Determinants, Paths, and Plane Partitions

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Ira M. Gessel[1]
Department of Mathematics
Brandeis University
Waltham, MA 02254

X. G. Viennot
Département de Mathématiques
Université de Bordeaux I
33405 Talence, France

1. Introduction

In studying representability of matroids, Lindström [42] gave a combinatorial interpretation to certain determinants in terms of disjoint paths in digraphs. In a previous paper [25], the authors applied this theorem to determinants of binomial coefficients. Here we develop further applications. As in [25], the paths under consideration are lattice paths in the plane. Our applications may be divided into two classes: first are those in which a determinant is shown to count some objects of combinatorial interest, and second are those which give a combinatorial interpretation to some numbers which are of independent interest. In the first class are formulas for various types of plane partitions, and in the second class are combinatorial interpretations for Fibonomial coefficients, Bernoulli numbers, and the less-known Salié and Faulhaber numbers (which arise in formulas for sums of powers, and are closely related to Genocchi and Bernoulli numbers).

Other enumerative applications of disjoint paths and related methods can be found in [14], [26], [19], [51–54], [57], and [67].

2. Lindström's theorem

Let D be an acyclic digraph. D need not be finite, but we assume that there are only finitely many paths between any two vertices. Let k be a fixed positive integer. A *k-vertex* is a k-tuple of vertices of D. If $\mathbf{u} = (u_1, \ldots, u_k)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_k)$ are k-vertices of D, a *k-path* from $\mathbf{u}$ to $\mathbf{v}$ is a k-tuple $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ such that A_i is a path from u_i to v_i. The k-path $\mathbf{A}$ is *disjoint* if the paths A_i are vertex-disjoint. Let S_k be the set of permutations of $\{1, 2, \ldots, k\}$. Then for $\pi \in S_k$, by $\pi(\mathbf{v})$ we mean the k-vertex $\bigl(v_{\pi(1)}, \ldots, v_{\pi(k)}\bigr)$.

Let us assign a weight to every edge of D. We define the weight of a path to be the product of the weights of its edges and the weight of a k-path to be the product of the weights of its components. Let $\mathsf{P}(u_i, v_j)$ be the set of paths from u_i to v_j and let $P(u_i, v_j)$ be the sum of their weights. Define $\mathsf{P}(\mathbf{u}, \mathbf{v})$ and $P(\mathbf{u}, \mathbf{v})$ analogously for k-paths from $\mathbf{u}$ to $\mathbf{v}$. Let $\mathsf{N}(\mathbf{u}, \mathbf{v})$ be the subset of $\mathsf{P}(\mathbf{u}, \mathbf{v})$ of disjoint paths and let $N(\mathbf{u}, \mathbf{v})$ be the sum of their weights. It is clear that for any permutation π of $\{1, 2, \ldots, k\}$,

$$P\bigl(\mathbf{u}, \pi(\mathbf{v})\bigr) = \prod_{i=1}^{k} P\bigl(u_i, v_{\pi(i)}\bigr) \tag{2.1}$$

We use the notation $|m_{ij}|_r^s$ to denote the determinant of the matrix $(m_{ij})_{i,j=r,\ldots,s}$.

Theorem 1. (Lindström [42])

$$\sum_{\pi \in S_k} (\operatorname{sgn} \pi) N\bigl(\mathbf{u}, \pi(\mathbf{v})\bigr) = |P(u_i, v_j)|_1^k .$$

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