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Spanning Forests of a Digraph and Their Applications[1]

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Abstract—We study spanning diverging forests of a digraph and related matrices. It is shown that the normalized matrix of out forests of a digraph coincides with the transition matrix in a specific observation model for Markov chains related to the digraph. Expressions are given for the Moore-Penrose generalized inverse and the group inverse of the Kirchhoff matrix. These expressions involve the matrix of maximum out forests of the digraph. Every matrix of out forests with a fixed number of arcs and the normalized matrix of out forests are represented as polynomials of the Kirchhoff matrix; with the help of these identities, new proofs are given for the matrix-forest theorem and some other statements. A connection is specified between the forest dimension of a digraph and the degree of an annihilating polynomial for the Kirchhoff matrix. Some accessibility measures for digraph vertices are considered. These are based on the enumeration of spanning forests.

1. INTRODUCTION

Directed graphs provide a simple and universal tool to model connection structures. It is not accidental that the first systematic monograph in the theory of digraphs [1] was titled "Structural Models: An Introduction to the Theory of Directed Graphs." Digraphs frequently serve to model processes that can proceed in the direction of arcs. Physical transference, service, control, transmission of influences, ideas, innovations, and diseases are examples of such processes. If a process can start from a number of vertices and ends with the inclusion of all vertices, then the process can be modelled by the family of out forests (i.e., spanning diverging forests) of the digraph. The enumeration of all out forests allows one to determine the typical roles of the vertices in the process: one vertex is a typical starting point, another vertex is a typical intermediate point, some vertex is a typical terminating point of the process, etc. If an initial (weighted) digraph imposes some measure on the said processes, then the "role profile" of each vertex can be expressed numerically. Moreover, an exact answer can be given to the following important question: how likely is it that the process initiated at vertex j arrives at vertex i. It is not surprising that out forests of a digraph turn out to be closely related with Markov chains realizable on the digraph.

The study of out forests has been started in [2]. Generally, they were given less attention in the literature, than that given to spanning diverging trees (out arborescences), which exist only for a narrow class of digraphs. We mention in this connection [3–11], where still undirected forests were considered in most cases. The maximum out forests (i.e., out forests with the greatest possible number of arcs) of a digraph were studied in [12,13]. It was established that the normalized matrix of such forests coincides with the matrix of limiting probabilities of every Markov chain *related* to the given digraph. Some results on spanning forests of directed and undirected multigraphs were given in [14,15].

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In this paper, we study the normalized matrix of out forests (which has been also termed the matrix of relative forest accessibilities and the matrix of forest proximities) and the matrices of forests with fixed numbers of arcs.

2. NOTATION AND SOME EARLIER RESULTS

In the terminology, we mainly follow [1, 16]. Suppose that $\mathit{\Gamma}$ is a weighted digraph without loops, $V(\mathit{\Gamma}) = \{1, \ldots, n\}$ $(n > 1)$ is its set of vertices, and $E(\mathit{\Gamma})$ its set of arcs. The weights of all arcs are supposed to be strictly positive. *A subgraph* of a digraph $\mathit{\Gamma}$ is a digraph whose vertices and arcs belong to the sets of vertices and arcs of $\mathit{\Gamma}$; the weights of subgraph's arcs are the same as in $\mathit{\Gamma}$. *A restriction* of $\mathit{\Gamma}$ to $V' \subset V(\mathit{\Gamma})$ is a digraph whose arc set contains all the arcs in $E(\mathit{\Gamma})$ that have both incident vertices in V'. *A spanning subgraph* of $\mathit{\Gamma}$ is a subgraph with vertex set $V(\mathit{\Gamma})$. The *indegree* $\mathrm{id}(w)$ of vertex w is the number of arcs that come to w, *outdegree* $\mathrm{od}(w)$ of vertex w is the number of arcs that come from w. A vertex w will be called *undominated* if $\mathrm{id}(w)$=0 and *dominated* if $\mathrm{id}(w) \geq 1$. A vertex w is *isolated* if $\mathit{\Gamma}$ contains no arcs incident to w.

A *route* in a digraph is an alternating sequence of vertices and arcs $w_0, e_1, w_1, \ldots, e_k, w_k$ with every arc e_i being (w_{i-1}, w_i). If every arc e_i is either (w_{i-1}, w_i) or (w_i, w_{i-1}), then the sequence is called a *semiroute*. A *path* in a digraph is a route all whose vertices are different. A *circuit* is a route with $w_0 = w_k$, the other vertices being distinct and different from w_0. A vertex w *is reachable* from a vertex z in $\mathit{\Gamma}$ if $w = z$ or $\mathit{\Gamma}$ contains a path from z to w. A *semicircuit* is an alternating sequence of distinct vertices and arcs, $w_0, e_1, w_1, \ldots, e_k, w_0$, where every arc e_i is either (w_{i-1}, w_i) or (w_i, w_{i-1}) and all vertices $w_0, \ldots, w_{k-1}$ are different. The restriction of $\mathit{\Gamma}$ to any maximal subset of vertices connected by semiroutes is called a *weak component* of $\mathit{\Gamma}$. Let $E = (\varepsilon_{ij})$ be the matrix of arc weights. Its entry ε_{ij} is zero if and only if there is no arc from vertex i to vertex j in $\mathit{\Gamma}$. If $\mathit{\Gamma}'$ is a subgraph of $\mathit{\Gamma}$, then the weight of $\mathit{\Gamma}'$, $\varepsilon(\mathit{\Gamma}')$, is the product of the weights of all its arcs; if $\mathit{\Gamma}'$ does not contain arcs, then $\varepsilon(\mathit{\Gamma}') = 1$. The weight of a nonempty set of digraphs $\mathcal{G}$ is defined as follows:

$$\varepsilon(\mathcal{G}) = \sum_{H \in \mathcal{G}} \varepsilon(H);$$

the weight of the empty set is 0.

The *Kirchhoff matrix* [17] of a weighted digraph $\mathit{\Gamma}$ is the $n \times n$-matrix $L = L(\mathit{\Gamma}) = (\ell_{ij})$ with elements $\ell_{ij} = -\varepsilon_{ji}$ when $j \neq i$ and $\ell_{ii} = -\sum\limits_{k \neq i} \ell_{ik}$, $i, j = 1, \ldots, n$.

A *diverging tree* is a digraph without semicircuits that has a vertex (called the *root*) from which every vertex is reachable. The indegree of every non-root vertex of a diverging tree is 1. If w is the root, then $\mathrm{id}(w) = 0$. A *converging tree* is a digraph without semicircuits that has a vertex (called the *sink*) reachable from every vertex.

A *diverging forest* (*converging forest*) is a digraph without circuits such that $\mathrm{id}(w) \leq 1$ (respectively, $\mathrm{od}(w) \leq 1$) for every vertex w. An *out forest* (*in forest*) of a digraph $\mathit{\Gamma}$ is any its spanning diverging (respectively, converging) forest.

The weak components of diverging forests (converging forests) are diverging trees (respectively, converging trees).

Definition 1. An out forest F of a digraph $\mathit{\Gamma}$ is called a *maximum out forest* of $\mathit{\Gamma}$ if $\mathit{\Gamma}$ has no out forest with a greater number of arcs than in F. An in forest F of a digraph $\mathit{\Gamma}$ is a *maximum in forest* of $\mathit{\Gamma}$ if $\mathit{\Gamma}$ has no in forest with a greater number of arcs than in F.

Obviously, every maximum out forest of $\mathit{\Gamma}$ has the minimum possible number of weak components (out trees); this number will be called the *out forest dimension* of the digraph and denoted by v. The number of arcs in any maximum out forest is obviously $n - v$. The number of weak components of every maximum in forest will be called the *in forest dimension* of the digraph and denoted by v'. Obviously, for every digraph, $v, v' \in \{1, \dots, n\}$.

If a digraph Γ_1 is obtained from Γ by the reversal of all arcs, then the out forests in Γ naturally correspond to the in forests in Γ_1 and vice versa. Therefore, the out forest dimension and in forest dimension of Γ are respectively equal to the in forest dimension and out forest dimension of Γ_1.

The following proposition states that the dimensions v and v' of a digraph are not connected, except for the case where $v = n$ and $v' = n$.

Proposition 1. 1. *Let $k, k' \in \{1, \dots, n-1\}$. Then there exists a digraph on n vertices such that $v = k$ and $v' = k'$.*

2. *For every digraph Γ on n vertices, $v = n \Leftrightarrow v' = n \Leftrightarrow E(\Gamma) = \emptyset$.*

The proofs are given in the Appendix.

Throughout the paper, we mainly deal with diverging forests. However, all the results have counterparts formulated in terms of converging forests. Simple properties of out forests have been studied in [13] (Section 3). We do not cite them here and only confine ourselves to the following

Proposition 2. *If i and j belong to different trees in a maximum out forest F of a digraph Γ, and j is a root in F, then Γ contains no paths from i to j.*

Let us adduce some definitions and results from [13] which are frequently used below.

Definition 2. A nonempty subset of vertices $K \subseteq V(\Gamma)$ of digraph Γ is an *undominated knot*[2] in Γ iff all the vertices that belong to K are mutually reachable and there are no arcs (w_j, w_i) such that $w_j \in V(\Gamma) \setminus K$ and $w_i \in K$.

Suppose that $\widetilde{K} = \bigcup\limits_{i=1}^{u} K_i$, where $K_1, \dots, K_u$ are all the undominated knots of Γ, and K_i^+ is the set of all vertices reachable from K_i and unreachable from the other undominated knots. For any undominated knot K of Γ, denote by Γ_K the restriction of Γ to K and by Γ_{-K} the subgraph with vertex set $V(\Gamma)$ and arc set $E(\Gamma) \setminus E(\Gamma_K)$. For a fixed K, $\mathcal{T}$ will designate the set of all spanning diverging trees of Γ_K and $\mathcal{P}$ will be the set of all maximum out forests of Γ_{-K}. By $\mathcal{T}^k$, $k \in K$, we denote the subset of $\mathcal{T}$ consisting of all trees that diverge from k, and by $\mathcal{P}^{K \to i}$, $i \in V(\Gamma)$, the set of all maximum out forests of Γ_{-K} such that i is reachable from some vertex that belongs to K in these forests.

By $\mathcal{F}(\Gamma) = \mathcal{F}$ and $\mathcal{F}_k(\Gamma) = \mathcal{F}_k$ we denote the set of all out forests of Γ and the set of all out forests of Γ with k arcs, respectively; $\mathcal{F}_k^{i \to j}$ will designate the set of all out forests with k arcs where j belongs to a tree diverging from i.

Definition 3. The matrix $\bar{J} = (\bar{J}_{ij}) = \sigma^{-1} Q_{n-v}$, where $\sigma = \varepsilon(\mathcal{F}_{n-v})$, $Q_{n-v} = (q_{ij}) = (\varepsilon(\mathcal{F}_{n-v}^{j \to i}))$, will be called the *normalized matrix of maximum out forests* of a digraph.

Theorem 1. *Suppose that Γ is an arbitrary digraph and K is an undominated knot in Γ. Then the following statements are true.*

1. *$\bar{J}$ is a stochastic matrix*: $\bar{J}_{ij} \ge 0,\ \sum\limits_{k=1}^{n} \bar{J}_{ik} = 1,\ i, j = 1, \dots, n.$

2. $\bar{J}_{ij} \neq 0 \ \Leftrightarrow\ $ *($j \in \widetilde{K}$ and i is reachable from j in Γ).*

[2] In [2], undominated knots are called W-bases.

3. *Suppose that* $j \in K$. *For any* $i \in V(\Gamma)$, $\bar{J}_{ij} = \varepsilon(\mathcal{T}^j)\varepsilon(\mathcal{P}^{K\to i})/\varepsilon(\mathcal{F}_{n-v})$. *Furthermore, if* $i \in K^+$, *then* $\bar{J}_{ij} = \bar{J}_{jj} = \varepsilon(\mathcal{T}^j)/\varepsilon(\mathcal{T})$.

4. $\sum_{j\in K} \bar{J}_{jj} = 1$. *In particular, if* j *is an undominated vertex, then* $\bar{J}_{jj} = 1$.

5. *If* $j_1, j_2 \in K$, *then* $\bar{J}_{\cdot j_2} = (\varepsilon(\mathcal{T}^{j_2})/\varepsilon(\mathcal{T}^{j_1}))\bar{J}_{\cdot j_1}$, *i.e., the* j_1 *and* j_2 *columns of* $\bar{J}$ *are proportional.*

Theorem 2. *For every weighted digraph,* $\bar{J}$ *is idempotent:* $\bar{J}^2 = \bar{J}$.

Theorem 3. *For every weighted digraph,* $L\bar{J} = \bar{J}L = 0$.

Theorem 4 (a parametric version of the matrix-forest theorem). *For any weighted multidigraph* Γ *with positive weights of arcs and any* $\tau > 0$, *there exists the matrix* $Q(\tau) = (I + \tau L(\Gamma))^{-1}$ *and*

$$Q(\tau) = \frac{1}{s(\tau)} \sum_{k=0}^{n-v} \tau^k Q_k, \tag{1}$$

where

$$s(\tau) = \sum_{k=0}^{n-v} \tau^k \varepsilon(\mathcal{F}_k), \quad Q_k = (q_{ij}^k), \quad q_{ij}^k = \varepsilon(\mathcal{F}_k^{j\to i}), \quad k = 0, \ldots, n-v, \quad i, j = 1, \ldots, n. \tag{2}$$

Definition 4. The matrix Q_k, $k = 0, \ldots, n-v$, will be called the *matrix of out forests of* Γ *with* k *arcs.*

Theorem 4 represents $(I + \tau L)^{-1}$ via the matrices of out forests with various numbers of arcs.

Definition 5. The matrices $Q(\tau) = (I + \tau L)^{-1}$, $\tau > 0$, will be called the *normalized matrices of out forests* of a digraph.

In [14], the matrices $Q(\tau) = (I + \tau L)^{-1}$ were referred to as the matrices of relative forest accessibilities of a digraph. In Section 4, $Q(\tau)$ are expressed as polynomials of L (Corollary from Theorem 8).

Theorem 5. *For every weighted digraph* Γ, $\lim_{\tau\to\infty} Q(\tau) = \lim_{\tau\to\infty} (I + \tau\, L)^{-1} = \bar{J}$.

3. MATRICES OF OUT FORESTS AND TRANSITION PROBABILITIES OF MARKOV CHAINS

It has been shown in [13] that the matrix of Cesàro limiting probabilities of a Markov chain coincides with the normalized matrix $\bar{J}$ of maximum out forests of any digraph related to this Markov chain. Now we give a Markov chain interpretation for the normalized matrices of out forests $Q(\tau)$ with any $\tau > 0$.

Definition 6 [13]. A homogeneous Markov chain with set of states $\{1, \ldots, n\}$ and transition probability matrix P is *related to a weighted digraph* Γ iff there exists $\alpha \neq 0$ such that

$$P = I - \alpha\, L(\Gamma). \tag{3}$$

Let Γ be a weighted digraph. Consider an arbitrary Markov chain *related to* Γ and the following observation model.

The geometric model of random observation. Suppose that a Bernoulli trial is performed at the point of time $t = 0$ with success probability q $(0 < q < 1)$. In case of success, $t = 0$ becomes the epoch of observation. Otherwise, Bernoulli trials are performed at $t = 1, 2, \ldots$—to the point of the first success. This point becomes the epoch of observation.

This model determines a discrete probability distribution $p(k)$ of the epoch of observation on the set $\{0, 1, 2, \ldots\}$. This is obviously the geometric distribution (which gives the name of the model) with parameter q:

$$p(k) = q(1-q)^k, \quad k = 1, 2, \ldots \tag{4}$$

Consider Markov chain multistep transitions in a *random number of steps*: from the initial state at $t = 0$ to the state at the random epoch of observation distributed geometrically with parameter q.

Suppose that $\widetilde{P}(\alpha, q) = \left(\widetilde{p}_{ij}(\alpha, q)\right)$ is the matrix of unconditional probabilities for such multistep transitions: from the initial state to the state at the epoch of observation.

Theorem 6. *For any weighted digraph, any* $\tau > 0$ *and any Markov chains related to the weighted digraph,*

$$Q(\tau) = \widetilde{P}(\alpha, q)$$

holds, where

$$q = (\tau/\alpha + 1)^{-1}. \tag{5}$$

Theorem 6 provides an interpretation for the normalized matrix $Q(\tau)$ of out forests in terms of Markov chain transition probabilities. Conversely, for any Markov chain, the transition probabilities in the geometric observation model can be interpreted in terms of diverging forests of the corresponding digraphs.

The following corollary stresses the arbitrariness of Markov chains in Theorem 6.

Corollary 1 from Theorem 6. *For every Markov chain, every success probability* $q \in]0, 1[$ *in the geometric observation model, and every digraph related to the Markov chain,*

$$\widetilde{P}(\alpha, q) = Q(\tau)$$

holds, where $\tau = (q^{-1} - 1)\alpha$.

Corollary 2 from Theorem 6.

$$\lim_{q \to +0} \widetilde{P}(\alpha, q) = \bar{J} = \lim_{k \to \infty} \frac{1}{k} \sum_{p=0}^{k-1} P^k. \tag{6}$$

By Corollary 2 from Theorem 6, at a vanishingly small success probability q, the transition probabilities in the geometric observation model are given by the matrix $\bar{J}$ of maximum out forests of any weighted digraph to which this chain is related.

4. REPRESENTATIONS OF FOREST MATRICES VIA THE KIRCHHOFF MATRIX AND THEIR CONSEQUENCES

In this section, we represent the matrices Q_k of out forests with k arcs as polynomials of the Kirchhoff matrix L (Theorem 7). This allows one to obtain alternative proofs of Theorems 2–4 and to represent the matrix $Q(\tau) = (I + \tau L)^{-1}$ as a polynomial of L (Theorem 8). Proposition 4 gives an easy way to calculate $Q_k,\ k = 1, \ldots, n - v$, and $\bar{J}$.

By σ_k we denote the total weight of all out forests of Γ with k arcs: $\sigma_k = \varepsilon(\mathcal{F}_k),\ k = 0, \ldots, n-v$.

Proposition 3. *For any weighted digraph and any* $k = 0, \ldots, n - v$,

$$Q_{k+1} = \sigma_{k+1} I - L Q_k. \tag{7}$$

Observe that since the weight of the empty set is 0, we have $Q_{n-v+1} = 0$ and $\sigma_{n-v+1} = 0$.

Taking the traces on the left-hand side and the right-hand side of (7) and using the fact that

$$\operatorname{tr}(Q_k) = (n - k)\sigma_k, \quad k = 0, \ldots, n - v + 1 \tag{8}$$

(because every out forest with k arcs has $n - k$ roots), we deduce

$$\sigma_{k+1} = \frac{\operatorname{tr}(L Q_k)}{k + 1}, \quad k = 0, \ldots, n - v. \tag{9}$$

Substituting (9) in (7) provides

Proposition 4. *For every weighted digraph,*

$$Q_{k+1} = \frac{\operatorname{tr}(L Q_k)}{k + 1} I - L Q_k, \quad k = 0, \ldots, n - v. \tag{10}$$

Identity (10) enables one to recursively determine the matrices $Q_k,\ k = 0, \ldots, n - v$, and $\bar{J}$, starting with $Q_0 = I$. Note that this procedure essentially coincides with Faddeev's algorithm [19] for the computation of the characteristic polynomial as applied to L. Thus, the matrices involved in Faddeev's method are precisely Q_k.

From Proposition 3, it follows

Theorem 7. *For any weighted digraph and any* $k = 0, \ldots, n - v$,

$$Q_k = \sum_{i=0}^{k} \sigma_{k-i} (-L)^i. \tag{11}$$

Corollary 1 from Theorem 7. *For every weighted digraph, matrices* Q_k, $k = 0, \ldots, n - v$, *commute with all matrices with which* L *commutes, in particular, with* L, $\bar{J}$, $Q(\tau)$, *and each other.*

Lemma 1. *For any* $k = 0, \ldots, n - v$, *every row sum of* $L Q_k$ *is* 0.

From Proposition 3 and Lemma 1, it follows

Proposition 5. *The matrices* $L Q_k$, $k = 0, \ldots, n-v$, *are the Kirchhoff matrices of some weighted digraphs.*

Corollary 2 from Theorem 7. *For any weighted digraph,* $LQ_{n-v} = Q_{n-v}L = 0.$

In view of Definition 3, this corollary is equivalent to Theorem 3. Thus, we get a new proof of this theorem.

Consider the matrices
$$\bar{J}_k = \sigma_k^{-1} Q_k, \quad k = 0, \ldots, n-v. \tag{12}$$
In particular, $\bar{J}_0 = I$ and $\bar{J}_{n-v} = \bar{J}$.

Making use of the last corollary, we obtain

Corollary 3 from Theorem 7. *For any* $k \in \{1, \ldots, n-v\}$, $\bar{J}_k \bar{J} = \bar{J} \bar{J}_k = \bar{J}$. *In particular,* $\bar{J}_{n-v}\,\bar{J} = \bar{J}^2 = \bar{J}$. *Moreover,* $Q(\tau)\bar{J} = \bar{J}Q(\tau) = \bar{J}$ *for every* $\tau > 0$.

This corollary provides a new proof of Theorem 2.

By virtue of Proposition 3 and Corollary 1 from Theorem 7, the matrices $\bar{J}_k$ are connected as follows:
$$\bar{J}_{k+1} = I - \frac{\sigma_k}{\sigma_{k+1}} \bar{J}_k L, \quad k = 0, \ldots, n-v-1, \tag{13}$$
and, by Lemma 1, each their row is unity. The entries of $\bar{J}_k$ are nonnegative by definition, thus, we obtain

Proposition 6. *For every weighted digraph* Γ, *matrices* $\bar{J}_k$, $k = 0, \ldots, n-v$, *are stochastic.*

Completing Proposition 3 with the obvious equality $Q_0 = I = \sigma_0 I$ gives
$$\begin{cases} Q_0 = \sigma_0 I, \\ Q_1 + LQ_0 = \sigma_1 I, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ Q_{n-v} + LQ_{n-v-1} = \sigma_{n-v} I. \end{cases} \tag{14}$$

Add up these equations and, using Corollary 2 from Theorem 7, substitute $(I+L)Q_{n-v}$ for Q_{n-v}:
$$(I+L)Q_0 + (I+L)Q_1 + \ldots + (I+L)Q_{n-v} = \Big(\sum_{k=0}^{n-v} \sigma_k\Big) I.$$

Making use of the nonsingularity of $I+L$ (Theorem 4) and the notation $s = \varepsilon(\mathcal{F}) = \sum_{k=0}^{n-v} \sigma_k$, we obtain
$$\sum_{k=0}^{n-v} Q_k = s(I+L)^{-1}, \tag{15}$$
which provides

Corollary from Proposition 3. *For any weighted digraph,*
$$Q(1) = (I+L)^{-1} = s^{-1} \sum_{k=0}^{n-v} Q_k.$$

This statement coincides with the matrix-forest theorem for digraphs [14] and with Theorem 4 in the case of $\tau = 1$. Accordingly, we obtain a new proof of the matrix-forest theorem.

By means of Theorem 7, the matrices $Q(\tau) = (I+\tau L)^{-1} = s^{-1} \sum_{k=0}^{n-v} \tau^k Q_k$ including $Q(1) = (I+L)^{-1}$ can be represented as polynomials of L.

Theorem 8. *For any weighted digraph,*

$$Q(1) = (I + L)^{-1} = s^{-1} \sum_{i=0}^{n-v} s_{n-v-i}(-L)^i, \tag{16}$$

where $s_k = \sum\limits_{j=0}^{k} \sigma_j$ *is the total weight of out forests of* Γ *with at most* k *arcs,* $k = 0, \ldots, n - v$.

Corollary from Theorem 8. *For any weighted digraph and any* $\tau > 0$,

$$Q(\tau) = (I + \tau L)^{-1} = s^{-1}(\tau) \sum_{i=0}^{n-v} s_{n-v-i}(\tau)(-\tau L)^i, \tag{17}$$

where $s_k(\tau) = \sum\limits_{j=0}^{k} \tau^j \sigma_j$, $k = 0, \ldots, n - v$.

Note that $s(I+L)^{-1}$ is the adjugate (the transposed matrix of cofactors) of $I+L$; $s(\tau)\,(I+\tau L)^{-1}$ is the same for $I+\tau L$. Theorem 8 and the above corollary provide representations for these matrices as polynomials of L:

$$\begin{aligned} s\,(I + L)^{-1} &= \sum_{i=0}^{n-v} s_{n-v-i}(-L)^i, \\ s(\tau)\,(I + \tau L)^{-1} &= \sum_{i=0}^{n-v} s_{n-v-i}(\tau)\,(-\tau L)^i. \end{aligned} \tag{18}$$

Remark 1. Since LQ_k the is Kirchhoff matrix of some weighted digraph (Proposition 5), all its principal minors are nonnegative (by Theorem 6 in [2]). Therefore, all LQ_k are singular M-matrices (see, e.g., item (A1) of Theorem 4.6 in [20]). Alternatively, this can be concluded from the nonnegativity of the real parts of the eigenvalues (see Proposition 9 below) and the nonpositivity of off-diagonal elements of L (item (F12) of Theorem 4.6 in [20]). It follows from the representation $\sigma_{k+1} I - Q_{k+1} = LQ_k$ (Proposition 3) of the singular M-matrix LQ_k that $\sigma_{k+1} = \rho(Q_{k+1})$, i.e., σ_{k+1} is the spectral radius of Q_{k+1}, $k = 0, \ldots, n - v - 1$. This also follows from Proposition 6 (see (12)).

5. ON SOME LINEAR TRANSFORMATIONS RELATED TO DIGRAPHS

For a matrix $A \in \mathbb{R}^{n\times n}$, by $\mathbf{A}$ we denote the linear transformation $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^n$ induced by A with respect to the standard basis of $\mathbb{R}^n$: $\mathbf{A}(\mathbf{x}) = A\mathbf{x}$. $\mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A})$ will designate the range and the null space of $\mathbf{A}$, respectively.

As has been seen in [13], the dimensions of $\mathcal{R}(\bar{\mathbf{J}})$ and $\mathcal{R}(\mathbf{L}^T)$ are v and $n - v$, respectively. Furthermore, $\mathcal{R}(\mathbf{L}^T) \cap \mathcal{R}(\bar{\mathbf{J}}) = \{\mathbf{0}\}$ and, since the dimensions of $\mathcal{R}(\mathbf{L}^T)$ and $\mathcal{R}(\bar{\mathbf{J}})$ sum to n, $\mathbb{R}^n$ decomposes to the direct sum of $\mathcal{R}(\mathbf{L}^T)$ and $\mathcal{R}(\bar{\mathbf{J}})$:

$$\mathbb{R}^n = \mathcal{R}(\mathbf{L}^T) \dot{+} \mathcal{R}(\bar{\mathbf{J}}). \tag{19}$$

Since $L\bar{J} = 0$ (Theorem 3), we get $\mathcal{N}(\mathbf{L}) = \mathcal{R}(\bar{\mathbf{J}})$ and $\mathcal{N}(\bar{\mathbf{J}}^T) = \mathcal{R}(\mathbf{L}^T)$, thus, the sum (19) is orthogonal.

Similarly, in view of $\bar{J}L = 0$, the orthogonal decomposition

$$\mathbb{R}^n = \mathcal{R}(\mathbf{L}) \dot{+} \mathcal{R}(\bar{\mathbf{J}}^T)$$

holds along with $\mathcal{R}(\mathbf{L}) \cap \mathcal{R}(\bar{\mathbf{J}}^T) = \{\mathbf{0}\}$, $\mathcal{N}(\bar{\mathbf{J}}) = \mathcal{R}(\mathbf{L})$, and $\mathcal{N}(\mathbf{L}^T) = \mathcal{R}(\bar{\mathbf{J}}^T)$.

In accordance with (19), every vector $\mathbf{u} \in \mathbb{R}^n$ is uniquely represented as $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1 \in \mathcal{R}(\mathbf{L}^T) = \mathcal{N}(\bar{\mathbf{J}}^T)$ and $\mathbf{u}_2 \in \mathcal{R}(\bar{\mathbf{J}}) = \mathcal{N}(\mathbf{L})$. For every $\mathbf{u} \neq \mathbf{0}$, we have $(L + \bar{J}^T)\mathbf{u} = (L + \bar{J}^T)\mathbf{u}_1 + (L + \bar{J}^T)\mathbf{u}_2 = L\mathbf{u}_1 + \bar{J}^T\mathbf{u}_2$. If $L\mathbf{u}_1 + \bar{J}^T\mathbf{u}_2 = 0$, then, since $\mathcal{R}(\mathbf{L}) \cap \mathcal{R}(\bar{\mathbf{J}}^T) = \{\mathbf{0}\}$, we have $L\mathbf{u}_1 = \bar{J}^T\mathbf{u}_2 = \mathbf{0}$, whence, by $\mathcal{N}(\mathbf{L}) \cap \mathcal{N}(\bar{\mathbf{J}}^T) = \{\mathbf{0}\}$, $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$ results. Therefore, the dimension of the range (rank) of $\mathbf{Z} = \mathbf{L} + \bar{\mathbf{J}}^T$ is n. Thus, we obtain

Theorem 9. *For any weighted digraph Γ, the matrix $Z = L + \bar{J}^T$ is nonsingular.*

We will also need the nonsingularity of $L + \bar{J}$.

Theorem 10. *For any weighted digraph Γ, the matrix $L + \bar{J}$ is nonsingular.*

Corollary from Theorem 10. *For any weighted digraph and any $\alpha \neq 0$, the matrix $L + \alpha\,\bar{J}$ is nonsingular.*

It follows from $\bar{J}^2 = \bar{J}$ (Theorem 2) that every nonzero columns of $\bar{J}$ is an eigenvector of $\bar{J}$ associated with the eigenvalue 1. Hence, for any $\mathbf{u} \in \mathcal{R}(\bar{\mathbf{J}})$, $\bar{\mathbf{J}}(\mathbf{u}) = \mathbf{u}$ holds, therefore, $\mathcal{R}(\bar{\mathbf{J}})$ is exactly the subspace of fixed vectors of $\bar{\mathbf{J}}$.

6. THE MOORE-PENROSE AND GROUP INVERSES OF THE KIRCHHOFF MATRIX

In this section, we obtain some expressions for the Moore-Penrose generalized inverse and the group inverse of the Kirchhoff matrix L. The *Moore-Penrose generalized inverse* of a rectangular complex matrix A is the unique matrix X such that

(1) $AXA = A$,

(2) $XAX = X$,

(3) $(AX)^* = AX$,

(4) $(XA)^* = XA$,

where $(AX)^*$ and $(XA)^*$ are the conjugate transposes (Hermitian adjoints) of AX and XA, respectively.

For any matrix A, the Moore-Penrose generalized inverse, A^+, does exist and is unique. If A is nonsingular, then A^+ coincides with A^{-1}.

The Moore-Penrose inverses are of theoretical and practical interest. The latter is because A^+ provides the normal pseudosolution of the inconsistent equation $A\mathbf{x} = \mathbf{b}$: it is $\mathbf{x} = A^+\mathbf{b}$. The normal pseudosolution is a vector of the minimum length that minimizes the length of $A\mathbf{x} - \mathbf{b}$ (the minimum norm least-squares solution). As applied to Laplacian matrices, such solutions, among others, were considered for some preference aggregation problems (more specifically, estimation from paired comparisons) [23], in constructing geometrical representations for systems modelled by graphs [24], in the analysis of social networks, and cluster analysis.

The group inverses are no less important (see, e.g., [22]). A matrix X is the *group inverse* of a square matrix A, if X satisfies the conditions (1) and (2) in the definition of Moore-Penrose generalized inverse and also

(5) $AX = XA$.

The group inverse of A is denoted by $A^{\#}$. Generally, group inverses need not exist, but if such a matrix exists, then it is unique, but $A^+ = A^{\#}$ is not necessary.

If L is symmetric (in particular, this is the case for symmetric digraphs, which can be identified with undirected graphs), then the matrix $(L + \alpha\,\bar{J})^{-1} - \alpha^{-1}\,\bar{J}$ (with any $\alpha > 0$) is [15] the Moore-

Penrose generalized inverse and the group inverse of L. Moreover, the latter is true for every digraph.

Theorem 11. *For every weighted digraph and any $\alpha \neq 0$,*

$$L^{\#} = (L + \alpha \bar{J})^{-1} - \alpha^{-1} \bar{J} \tag{20}$$

and

$$L^{\#} L = L L^{\#} = I - \bar{J}.$$

As well as in the case of undirected graphs, $L^{\#} = (\ell_{ij}^{\#})$ can be obtained via a passage to the limit.

Proposition 7. *For every weighted digraph,*

$$L^{\#} = \lim_{\tau \to \infty} \tau \Big(Q(\tau) - \bar{J} \Big).$$

We now express $L^{\#}$ in terms of the normalized matrices of out forests $\bar{J}_{n-v-1}$ and $\bar{J}_{n-v} = \bar{J}$ (see (12)). The following proposition is an analogue of Theorem 3 in [15].

Proposition 8. *For every weighted digraph,*

$$L^{\#} = \frac{\sigma_{n-v-1}}{\sigma_{n-v}} \left(\bar{J}_{n-v-1} - \bar{J} \right).$$

Because of the nonsymmetry of $I - \bar{J} = L^{\#} L = L L^{\#}$, $L^{\#}$ is not generally the Moore-Penrose generalized inverse of L for digraphs. To obtain an explicit formula for L^{+}, consider the matrix $Z = L + \bar{J}^{T}$ which, by Theorem 9, is nonsingular. Using the identity $L\bar{J} = 0$ (Theorem 3), we obtain

$$(Z^{T})^{-1} Z^{-1} = (Z Z^{T})^{-1} = (\bar{J}^{T} \bar{J} + L L^{T})^{-1}.$$

Lemma 2. *For every weighted digraph, $(ZZ^{T})^{-1}$ commutes with LL^{T} and $\bar{J}^{T}\bar{J}$.*

Matrices LL^{T}, $\bar{J}^{T}\bar{J}$, and $(ZZ^{T})^{-1}$ are symmetric. The product of two symmetric matrices is symmetric iff they are commuting [25]. This implies the following corollary.

Corollary from Lemma 2. *For every weighted digraph, the matrices $LL^{T}(ZZ^{T})^{-1}$ and $\bar{J}^{T}\bar{J}(ZZ^{T})^{-1}$ are symmetric.*

These facts are useful for the proof of the following theorem.

Theorem 12. *For every weighted digraph, the matrix $L^{T}(ZZ^{T})^{-1} = L^{T}(\bar{J}^{T}\bar{J} + LL^{T})^{-1}$ is the Moore-Penrose generalized inverse of L.*

7. ON THE GERŠGORIN REGION AND ANNIHILATING POLYNOMIALS FOR THE KIRCHHOFF MATRIX

By the Geršgorin theorem (see, e.g., [25]), the eigenvalues of a matrix A belong to the union $G(A)$ of n discs:

$$G(A) = \bigcup_{i=1}^{n} \Big\{ z \in \mathbb{C} \,\Big|\, |z - a_{ii}| \le R_i'(A) \Big\}, \tag{21}$$

where $\mathbb{C}$ is the complex field and $R'_i(A) = \sum_{j\neq i} |a_{ij}|$, $i = 1,\dots,n$, are the deleted absolute row sums of A.

Since $R'_i(L) = \ell_{ii}$ holds, (21) can be represented as follows:

$$G(L) = \bigcup_{i=1}^{n} \Big\{ z \in \mathbb{C} \;\Big|\; |z - \ell_{ii}| \le \ell_{ii} \Big\}. \tag{22}$$

Hence, we have

Proposition 9. (1) *The real part of every eigenvalue of* L *is nonnegative*: *every Geršgorin disc belongs to the right coordinate half-plane*;

(2) *the intersection of all Geršgorin discs contains zero*;

(3) $G(L) = \Big\{ (z+1) \max\limits_{1\le i\le n} \ell_{ii} \;\Big|\; |z| \le 1 \Big\}$.

Obviously, the intersection of all Geršgorin discs consists of zero iff the digraph contain an undominated vertex.

Consider the characteristic polynomial of L:

$$p_L(\lambda) = \sum_{i=0}^{n} (-1)^i E_i(L) \lambda^{n-i},$$

where $E_i(L)$ is the sum of all principal minors of order i. By Theorem 6 in [2], $E_i(L) = \sigma_i$ for every $i = 1,\dots,n$. Since every principal minor of order greater than $n-v$ is zero, we have $p_L(\lambda) = (-1)^{n-v}\lambda^v \sum_{i=0}^{n-v} \sigma_i\,(-\lambda)^{n-v-i} = \lambda^v \sum_{i=0}^{n-v} (-1)^i \sigma_i\,\lambda^{n-v-i}$.

Proposition 10. $p'_L(\lambda) = \lambda \sum_{i=0}^{n-v} \sigma_{n-v-i}\,(-\lambda)^i$ *is an annihilating polynomial for* L.

8. ACCESSIBILITY VIA FORESTS AND DENSE FORESTS IN DIGRAPHS

8.1. Forest accessibility

The entries of $Q(\tau)$ measure the proximity of the vertices of an undirected multigraph [14, 15]. The matrix $\bar{J}^T = \lim_{\tau\to\infty} Q^T(\tau)$ was analyzed in [13] as the matrix of limiting accessibilities of a multidigraph. Here, we study the matrix $P_1(\tau) = Q^T(\tau)$ with $\tau > 0$ as an accessibility measure for digraph vertices. By Theorem 4, the (i,j)-entry of this matrix is the total weight of out forests that "connect" i with j in the digraph where the weights of all arcs are multiplied by τ. Along with $P_1(\tau)$, we consider the matrix of in forests $P_2(\tau)$. Its (i,j)-entry is the total weight of in forests (of the modified digraph) where j is a sink and i belongs to a tree converging to j.

The following definition is formulated for an arbitrary vertex accessibility measure (formally, every square matrix of order n or, more precisely, the corresponding matrix-valued function of a digraph can be considered as such a measure). A measure P_2 is called to be *dual* to a measure P_1 if under the reversal of all arcs in an arbitrary digraph (provided that the weights of the arcs are preserved), the matrix of P_2 for the modified digraph coincides with P_1^T calculated for the initial digraph. It follows from this definition that P_2 is dual to P_1 if and only if P_1 is dual to P_2. In [15], three self-dual accessibility measures were studied.

Let us check the satisfaction of the characteristic conditions listed below for $P_1(\tau)$ and $P_2(\tau)$. Triangle inequality for accessibility measures requires the symmetry of the corresponding matrix (see, e.g., [26]). For that reason, we will check this condition for $P_3(\tau) = (P_1(\tau) + P_2(\tau) + P_1^T(\tau) + P_2^T(\tau))/4$.

Nonnegativity. For any digraph Γ, $p_{ij} \geq 0$, $i, j \in V(\Gamma)$.

Diagonal maximality. For any digraph Γ and any distinct $i, j \in V(\Gamma)$,
(1) $p_{ii} > p_{ij}$ and
(2) $p_{ii} > p_{ji}$ hold.

Disconnection condition. For any digraph Γ and any $i, j \in V(\Gamma)$, $p_{ij} = 0$ if and only if j is unreachable from i.

Triangle inequality for accessibility measures. For any digraph Γ and any $i, j, k \in V(\Gamma)$, $p_{ij} + p_{ik} - p_{jk} \leq p_{ii}$ holds. If, in addition, $j = k$ and $i \neq j$, then the inequality is strict.

Transit property. For any digraph Γ and any $i, k, t \in V(\Gamma)$, if Γ includes a path from i to k, $i \neq k \neq t$, and every path from i to t contains k, then (1) $p_{ik} > p_{it}$; (2) $p_{kt} > p_{it}$.

Monotonicity. Suppose that the weight of some arc ε_{kt}^p in a digraph Γ increases. Then:
(1) $\Delta p_{kt} > 0$ and for any $i, j \in V(\Gamma)$, $(i, j) \neq (k, t)$ implies $\Delta p_{kt} > \Delta p_{ij}$;
(2) For any $i \in V(\Gamma)$, if there is a path from k to t, and each path from k to i includes t, then (a) $\Delta p_{kt} > \Delta p_{ki}$ and (b) $\Delta p_{ki} > \Delta p_{ti}$;
(3) For any $i \in V(\Gamma)$, if there is a path from i to k and every path from i to t includes k, then (a) $\Delta p_{kt} > \Delta p_{it}$ and (b) $\Delta p_{it} > \Delta p_{ik}$.

The results of testing $P_1(\tau)$, $P_2(\tau)$, and $P_3(\tau)$ are collected in the following proposition.

Proposition 11. *The measures $P_1(\tau)$ and $P_2(\tau)$ are dual to each other for every $\tau > 0$. They satisfy nonnegativity, reversal property, disconnection condition, the first part of item* 1, *and item* 2 *of monotonicity. Moreover, $P_1(\tau)$ satisfies items* 1 *of diagonal maximality and transit property; $P_2(\tau)$ satisfies items* 2 *of these conditions. With respect to the remaining statements of monotonicity, $P_1(\tau)$ satisfies items* 2 *and* 3*b, whereas $P_2(\tau)$ satisfies items* 3 *and* 2*b, and they both violate the second part of item* 1. *Furthermore, $P_1(\tau)$ breaks item* 3*a of monotonicity and item* 2 *of transit property, whereas $P_2(\tau)$ breaks item* 2*a of monotonicity and item* 1 *of transit property. Triangle inequality for $P_3(\tau)$ is not satisfied.*

As was noted in [13], the limiting accessibility $P = \bar{J}^T$ of a digraph does not completely correspond to the general concept of proximity. Notice that *disconnection condition*, which is satisfied for the limiting accessibility in one side only, is completely fulfilled for $P_1(\tau)$ and $P_2(\tau)$. Moreover, $P_1(\tau)$ and $P_2(\tau)$ obey a number of conditions which are satisfied by the limiting accessibility in the nonstrict form only.[3]

8.2. Accessibility via dense forests

Now we consider a measure which is intermediate between the limiting accessibility (which depends on Q_{n-v} only) and the forest accessibility $Q(\tau)$ (which is a weighted sum of all matrices Q_k). This new measure is determined by the matrices Q_{n-v-1} and Q_{n-v} (or, equivalently, by the matrices

[3] By the nonstrict form of a condition we mean the result of substituting nonstrict inequalities ($\geq$ and $\leq$) for the strict ones ($>$ and $<$) in it.

$\bar{J}_{n-v-1}$ and $\bar{J}_{n-v} = \bar{J}$, which also determine $L^{\#}$ as stated in Proposition 8). This measure can be also obtained by the inversion of $L + \alpha\,\bar{J}$ with some values of α.

Thus, consider the matrices $R(\alpha) = (r_{ij}) = (L + \alpha\,\bar{J})^{-1}$ with $\alpha > 0$. Using Theorem 11 and Proposition 8, we have

$$(L + \alpha\,\bar{J})^{-1} = L^{\#} + \alpha^{-1}\,\bar{J} = \frac{\sigma_{n-v-1}}{\sigma_{n-v}}\bar{J}_{n-v-1} + \left(\alpha^{-1} - \frac{\sigma_{n-v-1}}{\sigma_{n-v}}\right)\bar{J}. \tag{23}$$

If $0 < \alpha < \frac{\sigma_{n-v}}{\sigma_{n-v-1}}$, then, by (23), $(L + \alpha\,\bar{J})^{-1}$ is the sum of Q_{n-v-1} and Q_{n-v} with positive coefficients. Spanning rooted forests (of an undirected multigraph) with $n - v$ or $n - v - 1$ arcs are called in [15] *dense forests*, and the undirected counterpart of the accessibility measure (23) with $0 < \alpha < \frac{\sigma_{n-v}}{\sigma_{n-v-1}}$ is called *accessibility via dense forests*.

Consider two accessibility measures for digraphs: $P_1(\alpha) = R^{T}(\alpha)$, *accessibility via dense diverging forests* and $P_2(\alpha)$, *accessibility via dense converging forests*.

An important property of the set of dense diverging forests is as follows.

Proposition 12. 1. *For any vertex $i \in V(\Gamma)$, there exists an out forest in $\mathcal{F}_{n-v-1}$ where i is a root.* 2. *For any path (chain subgraph) in Γ, there exists an out forest in $\mathcal{F}_{n-v-1} \cup \mathcal{F}_{n-v}$ that contains this path.*

A similar proposition is true for converging forests. At the same time, the set of maximum out forests $\mathcal{F}_{n-v}$ and the set of maximum in forests do not have this property. For example, on Fig. 1 in [13], no maximum out forest contains arc $(4, 2)$.

We now test $P_1(\alpha)$ and $P_2(\alpha)$. Similar to the previous consideration, triangle inequality for accessibility measures will be checked for the index $P_3(\alpha) = (P_1(\alpha) + P_1^{T}(\alpha) + P_2(\alpha) + P_2^{T}(\alpha))/4$, since this inequality requires the symmetry of the corresponding matrix.

Proposition 13. *For any $\alpha \in]\,0, \sigma_{n-v}/\sigma_{n-v-1}[$, the measures $P_1(\alpha)$ and $P_2(\alpha)$ are dual to each other. They satisfy nonnegativity and disconnection condition. Moreover, the nonstrict versions of items 1 of diagonal maximality and transit property are satisfied by $P_1(\alpha)$, and items 2 of these conditions by $P_2(\alpha)$. Both measures violate monotonicity. Triangle inequality for accessibility measures is not true for $P_3(\alpha)$.*

CONCLUSION

The normalized matrices of out forests are stochastic and determine the transition probabilities in the geometric observation model applied to the Markov chains related to the digraph under consideration. Various matrices of forests can be represented by simple polynomials of the Kirchhoff matrix. The Moore-Penrose generalized inverse L^{+} and the group inverse $L^{\#}$ of the Kirchhoff matrix L can be explicitly represented via L and the normalized matrix $\bar{J}$ of digraph's maximum out forests. The matrices of diverging and converging forests characterize the pairwise accessibility of vertices. These and other results enable one to consider the matrices of spanning forests as a useful tool for the analysis of digraph's structure.

APPENDIX

Proof of Proposition 1. 1. At first, let $k \le k' < n$. Γ is constructed as follows. We draw a diverging star rooted at the first vertex and having $k' - k$ leaf vertices. Also, we draw a path diverging from the root of the star and containing, in addition to the root, $n - k' \ge 1$ vertices that are not included in the star. The remaining $k-1$ vertices are left isolated. Then $v = 1+(k-1) = k$ and $v' = 1+(k'-k)+(k-1) = k'$, as required.

If $k' \le k < n$, then we draw a star converging to the first vertex and having $k-k'$ other vertices. Also, we draw a path diverging to the center of the star and containing, in addition to the sink, $n-k \ge 1$ vertices that are not included in the star. The remaining $k'-1$ vertices are left isolated. As well as in the first case, we have $v = k$ and $v' = k'$. The second statement is obvious.

Proof of Theorem 6. Since the spectral radius of P is 1, we have

$$\sum_{k=0}^{\infty}\Big((1-q)P\Big)^k = \Big(I-(1-q)P\Big)^{-1}.$$

Using the formula of total probability and equations (3)–(5), we obtain

$$\begin{aligned}\widetilde{P}(\alpha,q) &= \sum_{k=0}^{\infty} p(k)P^k = \sum_{k=0}^{\infty} q(1-q)^k P^k = q\Big(I-(1-q)P\Big)^{-1} \\ &= q\Big(I-(1-q)(I-\alpha L)\Big)^{-1} = q\Big(qI+(1-q)\alpha L\Big)^{-1} \\ &= \left(I+\frac{(1-q)\alpha}{q}L\right)^{-1} = (I+\tau L)^{-1} = Q(\tau).\end{aligned} \tag{24}$$

Proof of Proposition 3. In [13], we used the notion of weighted 2-digraph: it is a multidigraph with arc multiplicities no more than two. The weight of a 2-digraph is the product of the weights of its arcs. For a weighted digraph H and its vertices $u, w \in V(H)$, by $H+(u,w)$ we denote the 2-digraph with vertex set $V(H)$ and the arc multiset obtained from $E(H)$ by the increment of the multiplicity of (u,w) by 1. Similarly, if H is a 2-digraph and $u, w \in V(H)$, then by $H' = H-(u,w)$ we denote the 2-digraph that differs from H in the multiplicity of (u,w) only: $n'((u,w)) = \max(n((u,w))-1, 0)$.

Now introduce the following notation. Let $\mathcal{F}_k^{j\to s}+(\ell,i) = \{F_k^{j\to s}+(\ell,i) \mid F_k^{j\to s} \in \mathcal{F}_k^{j\to s}\}$. For all $i \ne j$, by $\mathcal{F}_{k,i}^{j\to s}$ denote the set of out forests with k arcs where i is a root and s belongs to a tree diverging from j. Obviously, in such digraphs, s is unreachable from i whenever $i \ne j$. By $\mathcal{F}_{k,\bar{i}}^{j\to s}$ we denote the set of all forests with k arcs where i *is not* a root, and s belongs to a tree diverging from j. These definitions induce the matrices $(q_{sj,\bar{i}}^k)$ and $(q_{sj,i}^k)$ with elements

$$\begin{aligned} q_{sj,\bar{i}}^k &= \varepsilon(\mathcal{F}_{k,\bar{i}}^{j\to s}), \\ q_{sj,i}^k &= \varepsilon(\mathcal{F}_{k,i}^{j\to s}), \quad s, j = 1,\dots,n. \end{aligned}$$

Denote by $q_{\bar{i}j}^k$ the total weight of all diverging forests where i is unreachable from j. For all $i, j \in V(\varGamma)$, $q_{jj}^k = q_{ij}^k + q_{\bar{i}j}^k$ holds.

Lemma 3. *For any weighted digraph and all* $i, j = 1,\dots,n$, *we have*:

(1) *If* $s \ne p$, $i \ne j$, *and* $(s,i),(p,i) \in E(\varGamma)$, *then* $(\mathcal{F}_{k,i}^{j\to s}+(s,i)) \bigcap (\mathcal{F}_{k,i}^{j\to p}+(p,i)) = \emptyset$;

(2) $\bigcup_{(s,i)\in E(\varGamma)} (\mathcal{F}_{k,i}^{j\to s}+(s,i)) = \mathcal{F}_{k+1}^{j\to i}$.

Proof of Lemma 3. The first statement follows from the definition of out forest. Let us prove the second statement. Suppose that $(s,i) \in E(\varGamma)$ and $\mathcal{F}_{k,i}^{j\to s} \ne \emptyset$. Then s is unreachable from i in every forest $F_k \in \mathcal{F}_{k,i}^{j\to s}$. After the addition of (s,i) to F_k, we obtain a diverging forest with $k+1$ arcs where i is reachable from j, i.e., $F_k + (s,i) \in \mathcal{F}_{k+1}^{j\to i}$.

Suppose now that $F_{k+1} \in \mathcal{F}_{k+1}^{j\to i}$. Let (p,i) be the unique arc directed to i in F_{k+1}. Since $F_{k+1} - (p,i) \in \mathcal{F}_{k,i}^{j\to p}$, we have $F_{k+1} \in \mathcal{F}_{k,i}^{j\to p} + (p,i) \subseteq \bigcup\limits_{(s,i)\in E(\Gamma)} (\mathcal{F}_{k,i}^{j\to s} + (s,i))$.

Corollary from Lemma 3. For any digraph and all $i,j = 1,\dots,n$ and $k = 0,\dots,n-v$,

$$q_{ij}^{k+1} = \varepsilon(\mathcal{F}_{k+1}^{j\to i}) = \varepsilon\Big(\bigcup_{(s,i)\in E(\Gamma)} (\mathcal{F}_{k,i}^{j\to s} + (s,i))\Big) = \sum_{(s,i)\in E(\Gamma)} \varepsilon(\mathcal{F}_{k,i}^{j\to s} + (s,i)) = \sum_{(s,i)\in E(\Gamma)} \varepsilon_{si} q_{sj,i}^{k}.$$

We now continue proving Proposition 3. Let $LQ_k = (a_{ij}^k)$. Then for all $i = 1,\dots,n$,

$$\begin{aligned} a_{ii}^k &= \sum_{s=1}^n \ell_{is} q_{si}^k = \sum_{s\neq i} \ell_{is} q_{si}^k + \ell_{ii} q_{ii}^k = \sum_{s\neq i} \ell_{is} q_{si}^k - \sum_{s\neq i} \ell_{is} q_{ii}^k = \sum_{s=1}^n \varepsilon_{si}(q_{ii}^k - q_{si}^k) \\ &= \sum_{s=1}^n \varepsilon_{si}(q_{\bar{s}i}^k + q_{si}^k - q_{si}^k) = \sum_{s=1}^n \varepsilon_{si} q_{\bar{s}i}^k = \sigma_{k+1} - q_{ii}^{k+1} \geq 0. \end{aligned} \tag{25}$$

The statement of Proposition 3 with respect to the diagonal entries of LQ_k is proved.

For $j \neq i$ we have

$$\begin{aligned} a_{ij}^k &= \sum_{s=1}^n \ell_{is} q_{sj}^k = \sum_{s\neq i} \ell_{is} q_{sj}^k - \sum_{s\neq i} \ell_{is} q_{ij}^k = \sum_{s=1}^n \varepsilon_{si} q_{ij}^k - \sum_{s=1}^n \varepsilon_{si} q_{sj}^k \\ &= \sum_{s=1}^n \varepsilon_{si} q_{ij}^k - \sum_{s=1}^n \varepsilon_{si}(q_{sj,\bar{i}}^k + q_{sj,i}^k) = \sum_{s=1}^n \varepsilon_{si} q_{ij}^k - \sum_{s=1}^n \varepsilon_{si} q_{sj,\bar{i}}^k - \sum_{s=1}^n \varepsilon_{si} q_{sj,i}^k \\ &= \varepsilon(\mathcal{G}_1) - \varepsilon(\mathcal{G}_2) - \sum_{s=1}^n \varepsilon_{si} q_{sj,i}^k, \end{aligned} \tag{26}$$

where $\varepsilon(\mathcal{G}_1) = \sum_{s=1}^n \varepsilon_{si} q_{ij}^k$, $\varepsilon(\mathcal{G}_2) = \sum_{s=1}^n \varepsilon_{si} q_{sj,\bar{i}}^k$, $\mathcal{G}_1$ is the multiset of 2-subgraphs obtained by the addition of all arcs $(s,i) \in E(\Gamma)$ to all possible forests in $\mathcal{F}_k^{j\to i}$, and $\mathcal{G}_2$ is the multiset of 2-subgraphs obtained by the addition of all arcs $(s,i) \in E(\Gamma)$ of Γ to all possible forests in $\mathcal{F}_{k,\bar{i}}^{j\to s}$. The multiset $\mathcal{G}_1$ consists of the pairs $(H, n_1(H))$ where $n_1(H) \geq 1$ is the multiplicity of H in $\mathcal{G}_1$; $\mathcal{G}_2$ consists of the pairs $(H, n_2(H))$ where $n_2(H)$ is the multiplicity of H in $\mathcal{G}_2$. The notation $H \in \mathcal{G}_1$ means here $n_1(H) > 0$.

Prove that $\mathcal{G}_1 = \mathcal{G}_2$. In every $H \in \mathcal{G}_1$, two arcs are directed to i: (s,i) and (p,i) such that $(p,i) \in E(F_k^{j\to i})$, since $j \neq i$. Consider the digraph $H - (p,i)$. It is a forest where i is not a root and j is the root of the tree that contains p. Then $H - (p,i) \in \mathcal{F}_{k,\bar{i}}^{j\to p}$ and $H \in \mathcal{F}_{k,\bar{i}}^{j\to p} + (p,i) \subseteq \mathcal{G}_2$. Conversely, suppose that $H = F_{k,\bar{i}}^{j\to s} + (s,i) \in \mathcal{G}_2$. Consider $H - (p,i)$ such that $(p,i) \in E(F_{k,\bar{i}}^{j\to s})$. We have $H - (p,i) \in \mathcal{F}_k^{j\to i}$, hence, $H = F_k^{j\to i} + (p,i) \in \mathcal{G}_1$. For the multisets $\mathcal{G}_1$ and $\mathcal{G}_2$, the multiplicities of all elements, $n_1(H)$ and $n_2(H)$, do not exceeded two. Otherwise, $H - (s,i)$ would not be a forest. Prove that for any H, $n_1(H) = 2$ iff $n_2(H) = 2$. Suppose that $H \in \mathcal{G}_1$ and $n_1(H) = 2$. Then there are vertices $s_1, s_2 \in V(\Gamma)$ such that $s_1 \neq s_2$, $E(H)$ contains (s_1,i) and (s_2,i), and $\{H - (s_1,i),\ H - (s_2,i)\} \subseteq \mathcal{F}_k^{j\to i}$. Since s_1 is reachable from j in $H - (s_2,i)$, s_1 is reachable from j in H too. Therefore, s_1 is reachable from j in $H - (s_1,i)$ as well. Similarly, s_2 is reachable from j in $H - (s_2,i)$. We obtain $H - (s_1,i) \subseteq \mathcal{F}_{k,\bar{i}}^{j\to s_1}$ and $H - (s_2,i) \subseteq \mathcal{F}_{k,\bar{i}}^{j\to s_2}$, consequently, $n_2(H) = 2$.

Suppose that $H \in \mathcal{G}_2$ and $n_2(H) = 2$. Then for some distinct vertices s_1 and s_2, there exist $F_{k,\bar{i}}^{j\to s_1} \in \mathcal{F}_{k,\bar{i}}^{j\to s_1}$ and $F_{k,\bar{i}}^{j\to s_2} \in \mathcal{F}_{k,\bar{i}}^{j\to s_2}$ such that $H = F_{k,\bar{i}}^{j\to s_1} + (s_1,i) = F_{k,\bar{i}}^{j\to s_2} + (s_2,i)$. Note that

(A) $F_{k,\bar{i}}^{j\to s_1}$ contains the arc (s_2, i), therefore, s_2 is not reachable from i in this forest. Then (B) s_2 is reachable from j in $F_{k,\bar{i}}^{j\to s_1}$, since otherwise s_2 would be reachable from j in $H = F_{k,\bar{i}}^{j\to s_1} + (s_1, i)$, which is false. By (A) and (B), $F_{k,\bar{i}}^{j\to s_1} \in \mathcal{F}_k^{j\to i}$, and similarly, $F_{k,\bar{i}}^{j\to s_2} \in \mathcal{F}_k^{j\to i}$. Hence, $n_1(H) = 2$.

By (26) and Corollary from Lemma 3, at $j \neq i$ we have

$$a_{ij}^k = -\sum_{s\neq i} \varepsilon_{si} q_{sj,i}^k = -q_{ij}^{k+1} \leq 0. \tag{27}$$

Proposition 3 is proved.

Proof of Theorem 7. Let us represent (14) as follows:

$$\begin{cases} Q_0 = \sigma_0\, I = I, \\ Q_1 = \sigma_1\, I - LQ_0, \\ \dots\dots\dots\dots\dots\dots\dots \\ Q_k = \sigma_k\, I - LQ_{k-1}, \\ \dots\dots\dots\dots\dots\dots\dots \\ Q_{n-v} = \sigma_{n-v}\, I - LQ_{n-v-1}. \end{cases} \tag{28}$$

By substituting each equation in the subsequent one, for all $k = 2, \dots, n - v$ we have

$$\begin{aligned} Q_k &= \sigma_k\, I - L\Big(\sigma_{k-1}\, I - \ldots - L(\sigma_1\, I - L\sigma_0\, I)\ldots\Big) \\ &= \sigma_k\, I - \sigma_{k-1}\, L + \ldots + (-1)^k \sigma_0\, L^k = \sum_{i=0}^{k} \sigma_{k-i}\, (-L)^i. \end{aligned}$$

Theorem 7 is proved.

Proof of Lemma 1. The ith row sum of $LQ_k = (a_{ij}^k)$ is

$$\sum_{j=1}^n a_{ij}^k = \sum_{j=1}^n \sum_{s=1}^n \ell_{is} q_{sj}^k = \sum_{s=1}^n \ell_{is} \sum_{j=1}^n q_{sj}^k = \sum_{s=1}^n \ell_{is} \sigma_k = 0.$$

Proof of Corollary 1 from Theorem 7. Multiplying both sides of (11) by any matrix that commutes with L and using distributivity and associativity of matrix operations, we get the required statement.

Proof of Corollary 2 from Theorem 7. Consider (25) at $k = n - v$. By virtue of Proposition 2, for every $(s, i) \in E(\Gamma)$, $\varepsilon_{si} q_{\bar{s}i}^{n-v} = 0$ holds, i.e., $a_{ii}^{n-v} = 0$ for all $i \in \{1, \dots, n\}$. In this way, Corollary 2 is derived from $a_{ii}^{n-v} = 0$, inequality (27), Lemma 1, and Corollary 1 from Theorem 7.

Proof of Corollary 3 from Theorem 7. Postmultiplying both sides of (11) by $\bar{J}$ and using $L\bar{J} = 0$ (Corollary 2 from Theorem 7) provides $Q_k \bar{J} = \sigma_k\, \bar{J}$. By commutativity, $\bar{J}\bar{J}_k = \bar{J}_k \bar{J} = \bar{J}$ holds. Using Theorem 4, we also get $Q(\tau)\bar{J} = \bar{J}Q(\tau) = \bar{J}$ for any $\tau > 0$.

Proof of Theorem 8. Substituting (11) in (15) gives

$$\begin{aligned} s\,(I + L)^{-1} &= \sum_{k=0}^{n-v} \sum_{i=0}^{k} \sigma_{k-i} (-L)^i = \sum_{i=0}^{n-v} \sum_{k=i}^{n-v} \sigma_{k-i} (-L)^i \\ &= \sum_{i=0}^{n-v} \sum_{j=0}^{n-v-i} \sigma_j (-L)^i = \sum_{i=0}^{n-v} s_{n-v-i} (-L)^i. \end{aligned}$$

Proof of Corollary from Theorem 8. Observe that the digraph resulting from $\mathit{\Gamma}$ by multiplying the weights of all arcs by τ has the Kirchhoff matrix τL, and its total weight of out forests with j arcs is $\tau^j\sigma_j$. Hence, the required statement follows from Theorem 8.

Proof of Theorem 10. We first prove the following lemma.

Lemma 4. *If a maximum out forest of a digraph $\mathit{\Gamma}$ is a tree, i.e., the out forest dimension of $\mathit{\Gamma}$ is* 1, *then $L+\bar{J}$ is nonsingular.*

Proof of Lemma 4. Assume, on the contrary, that $\det(L+\bar{J})=0$. Then there exists a vector $\mathbf{b}=(b_1,\ldots,b_n)^T\neq\mathbf{0}$ such that $(L+\bar{J})^T\mathbf{b}=\mathbf{0}$, where $\mathbf{0}=(0,\ldots,0)^T$.

Since, for every digraph of out forest dimension 1, every column of $\bar{J}$ consists of equal entries (item 3 of Theorem 1), we obtain

$$\begin{aligned}(L+\bar{J})^T\mathbf{b}&=\left\|\begin{matrix}b_1\ell_{11}+\ldots+b_n\ell_{n1}+b_1\bar{J}_{11}+\ldots+b_n\bar{J}_{n1}\\ b_1\ell_{12}+\ldots+b_n\ell_{n2}+b_1\bar{J}_{12}+\ldots+b_n\bar{J}_{n2}\\ \dotfill\\ b_1\ell_{1n}+\ldots+b_n\ell_{nn}+b_1\bar{J}_{1n}+\ldots+b_n\bar{J}_{nn}\end{matrix}\right\|\\ &=\left\|\begin{matrix}b_1\ell_{11}+\ldots+b_n\ell_{n1}+\bar{J}_{11}(b_1+\ldots+b_n)\\ b_1\ell_{12}+\ldots+b_n\ell_{n2}+\bar{J}_{12}(b_1+\ldots+b_n)\\ \dotfill\\ b_1\ell_{1n}+\ldots+b_n\ell_{nn}+\bar{J}_{1n}(b_1+\ldots+b_n)\end{matrix}\right\|=\left\|\begin{matrix}0\\0\\ \vdots\\0\end{matrix}\right\|.\end{aligned}\tag{29}$$

Adding up the components of the vectors that form the last equality and using the identities $\sum\limits_{k=1}^{n}\bar{J}_{1k}=1$ (item 1 of Theorem 1) and $\sum\limits_{k=1}^{n}\ell_{ik}=0$, $i=1,\ldots,n$, we deduce $b_1+\ldots+b_n=0$. Replacing the last equation of (29) with this equality, we obtain the following system of equations in $b_1,\ldots,b_n$:

$$\begin{cases}b_1\ell_{11}+\ldots+b_n\ell_{n1} &=0\\ b_1\ell_{12}+\ldots+b_n\ell_{n2} &=0\\ \dotfill\\ b_1\ell_{1n-1}+\ldots+b_n\ell_{nn-1} &=0\\ b_1+\ldots+b_n &=0.\end{cases}\tag{30}$$

Let M be the transposed matrix of coefficients of the system (30). Expand the determinant of M in the last column (which consists of ones). By the matrix-tree theorem for digraphs (see, e.g., [1, 16]), if there exists a spanning tree diverging from i, then the cofactor of ℓ_{in} in L—which equals the cofactor of the ith entry of the last column of M—is positive, whereas, in the opposite case, it is zero. By the hypothesis of this lemma, the maximum out forests of $\mathit{\Gamma}$ are diverging trees. Therefore, the cofactor of at least one entry of the last column of M is positive. Hence, $\det M$ expanded as above is also positive. Thereby, $\operatorname{rank} M=n$, whence the unique solution of the system (30) is $b_1=\cdots=b_n=0$. This contradiction proves Lemma 4.

Suppose that $V_1,\ldots,V_p$ are the vertex sets of the weak components of $\mathit{\Gamma}$. Without loss of generality, we assume that the vertices of V_1 are numbered first, the vertices of V_2 are numbered next, etc. (at any other numeration, the corresponding permutation of the rows and columns preserves the rank of $L+\bar{J}$). By virtue of Theorem $2'$ in [13], $L+\bar{J}$ is a block diagonal matrix with p blocks, which is expressed in the following manner [25]:

$$L+\bar{J}=\oplus\sum_{s=1}^{p}A_{ss},\tag{31}$$

where submatrix A_{ss} corresponds to the sth weak component. A block diagonal matrix $L+\bar{J}$ is nonsingular if and only if every A_{ss} is nonsingular, moreover,

$$\det(L+\bar{J}) = \prod_{s=1}^{p} \det A_{ss}. \tag{32}$$

Let Γ_s be the restriction of Γ to V_s, $s \in \{1,\dots,p\}$; $L^{(s)}, Q^{(s)}_{n-v}$, and $\bar{J}^{(s)}$ will be the matrices constructed for Γ_s. Then

$$A_{ss} = L^{(s)} + \bar{J}^{(s)}\,. \tag{33}$$

Indeed, $L^{(s)}$ coincides with the sth block of L, whereas, by item 5 of Theorem 1, the sth block of Q_{n-v} is proportional to $Q^{(s)}_{n-v}$ (the proportionality factor being the total weight of all forests in Γ_{-V_s}). Consequently, the sth block of $\bar{J}$ coincides with $\bar{J}^{(s)}$.

By virtue of (32) and (33), it is sufficient to prove the statement of the theorem for the case of $p=1$. Let Γ consist of a single weak component. Suppose, without loss of generality, that $V(\Gamma)$ is indexed in such a way that the first numbers are attached to the vertices in K_1, the subsequent numbers to the vertices in K_2, etc. The last numbers are given to the vertices in $V(\Gamma)\backslash\widetilde{K}$. By item 2 of Theorem 1, $L+\bar{J}$ is a block lower triangular matrix with $v+1$ blocks: K_1 corresponds to the first block, K_v corresponds to the vth block; $V(\Gamma)\backslash\widetilde{K}$ corresponds to the last block.

The determinant of the block triangular matrix $L+\bar{J}$ is the product of the determinants of its diagonal blocks, whereas its rank is no less than the sum of ranks of the diagonal blocks (see, e.g., [25]). Note that, as well as in the case of weak components of Γ (see above), the block of $L+\bar{J}$ corresponding to an undominated knot K_i, coincides with the matrix $L(\Gamma_{K_i})+\bar{J}(\Gamma_{K_i})$ constructed for Γ_{K_i}. To demonstrate this, it suffices to use item 3 of Theorem 1. Every maximum out forest of an undominated knot is a diverging tree (out arborescence). Therefore, the nonsingularity of the diagonal blocks of $L+\bar{J}$ that correspond to the undominated knots follows from Lemma 4.

The $(v+1)$st block of $L+\bar{J}$ coincides with the corresponding block of L, since the last block of $\bar{J}$ is zero (by item 2 of Theorem 1). The last block of L is nonsingular. Indeed, by Theorem 6 in [2], its determinant is the weight of the set of out forests in Γ where $\widetilde{K}$ is the set of roots. This weight is strictly positive, since the indicated set of forests is nonempty: such forests can be obtained from the maximum out forests of Γ by the removal of all arcs between the vertices within $\widetilde{K}$.

Thus, the diagonal blocks of $L+\bar{J}$ are nonsingular, hence, $L+\bar{J}$ is nonsingular. The theorem is proved.

Proof of Corollary from Theorem 10. First, we prove the following lemma.

Lemma 5. *For every weighted digraph Γ of out forest dimension* 1 *and any $\alpha \neq 0$, the matrix $L+\alpha\,\bar{J}$ is nonsingular.*

Proof of Lemma 5. This lemma is proved by the same argument as Lemma 4 with the only difference that the analogue of (29) takes here a more general form:

$$(L+\alpha\,\bar{J})^T\mathbf{b} = \begin{Vmatrix} b_1\ell_{11}+\ldots+b_n\ell_{n1}+\alpha\bar{J}_{11}(b_1+\ldots+b_n) \\ b_1\ell_{12}+\ldots+b_n\ell_{n2}+\alpha\bar{J}_{12}(b_1+\ldots+b_n) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ b_1\ell_{1n}+\ldots+b_n\ell_{nn}+\alpha\bar{J}_{1n}(b_1+\ldots+b_n) \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Vmatrix}.$$

To complete the proof of Corollary from Theorem 10, note that for any $\alpha \neq 0$, the matrix $L+\alpha\,\bar{J}$, as well as $L+\bar{J}$, is a block lower triangular matrix with $v+1$ blocks. By item 2 of

Theorem 1, its $(v+1)$st diagonal block coincides with the corresponding block of $L+\bar{J}$. Using Lemma 5, we conclude that the other diagonal blocks are also nonsingular.

Proof of Theorem 11. By Theorems 2 and 3, $(L+\bar{J})\,\bar{J}=\bar{J}$. Premultiplying both sides of this identity by $(L+\bar{J})^{-1}$ (which exists by Theorem 10), we have

$$\bar{J}=(L+\bar{J})^{-1}\,\bar{J}\,. \tag{34}$$

Similarly,

$$\bar{J}(L+\bar{J})^{-1}=\bar{J}\,. \tag{35}$$

Denote $\widetilde{Q}=(L+\bar{J})^{-1}-\bar{J}$. Using (34) and Theorems 2 and 3, we obtain

$$\widetilde{Q}L=(L+\bar{J})^{-1}L-\bar{J}\,L=(L+\bar{J})^{-1}(L+\bar{J}-\bar{J})=I-(L+\bar{J})^{-1}\,\bar{J}=I-\bar{J}\,. \tag{36}$$

Similarly,

$$L\widetilde{Q}=I-\bar{J}\,. \tag{37}$$

By (34) and Theorem 2,

$$\widetilde{Q}\,\bar{J}=(L+\bar{J})^{-1}\,\bar{J}-\bar{J}^2=0.$$

Consequently, for any $\alpha\neq 0$,

$$(\widetilde{Q}+\alpha^{-1}\,\bar{J})(L+\alpha\,\bar{J})=I-\bar{J}+\bar{J}=I.$$

Hence, $\widetilde{Q}+\alpha^{-1}\,\bar{J}=(L+\alpha\,\bar{J})^{-1}$ and

$$\widetilde{Q}=(L+\alpha\,\bar{J})^{-1}-\alpha^{-1}\,\bar{J} \tag{38}$$

for any $\alpha\neq 0$.

By (36) and (37), $L\widetilde{Q}=\widetilde{Q}L=I-\bar{J}$, thus $\widetilde{Q}=(L+\alpha\,\bar{J})^{-1}-\alpha^{-1}\,\bar{J}$ satisfies condition (5) in the definition of group inverse. Let us prove that conditions (1) and (2) (common with the definition of Moore-Penrose inverse) are also fulfilled. Making use of Theorems 2 and 3 and identities (35)–(38), we obtain

$$\begin{aligned}&L\widetilde{Q}L=L(I-\bar{J})=L,\\&\widetilde{Q}L\widetilde{Q}=(I-\bar{J})\widetilde{Q}=\widetilde{Q}-\bar{J}\,\widetilde{Q}=\widetilde{Q}-\bar{J}\,(L+\bar{J})^{-1}+\bar{J}^2=\widetilde{Q}-\bar{J}+\bar{J}=\widetilde{Q},\end{aligned}$$

which completes the proof.

Proof of Proposition 7. Using Theorems 2, 3, and 5 and identities $(I+\tau L)^{-1}\bar{J}=\bar{J}$ (Corollary 3 from Theorem 7) and (34), we obtain

$$\begin{aligned}&\left(\lim_{\tau\to\infty}\tau\left((I+\tau L)^{-1}-\bar{J}\right)+\bar{J}\right)(L+\bar{J})\\&=\lim_{\tau\to\infty}\tau\left((I+\tau L)^{-1}L+(I+\tau L)^{-1}\,\bar{J}-\bar{J}\,L-\bar{J}^2\right)+\bar{J}\,L+\bar{J}^2\\&=\lim_{\tau\to\infty}\tau(I+\tau L)^{-1}L+\bar{J}=\lim_{\tau\to\infty}(I+\tau L)^{-1}(I+\tau L-I)+\bar{J}\\&=I-\lim_{\tau\to\infty}(I+\tau L)^{-1}+\bar{J}=I.\end{aligned}$$

Postmultiplying the first and last expressions by $(L+\bar{J})^{-1}$ and using Theorem 11 we obtain the required equation.

Proof of Proposition 8. Using Proposition 7 and Theorem 4, for any $i, j \in V(\mathit{\Gamma})$, find the limit

$$\ell^{\#}_{ij} = \lim_{\tau\to\infty} \tau\Big(q_{ij}(\tau) - \bar{J}_{ij}\Big) = \lim_{\tau\to\infty} \tau \left(\frac{\sum\limits_{k=0}^{n-v} \tau^k \varepsilon(\mathcal{F}_k^{j\to i})}{\sum\limits_{k=0}^{n-v} \tau^k \sigma_k} - \bar{J}_{ij} \right)$$

$$= \lim_{\tau\to\infty} \frac{\sum\limits_{k=0}^{n-v} \tau^{k+1} \varepsilon(\mathcal{F}_k^{j\to i}) - \sum\limits_{k=0}^{n-v} \tau^{k+1}\sigma_k \bar{J}_{ij}}{\sum\limits_{k=0}^{n-v} \tau^k \sigma_k}$$

$$= \lim_{\tau\to\infty} \frac{\sum\limits_{k=0}^{n-v-1} \tau^{k+1} \varepsilon(\mathcal{F}_k^{j\to i}) + \tau^{n-v+1}\varepsilon(\mathcal{F}_{n-v}^{j\to i}) - \sum\limits_{k=0}^{n-v-1} \tau^{k+1}\sigma_k \bar{J}_{ij} - \tau^{n-v+1}\sigma_{n-v} \bar{J}_{ij}}{\sum\limits_{k=0}^{n-v} \tau^k \sigma_k}.$$

By the definition of $\bar{J}$, we have $\tau^{n-v+1}\varepsilon(\mathcal{F}_{n-v}^{j\to i}) - \tau^{n-v+1}\sigma_{n-v} \bar{J}_{ij} = 0$. Therefore, in view of $\sigma_{n-v} \neq 0$, this yields

$$\ell^{\#}_{ij} = \frac{\varepsilon(\mathcal{F}_{n-v-1}^{j\to i}) - \sigma_{n-v-1} \bar{J}_{ij}}{\sigma_{n-v}},$$

which completes the proof of Proposition 8.

Proof of Lemma 2. By virtue of the identity $\bar{J}L = 0$ (Theorem 3), matrices LL^T and $ZZ^T = \bar{J}^T\bar{J} + LL^T$ commute, i.e., $LL^T(\bar{J}^T\bar{J} + LL^T) = (\bar{J}^T\bar{J} + LL^T)LL^T = (LL^T)^2$. Premultiplying and postmultiplying both sides of the first equality by $(ZZ^T)^{-1} = (\bar{J}^T\bar{J} + LL^T)^{-1}$, we obtain the desired $(\bar{J}^T\bar{J} + LL^T)^{-1}LL^T = LL^T(\bar{J}^T\bar{J} + LL^T)^{-1}$. The second statement is proved similarly.

Proof of Theorem 12. Let $L^{(+)} = L^T(ZZ^T)^{-1}$. Prove that the following four conditions are satisfied:

(1) $LL^{(+)}L = L$;

(2) $L^{(+)}LL^{(+)} = L^{(+)}$;

(3) $LL^{(+)}$ is symmetric;

(4) $L^{(+)}L$ is symmetric.

Condition 1. Using Lemma 2 and the identity $\bar{J}L = 0$ (Theorem 3), we obtain

$$\begin{aligned} LL^{(+)}L &= LL^T(\bar{J}^T\bar{J} + LL^T)^{-1}L = ((LL^T + \bar{J}^T\bar{J}) - \bar{J}^T\bar{J})(\bar{J}^T\bar{J} + LL^T)^{-1}L \\ &= (I - \bar{J}^T\bar{J}(\bar{J}^T\bar{J} + LL^T)^{-1})L = L - (\bar{J}^T\bar{J} + LL^T)^{-1}\bar{J}^T\bar{J}L = L. \end{aligned}$$

Condition 2. Using Lemma 2 and the identity $L^T\bar{J}^T = 0$, we have

$$\begin{aligned} L^{(+)}LL^{(+)} &= L^{(+)}LL^T(\bar{J}^T\bar{J} + LL^T)^{-1} = L^{(+)}((LL^T + \bar{J}^T\bar{J}) - \bar{J}^T\bar{J})(\bar{J}^T\bar{J} + LL^T)^{-1} \\ &= L^{(+)}(I - \bar{J}^T\bar{J}(\bar{J}^T\bar{J} + LL^T)^{-1}) = L^{(+)} - L^T(\bar{J}^T\bar{J} + LL^T)^{-1}\bar{J}^T\bar{J}(\bar{J}^T\bar{J} + LL^T)^{-1} \\ &= L^{(+)} - L^T\bar{J}^T\bar{J}(\bar{J}^T\bar{J} + LL^T)^{-2} = L^{(+)}. \end{aligned}$$

Condition 3. By Corollary from Lemma 2, the matrix $LL^{(+)} = LL^T(\bar{J}^T\bar{J} + LL^T)^{-1}$ is symmetric.

Condition 4. Since $(\bar{J}^T\bar{J} + LL^T)^{-1}$ is symmetric, $L^{(+)}L = L^T(\bar{J}^T\bar{J} + LL^T)^{-1}L$ is symmetric as well (see, e.g., [25], Theorem 4.1.3).

Theorem 12 is proved.

Proof of Proposition 10. By Theorem 7, $Q_{n-v} = \sum\limits_{i=0}^{n-v} \sigma_{n-v-i} \, (-L)^i$. Using Corollary 2 from Theorem 7, we obtain

$$p'_L(L) = L \sum_{i=0}^{n-v} \sigma_{n-v-i} \, (-L)^i = LQ_{n-v} = 0.$$

Proof of Proposition 11. Under the reversal of all arcs in a digraph, diverging and converging forests change places. Therefore, $P_1(\tau)$ and $P_2(\tau)$ are dual to each other.

Nonnegativity and *disconnection condition* for $P_1(\tau)$ and $P_2(\tau)$ and also item 1 of *diagonal maximality* for $P_1(\tau)$ and item 2 of *diagonal maximality* for $P_2(\tau)$ are proved with the help of Theorem 4 by the same argument as the corresponding conditions in [14].

For any $i, k, t \in V(G)$, if $\mathit{\Gamma}$ contains a path from i to k, $i \neq k \neq t$, and every path from i to t includes k, then $\mathcal{F}^{i \to t} \subset \mathcal{F}^{i \to k}$. This inclusion and Theorem 4 imply the fulfillment of item 1 of *transit property* for $P_1(\tau)$. Item 2 of *transit property* for $P_2(\tau)$ is proved similarly.

It was established in [13] (in the proof of Proposition 17) that if the weight of some arc (k, t) is increased by $\Delta\varepsilon_{kt}$ and the weights of all other arcs are preserved, then the increments of the entries of the matrix $P_1(\tau) = \left(p_{ij}^{(1)}(\tau)\right)$ are expressed as follows:

$$\Delta \, p_{ij}^{(1)}(\tau) = \frac{p_{tj}^{(1)}(\tau)\left(p_{ik}^{(1)}(\tau) - p_{it}^{(1)}(\tau)\right)}{(\Delta\varepsilon_{kt}\,\tau)^{-1} + p_{tt}^{(1)}(\tau) - p_{tk}^{(1)}(\tau)}, \quad i, j = 1, \ldots, n. \tag{39}$$

By *reversal property*, the corresponding formula for $P_2(\tau) = \left(p_{ij}^{(2)}(\tau)\right)$ is

$$\Delta \, p_{ij}^{(2)}(\tau) = \frac{p_{ik}^{(2)}(\tau)\left(p_{tj}^{(2)}(\tau) - p_{kj}^{(2)}(\tau)\right)}{(\Delta\varepsilon_{kt}\,\tau)^{-1} + p_{kk}^{(2)}(\tau) - p_{tk}^{(2)}(\tau)}, \quad i, j = 1, \ldots, n, \tag{40}$$

where $p_{ij}^{(2)}(\tau)$ is the total weight of in forests wherein j belongs to a tree converging to i.

Transcribe (39) for $\Delta \, p_{kt}^{(1)}(\tau)$:

$$\Delta \, p_{kt}^{(1)}(\tau) = \frac{p_{tt}^{(1)}(\tau)\left(p_{kk}^{(1)}(\tau) - p_{kt}^{(1)}(\tau)\right)}{(\Delta\varepsilon_{kt}\,\tau)^{-1} + p_{tt}^{(1)}(\tau) - p_{tk}^{(1)}(\tau)}.$$

Since $p_{ii}^{(1)}(\tau) > p_{ij}^{(1)}(\tau)$ for all $i, j = 1, \ldots, n$, we deduce $\Delta \, p_{kt}^{(1)}(\tau) > 0$, i.e., $P_1(\tau)$ satisfies the first part of item 1 of *monotonicity*.

Now represent $\Delta \, p_{kt}^{(2)}(\tau)$ using (40):

$$\Delta \, p_{kt}^{(2)}(\tau) = \frac{p_{kk}^{(2)}(\tau)\left(p_{tt}^{(2)}(\tau) - p_{kt}^{(2)}(\tau)\right)}{(\Delta\varepsilon_{kt}\,\tau)^{-1} + p_{kk}^{(2)}(\tau) - p_{tk}^{(2)}(\tau)}.$$

Since $p_{ii}^{(2)}(\tau) > p_{ji}^{(2)}(\tau)$ for all $i, j = 1, \ldots, n$, we have $\Delta \, p_{kt}^{(2)}(\tau) > 0$, i.e., $P_2(\tau)$ also satisfies the first part of item 1 of *monotonicity*.

Transcribing (39) for $\Delta \, p_{kt}^{(1)}(\tau)$, $\Delta \, p_{ki}^{(1)}(\tau)$, and $\Delta \, p_{ti}^{(1)}(\tau)$ and using item 1 of diagonal maximality, we conclude that $P_1(\tau)$ satisfies item 2 of monotonicity. Comparing the expressions for $\Delta \, p_{it}^{(1)}(\tau)$ and $\Delta \, p_{ik}^{(1)}(\tau)$ and using item 1 of transit property, we obtain item 3b of monotonicity. Along the same lines, the statement of item 3a holds if and only if $\left(p_{kk}^{(1)}(\tau) - p_{kt}^{(1)}(\tau)\right) - \left(p_{ik}^{(1)}(\tau) - p_{it}^{(1)}(\tau)\right) > 0.$

Since triangle inequality for accessibility measures is violated by $P_1(\tau)$ (as well as by all asymmetric measures), item $3a$ of monotonicity is not true. Similarly, $P_2(\tau)$ satisfies items 3 and $2b$, but violates item $2a$ of monotonicity.

Consider the digraph with vertex set $\{i,j,k,t\}$, arc set $\{(i,k),(k,t),(t,j)\}$, and arc weights $\varepsilon(i,k)=4$, $\varepsilon(k,t)=1$, and $\varepsilon(t,j)=4$. Then $P_1(1)$ is given by

$$P_1(1)=\begin{array}{c|c} \begin{matrix} i & j & k & t \end{matrix} & \\ \left\|\begin{matrix} 1 & 0.32 & 0.8 & 0.4 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0.08 & 0.2 & 0.1 \\ 0 & 0.4 & 0 & 0.5 \end{matrix}\right\| & \begin{matrix} i \\ j \\ k \\ t \end{matrix} \end{array}. \tag{41}$$

Since $p_{it}^{(1)}(1) > p_{kt}^{(1)}(1)$, item 2 of transit property is not satisfied.

Compare the increments $\Delta\, p_{kt}^{(1)}(1)$ and $\Delta\, p_{ij}^{(1)}(1)$ for an arbitrary $\Delta\varepsilon_{kt} > 0$:

$$\Delta\, p_{kt}^{(1)} = \frac{p_{tt}^{(1)}(1)\left(p_{kk}^{(1)}(1) - p_{kt}^{(1)}(1)\right)}{(\Delta\varepsilon_{kt})^{-1} + p_{tt}^{(1)}(1) - p_{tk}^{(1)}(1)}, \qquad \Delta\, p_{ij}^{(1)} = \frac{p_{tj}^{(1)}(1)\left(p_{ik}^{(1)}(1) - p_{it}^{(1)}(1)\right)}{(\Delta\varepsilon_{kt})^{-1} + p_{tt}^{(1)}(1) - p_{tk}^{(1)}(1)}.$$

Since $p_{tt}^{(1)}(1)\left(p_{kk}^{(1)}(1) - p_{kt}^{(1)}(1)\right) = 0.5(0.2-0.1) = 0.05$, $p_{tj}^{(1)}(1)\left(p_{ik}^{(1)}(1) - p_{it}^{(1)}(1)\right) = 0.4(0.8 - 0.4) = 0.16$ and the common denominator is positive, $\Delta\, p_{kt}^{(1)}(1) < \Delta\, p_{ij}^{(1)}(1)$ follows, i.e., the second part of item 1 of monotonicity is not satisfied.

It is easy to verify that item 1 of transit property and the second part of item 1 of monotonicity are violated by $P_2(\tau)$ in the same example.

Let us show that $P_3(\tau)$ does not satisfy *triangle inequality for accessibility measures*. Consider the digraph with vertex set $\{i,j,k,t\}$, arc set $\{(i,j),(j,k),\ (k,t),(t,i)\}$, and arc weights $\varepsilon(i,j) = 1$, $\varepsilon(j,k)=10$, $\varepsilon(k,t)=10$, and $\varepsilon(t,i)=1$. Here, $P_3(1)$ is as follows:

$$P_3(1)=\begin{array}{c|c} \begin{matrix} i & j & k & t \end{matrix} & \\ \left\|\begin{matrix} 0.6302 & 0.2233 & 0.1693 & 0.2233 \\ 0.2233 & 0.3724 & 0.1823 & 0.2747 \\ 0.1693 & 0.1823 & 0.1146 & 0.1823 \\ 0.2233 & 0.2747 & 0.1823 & 0.3724 \end{matrix}\right\| & \begin{matrix} i \\ j \\ k \\ t \end{matrix} \end{array}. \tag{42}$$

Triangle inequality for accessibility measures is violated, because $p_{ki}^{(3)}(1) + p_{kj}^{(3)}(1) - p_{ij}^{(3)}(1) > p_{kk}^{(3)}(1)$. Symmetry is a necessary condition of triangle inequality for accessibility measures. Therefore, $P_1(\tau)$ and $P_2(\tau)$ do not satisfy this inequality either.

Proof of Proposition 12. 1. To construct the required forest, it suffices to take a maximum out forest and to delete any arc directed to i (if such an arc exists) or *any* arc in the opposite case.

2. For the given path (chain subgraph) and any maximum out forest in Γ, consider their join and remove all arcs of the out forest that are directed to the vertices of the path but do not belong to the path. The resulting subgraph contains neither circuits nor vertices with indegree greater than one, i.e., it is an out forest. It contains at least $n-v-1$ arcs, consequently it belongs to $\mathcal{F}_{n-v-1} \cup \mathcal{F}_{n-v}$.

Proof of Proposition 13. Under the reversal of all arcs in Γ, diverging and converging forests change places. Hence, $P_1(\alpha)$ and $P_2(\alpha)$ are dual to each other. *Disconnection condition* follows from item 2 of Proposition 12.

Nonnegativity follows from the fact that the entries of $\bar{J}_{n-v}$ and $\bar{J}_{n-v-1}$ are proportional to the weights of some sets and from the nonnegativity of the coefficients in (23).

Item 1 of *diagonal maximality* and item 1 of *transit property* for $P_1(\alpha)$ in the nonstrict form (as well as items 2 of these conditions for $P_2(\alpha)$ in the nonstrict form) follow from the nonstrict inclusion of the sets of forests that determine the entries of matrices Q_{n-v} and Q_{n-v-1} under comparison.

Diagonal maximality (in the strict form) is violated, for example, for digraphs with vertex set $\{j,i,k,t\}$, arc set $\{(j,i),(i,k)(k,t)\}$, and arc weights $\varepsilon(j,i)=4$, $\varepsilon(i,k)=1$, and $\varepsilon(k,t)=1$. The matrices of out forests Q_{n-v} and Q_{n-v-1}, the matrices of in forests S_{n-v} and S_{n-v-1}, and the matrices $P_1(\alpha)$ and $P_2(\alpha)$ at $\alpha=4/13$ are as follows:

$$Q_{n-v}=\begin{array}{cc} \begin{matrix} j & i & k & t\end{matrix} & \\ \left\|\begin{matrix}4&0&0&0\\4&0&0&0\\4&0&0&0\\4&0&0&0\end{matrix}\right\| & \begin{matrix}j\\i\\k\\t\end{matrix}\end{array},\quad Q_{n-v-1}=\begin{array}{cc} \begin{matrix} j & i & k & t\end{matrix} & \\ \left\|\begin{matrix}9&0&0&0\\8&1&0&0\\4&1&4&0\\0&1&4&4\end{matrix}\right\| & \begin{matrix}j\\i\\k\\t\end{matrix}\end{array},\quad P_1(\alpha)=\begin{array}{cc} \begin{matrix} j & i & k & t\end{matrix} & \\ \left\|\begin{matrix}3.25&3&2&1\\0&0.25&0.25&0.25\\0&0&1&1\\0&0&0&1\end{matrix}\right\| & \begin{matrix}j\\i\\k\\t\end{matrix}\end{array},$$

$$S_{n-v}=\begin{array}{cc} \begin{matrix} j & i & k & t\end{matrix} & \\ \left\|\begin{matrix}0&0&0&4\\0&0&0&4\\0&0&0&4\\0&0&0&4\end{matrix}\right\| & \begin{matrix}j\\i\\k\\t\end{matrix}\end{array},\quad S_{n-v-1}=\begin{array}{cc} \begin{matrix} j & i & k & t\end{matrix} & \\ \left\|\begin{matrix}1&4&4&0\\0&4&4&1\\0&0&4&5\\0&0&0&9\end{matrix}\right\| & \begin{matrix}j\\i\\k\\t\end{matrix}\end{array},\quad P_2(\alpha)=\begin{array}{cc} \begin{matrix} j & i & k & t\end{matrix} & \\ \left\|\begin{matrix}0.25&1&1&1\\0&1&1&1.25\\0&0&1&2.25\\0&0&0&3.25\end{matrix}\right\| & \begin{matrix}j\\i\\k\\t\end{matrix}\end{array}.$$

In this example, item 1 of transit property for $P_1(\alpha)$ is violated (since $p_{ik}^{(1)}(\alpha)=p_{it}^{(1)}(\alpha)$), and so is item 2 of transit property for $P_2(\alpha)$ (since $p_{ik}^{(2)}(\alpha)=p_{jk}^{(2)}(\alpha)$).

Let us demonstrate that *triangle inequality for accessibility measures* is not satisfied by $P_3(\alpha)=(P_1(\alpha)+P_1^T(\alpha)+P_2(\alpha)+P_2^T(\alpha))/4$ in this example. Indeed,

$$P=P_3(\alpha)=\begin{array}{cc} \begin{matrix} j & i & k & t\end{matrix} & \\ \left\|\begin{matrix}1.75&1&0.75&0.5\\1&0.625&0.3125&0.375\\0.75&0.3125&1&0.8125\\0.5&0.375&0.8125&2.125\end{matrix}\right\| & \begin{matrix}j\\i\\k\\t\end{matrix}\end{array},$$

and since $p_{ij}^{(3)}(\alpha)+p_{it}^{(3)}(\alpha)-p_{jt}^{(3)}(\alpha)=0.875>p_{ii}^{(3)}(\alpha)=0.625$, this condition is violated.

Monotonicity is not satisfied on undirected graphs (see Proposition 10 in [15]), hence, it is violated for digraphs also.

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