262.

[21] David E. Johnson and Jonny B. Johnson, *On enumerating the trees of the wheel and other special graphs*, Journal of the Franklin Institute **315** (1983), no. 2, 139–148.

[22] Alexander K. Kelmans, *Spanning trees of extended graphs*, Combinatorica **12** (1992), no. 1, 45–51.

[23] Richard W. Kenyon, James G. Propp, and David B. Wilson, *Trees and matchings*, Electronic Journal of Combinatorics (to appear).

[24] G. Kirchhoff, *Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird*, Poggendorf's Ann. Phys. Chem. (1847), 497–508.

[25] D. J. Kleitman and B. Golden, *Counting trees in a certain class of graphs*, American Mathematical Monthly **82** (1975), 40–44.

[26] Donald E. Knuth, *Another enumeration of trees*, Canadian Journal of Mathematics **20** (1968), 1077–1086.

[27] ———, *Aztec diamonds, checkerboard graphs, and spanning trees*, Electronic Journal of Combinatorics **3** (1996), no. 2, #R5, 13pp.

[28] Germain Kreweras, *Complexité et circuits Eulériens dans les sommes tensorielles de graphes*, Journal of Combinatorial Theory, Series B **24** (1978), 202–212.

[29] Xueliang Li and Fuji Zhang, *On the number of spanning trees and Eulerian tours in generalized de Bruijn graphs*, Discrete Mathematics **94** (1991), no. 3, 189–197.

[30] Sóstenes Lins, *On the fundamental group of* 3*-gems and a 'planar' class of* 3*-manifolds*, European Journal of Combinatorics **9** (1988), 291–305.

[31] Stephen B. Maurer, *Matrix generalisations of some theorems on trees, cycles and cocycles in graphs*, SIAM Journal on Applied Mathematics **30** (1976), no. 1, 143–148.

[32] J. W. Moon, *Counting labelled trees*, Canadian Mathematical Monographs 1, Canadian Mathematical Congress, 1970.

[33] Igor M. Pak and Alexander E. Postnikov, *Enumeration of spanning trees of certain graphs*, Communications of the Moscow Mathematical Society **45** (1990), no. 3, 220–221.

[34] H. Prüfer, *Neuer Beweis eines Satzes über Permutationen*, Archiv für Mathematik und Physik **27** (1918), no. 3, 142–144.

[35] Philipp Reinfeld, *Chromatic Polynomials and the Spectrum of the Kneser Graph*, CDAM Research Report Series **2** (2000).

[36] Jon A. Sjogren, *Cycles and spanning trees*, Mathematical Computer Modelling **15** (1991), no. 9, 87–102.

[37] J. J. Sylvester, *On the change of systems of independent variables*, Quarterly Journal on Pure and Applied Mathematics **1** (1857), 126–134.

[38] H. N. V. Temperley, Combinatorics: Proceedings of the British Combinatorial Conference 1973, London Mathematical Society Lecture Notes Series, no. 13, 1974, pp. 202–204.

[39] R. Vohra and L. Washington, *Counting spanning trees in the graphs of Kleitman and Golden and a generalization*, Journal of the Franklin Institute **318** (1984), no. 5, 349–355.

[40] Derek A. Waller, *General solution to the spanning tree enumeration problem in arbitrary multigraph joins*, IEEE Transactions on Circuits and Systems **23** (1976), no. 7, 467–469.

[41] William Earl Watkins, *The Laplacian matrix of a graph: unimodular congruence*, Linear and Multilinear Algebra **28** (1990), no. 1–2, 35–43.

[42] C. S. Yang and L. C. Han, *The number of spanning trees of the generalized hypercube network*, Mathematical Computer Modelling **17** (1993), no. 6, 97–107.

[43] Huaxiao Zhang, Fuji Zhang, and Qiongxiang Huang, *On the number of spanning trees and Eulerian tours in iterated line digraphs*, Discrete Applied Mathematics **73** (1997), no. 1, 59–67.

Lebenslauf

Persönliche Daten

	Martin Rubey, geb. Anderle Hagengasse 3/16 1150 Wien
	geboren am 2. August 1973 in Tulln, Niederösterreich
	verheiratet seit März 1999
	Tochter Miriam, geboren am 6. März 2000

Ausbildung

1979–1983	Volksschule Zieglergasse, 1070 Wien
1983–1991	Bundesrealgymnasium Rahlgasse, 1060 Wien
1989	Auslandsjahr in Australien (Koonung High School, Melbourne, Victoria)
Juni 1991	Matura mit gutem Erfolg bestanden
seit 1991	Studium an der Universität Wien zunächst Informatik und Logistik, seit 1992 Mathematik
Juni 1996–Mai 1997	Zivildienst (Landwirtschaftliche Betriebshilfe)

Berufserfahrung

März 1998–Mai 2000	Tätigkeit als Programmierer bei BOSCH Telecom im Auftrag von IVM Technical Consultants