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Matrix-Forest Theorems

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The Laplacian matrix of a graph G is $L(G) = D(G) - A(G)$, where $A(G)$ is the adjacency matrix and $D(G)$ is the diagonal matrix of vertex degrees. According to the Matrix-Tree Theorem, the number of spanning trees in G is equal to any cofactor of an entry of $L(G)$. A rooted forest is a union of disjoint rooted trees. We consider the matrix $W(G) = I + L(G)$ and prove that the (i, j)-cofactor of $W(G)$ is equal to the number of spanning rooted forests of G, in which the vertices i and j belong to the same tree rooted at i. The determinant of $W(G)$ equals the total number of spanning rooted forests, therefore the (i, j)-entry of the matrix $W^{-1}(G)$ can be considered as a measure of relative "forest-accessibility" of the vertex i from j (or j from i). These results follow from somewhat more general theorems we prove, which concern weighted multigraphs. The analogous theorems for (multi)digraphs are established. These results provide a graph-theoretic interpretation to the adjugate of the Laplacian characteristic matrix.

1. INTRODUCTION

Let G be a labeled graph on n vertices with adjacency matrix $A(G) = (a_{ij})$. The Laplacian (the Kirchhoff or the admittance) matrix of G is the n-by-n matrix $L(G) = (\ell_{ij})$ with $\ell_{ij} = -a_{ij}$ ($j \neq i$, $i, j = 1, \ldots, n$) and $\ell_{ii} = \sum_{j \neq i} a_{ij} = -\sum_{j \neq i} \ell_{ij}$ ($i = 1, \ldots, n$). According to the Matrix-Tree Theorem attributed to Kirchhoff (for its history, see [19]), any cofactor of an entry of $L(G)$ is equal to the number of spanning trees of G. Tutte (see [26]) has generalized this theorem to weighted multigraphs and multidigraphs. Bapat and Constantine [1] presented a version for graphs in which each edge is assigned a color. Merris [17] proposed an "edge version" of the Matrix-Tree Theorem and Moon [20] generalized it. Forman [9] considered the Kirchhoff theorem in a more general context of vector fields.

Another trend of literature studies the characteristic polynomial and the spectrum of the Laplacian matrix. For review of this literature we refer to [10, 11, 18]. We would like to mention here the research by Kelmans, who had published in 1965–1967 a series of results on the Laplacian characteristic polynomial and spectrum (see [13, 14], and the references therein), some of which were rediscovered later by other writers.

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