Discrete Mathematics 91 (1991) 277–282
North-Holland

A finite group attached to the laplacian of a graph

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Received 23 September 1988
Revised 30 January 1990

Abstract

Lorenzini, D.J., A finite group attached to the laplacian of a graph, Discrete Mathematics 91 (1991) 277–282.

Let $F = \text{diag}(\varphi_1, \ldots, \varphi_{n-1}, 0)$, $\varphi_1 \mid \cdots \mid \varphi_{n-1}$, denote the Smith normal form of the laplacian matrix associated to a connected graph G on n vertices. Let $\bar{h}$ denote the cardinal of the set $\{i \mid \varphi_i > 1\}$. We show that $\bar{h}$ is bounded by the number of independent cycles of G and we study some cases where these two integers are equal.

Let G be a connected graph with m edges, n vertices and adjacency matrix $A = (a_{ij})$. Let d_i denote the degree of the ith vertex and define the laplacian of G to be the matrix $M := D - A$ with $D = \text{diag}(d_1, \ldots, d_n)$. Let ${}^tJ = (1, \ldots, 1): \boldsymbol{Z}^n \to \boldsymbol{Z}$. We define $\Phi := \text{Ker}\, {}^tJ / \text{Im}\, M$, where M is thought of as a linear map $M: \boldsymbol{Z}^n \to \boldsymbol{Z}^n$. Let $\bar{h}$ denote the minimal number of generators of the group Φ. Let $\beta(G) = m - (n-1)$ be the number of independent cycles of G. In [**2**, 5.2] we showed that

$$\bar{h}(G) \leqslant \beta(G).$$

In the present paper, we recall two other descriptions of the group Φ and use them to characterize some families of graphs for which the equality $\bar{h}(G) = \beta(G)$ holds. We also give a new proof of the inequality $\bar{h}(G) \leqslant \beta(G)$.

The finite abelian group Φ can be described in terms of the Smith normal form $F = \text{diag}(\varphi_1, \ldots, \varphi_{n-1}, 0)$ of M (see [**2**, 1.4]). Any diagonal matrix $E = \text{diag}(e_1, \ldots, e_{n-1}, 0)$, row and column equivalent to M over the integers, induces an isomorphism

$$\Phi \cong \boldsymbol{Z}/e_1\boldsymbol{Z} \times \cdots \times \boldsymbol{Z}/e_{n-1}\boldsymbol{Z}.$$

The integers $\varphi_1 \mid \cdots \mid \varphi_{n-1}$ can be computed in the following way: $\varphi_i = \Delta_i / \Delta_{i-1}$ where $\Delta_0 = 1$ and Δ_i is the gcd of the determinants of the $i \times i$ minors of M. The

Elsevier Science Publishers B.V. (North-Holland)