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A finite group attached to the laplacian of a graph

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Abstract

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Let $F = \text{diag}(\varphi_1, \ldots, \varphi_{n-1}, 0)$, $\varphi_1 \mid \cdots \mid \varphi_{n-1}$, denote the Smith normal form of the laplacian matrix associated to a connected graph G on n vertices. Let $\bar{h}$ denote the cardinal of the set $\{i \mid \varphi_i > 1\}$. We show that $\bar{h}$ is bounded by the number of independent cycles of G and we study some cases where these two integers are equal.

Let G be a connected graph with m edges, n vertices and adjacency matrix $A = (a_{ij})$. Let d_i denote the degree of the ith vertex and define the laplacian of G to be the matrix $M := D - A$ with $D = \text{diag}(d_1, \ldots, d_n)$. Let ${}^tJ = (1, \ldots, 1) : \boldsymbol{Z}^n \to \boldsymbol{Z}$. We define $\Phi := \text{Ker}\, {}^tJ / \text{Im}\, M$, where M is thought of as a linear map $M : \boldsymbol{Z}^n \to \boldsymbol{Z}^n$. Let $\bar{h}$ denote the minimal number of generators of the group Φ. Let $\beta(G) = m - (n-1)$ be the number of independent cycles of G. In [**2**, 5.2] we showed that

$$\bar{h}(G) \leq \beta(G).$$

In the present paper, we recall two other descriptions of the group Φ and use them to characterize some families of graphs for which the equality $\bar{h}(G) = \beta(G)$ holds. We also give a new proof of the inequality $\bar{h}(G) \leq \beta(G)$.

The finite abelian group Φ can be described in terms of the Smith normal form $F = \text{diag}(\varphi_1, \ldots, \varphi_{n-1}, 0)$ of M (see [**2**, 1.4]). Any diagonal matrix $E = \text{diag}(e_1, \ldots, e_{n-1}, 0)$, row and column equivalent to M over the integers, induces an isomorphism

$$\Phi \cong \boldsymbol{Z}/e_1\boldsymbol{Z} \times \cdots \times \boldsymbol{Z}/e_{n-1}\boldsymbol{Z}.$$

The integers $\varphi_1 \mid \cdots \mid \varphi_{n-1}$ can be computed in the following way: $\varphi_i = \Delta_i / \Delta_{i-1}$ where $\Delta_0 = 1$ and Δ_i is the gcd of the determinants of the $i \times i$ minors of M. The

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integer $|\Phi| = \Delta_{n-1} = \varphi_1 \cdots \varphi_{n-1}$ is well known: it equals the number of spanning trees of G (see [**1**, 6.3]). The integer $\bar{h}(G)$ equals the cardinal of the set $\{i \mid \varphi_i > 1\} = \{i \mid \Delta_i > 1\}$. As we mentioned above, $\bar{h} \leqslant \beta$; obviously, $\bar{h} \leqslant n - 1$.

We say that a graph is *simple* if any two of its vertices are linked by at most one edge (i.e. $a_{ij} \leqslant 1$, $\forall i \neq j$). A *2-graph* is a graph with $d_i \geqslant 2$, $\forall i = 1, \ldots, n$.

Lemma. *If G is a simple 2-graph, then $\bar{h} \leqslant n - 3$, unless G is the complete graph on n vertices, in which case the Smith normal form of the laplacian of G is* $F = \operatorname{diag}(1, n, \ldots, n, 0)$. *In particular, the group Φ attached to a complete graph on n vertices is isomorphic to $(\mathbf{Z}/n\mathbf{Z})^{n-2}$.*

Proof. It is clear that $\Delta_1 = 1$ when G is simple. Since G is a 2-graph, we can find, for each vertex v_i, a minor of the form

$$\begin{pmatrix} -1 & -q \\ d_i & -1 \end{pmatrix} \quad \text{with } 1 \geqslant q \geqslant 0.$$

Assume that $\bar{h} = n - 2$, i.e. that $\Delta_2 > 1$: the case $q = 0$ is then excluded. In particular, for any i, Δ_2 divides $d_i + 1$. If a vertex v_j is not linked to v_i, we have a minor

$$\begin{pmatrix} d_i & 0 \\ 0 & d_j \end{pmatrix}$$

and it follows that Δ_2 divides $d_i d_j$. Hence $d_j + 1$ and $-d_j = d_i d_j - (d_i + 1)d_j$ are divisible by Δ_2. This cannot happen because we assumed $\Delta_2 \geqslant 2$.

In case G is the complete graph on n vertices, the 2×2 minors of M are of the form

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 \\ n-1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} n-1 & -1 \\ -1 & n-1 \end{pmatrix}$$

so that $\Delta_2 = n$. Since $\Delta_{n-1}(G) = n^{n-2}$, we conclude that $F = \operatorname{diag}(1, n, \ldots, n, 0)$. $\square$

Our next proposition classifies the graphs with $\bar{h} = \beta = n - 1$ and $\bar{h} = \beta = n - 2$. Let T be a tree with n vertices and denote by M its laplacian matrix. Let $\boldsymbol{T}$ be the graph corresponding to the matrix $2M$ and $\boldsymbol{T}'$ be a graph obtained from $\boldsymbol{T}$ by removing a unique edge.

Proposition 1. *Let G be a 2-graph.*

(i) *$\bar{h} = \beta = n - 1$ iff $G = \boldsymbol{T}$ for some tree T,*

(ii) *$\bar{h} = \beta = n - 2$ iff $G = \boldsymbol{T}'$ for some tree T.*

Proof. It is straightforward to check that $\beta(\boldsymbol{T}) = n - 1$ and $\beta(\boldsymbol{T}') = n - 2$. We computed $F(\boldsymbol{T}) = \operatorname{diag}(2, 2, \ldots, 2, 0)$ and $F(\boldsymbol{T}') = \operatorname{diag}(1, 2, \ldots, 2, 0)$ in [**2**, Corollary 2.2]. These facts can also be proven using the remark below.

Let G with $\bar{h} = \beta = n - 1$ and adjacency matrix $A = (a_{ij})$. We have

$$\sum_{i<j} a_{ij} = m = \beta - (n-1) = 2(n-1).$$

Since any connected graph has at least $(n-1)$ nonzero coefficients a_{ij} and $\Delta_1 \mid a_{ij}$, we get

$$2(n-1) = \sum_{i<j} a_{ij} \geq \Delta_1 \cdot (n-1).$$

Our assumption that $\bar{h} = n - 1$ implies that $\Delta_1 > 1$ and the above inequality forces $\Delta_1 = 2$. So $M = 2N$ with N associated to a tree.

Let G with $\bar{h} = \beta = n - 2$. Since

$$2n - 3 = m = \sum_{k<l} a_{kl} \geq \min\{a_{kl} \neq 0\} \cdot (n-1),$$

there must exist $i < j$ with $a_{ij} = 1$. Let $v_1, \ldots, v_n$ denote the vertices of G. Write $\{v_1, \ldots, v_n\} = \{v_i, v_j\} \sqcup A_0 \sqcup A_1 \sqcup B$, where A_0 is the set of vertices not linked to either v_i or v_j, A_1 is the set of vertices linked to exactly one of the vertices v_i or v_j and B is the set of vertices linked to both v_i and v_j. If $v_k \in A_0 \sqcup A_1$, say with $a_{ik} = 0$, every coefficient a_{ek} of the kth row of M is divisible by Δ_2 because of the existence of a minor of the form

$$\begin{pmatrix} 0 = a_{ik} & -1 = -a_{ij} \\ -a_{ek} & * \end{pmatrix}.$$

We claim that $m \geq 1 + 2b + \Delta_2 \cdot a$, where $a = |A_0 \sqcup A_1|$ and $b = |B|$. The vertices v_i and v_j are linked by one edge; each vertex in B defines two distinct edges. It is clear that each element of A_1 defines at least Δ_2 edges of G, distinct from each edge previously defined. Consider now the full subgraph of G spanned by the vertices in A_0. We get a disjoint union of connected components $A_0 = C_1 \sqcup \cdots \sqcup C_s$. Each C_i has at least $\Delta_2 \cdot (|C_i| - 1)$ distinct edges. Since each C_i is linked to the rest of the graph (G is connected), each C_i defines $\Delta_2 \cdot |C_i|$ distinct edges of G.

Our assumption that $\bar{h} = n - 2$ implies that $\Delta_2 > 1$. This implies that the inequality

$$m = 2(a + b + 2) - 3 \geq \Delta_2 \cdot a + 2b + 1$$

is an equality. In particular,

(1) $\Delta_2 = 2$.

(2) Each C_i is a 'double tree' linked by a double edge to a vertex in A_1.

(3) Each element of A_1 is linked to v_i or v_j by a double edge.

(4) Elements of B are linked only to v_i and v_j.

Let G' be the subgraph of G spanned by v_i, v_j and the elements of B. We have $\beta(G') = (2b+1) - [(b+2) - 1] = b$. Using [2, Theorem 2.1], we obtain an isomorphism

$$\Phi(G) \cong (\mathbf{Z}/2\mathbf{Z})^a \times \Phi(G').$$

Since by hypothesis, the minimal number of generators of $\Phi(G)$ equals $n-2 = a+b$, $\bar{h}(G')$ must at least equal $b = \beta(G')$. If $n' = b+2$ denotes the number of vertices of G', we see that G' has to satisfy $\bar{h}(G') = \beta(G') = n' - 2$. As shown in the previous lemma, G' has to be a complete graph unless $b = 0$. If $b > 0$, G' is a complete graph only when $b = 1$. But if $b = 1$, $F(G') = \text{diag}(1, 3, 0)$ and

$$\Phi(G) \cong (\mathbf{Z}/2\mathbf{Z})^{a-1} \times \mathbf{Z}/6\mathbf{Z}$$

would be generated by $n-3$ elements, a contradiction. Hence $b = 0$ and $G = \boldsymbol{T}'$ for some tree T. □

Remark. Let G and G' be any connected graphs. Let v denote a vertex of G and w a vertex of G'. Let Γ be the graph obtained as the union of G and G' with the vertex w identified to the vertex v. One checks easily that $\Phi(\Gamma) \cong \Phi(G) \times \Phi(G')$. In fact, if we number the vertices of G by $v_1, \ldots, v_s = v = w$ and the vertices of G' by $w = v_s, \ldots, v_{s+t}$, we see that the laplacian M of Γ is almost made up of two blocks. By adding all rows to the sth row and all columns to the sth column, M becomes equivalent to a matrix made up of two disjoint blocks; the Smith normal form of the top left (resp. bottom right) corner block can be used to compute the Smith normal form of the laplacian of G (resp. G').

The graphs of the form $\boldsymbol{T}$ or $\boldsymbol{T}'$ considered in the previous proposition can be generalized using the above process to give a more general family of graphs with $\bar{h} = \beta$. Let L_q denote the loop of length $q > 1$, the only connected graph with q vertices, q edges and all degrees equal to two. It is not hard to check that the group $\Phi(L_q)$ is cyclic of order q: in this case, $\bar{h} = \beta = 1$. We can then build up, using the above construction, 'trees of loops L_q' and obtain in this way connected graphs with $\bar{h} = \beta$.

Example. We give now an example of a simple graph with $\bar{h} = \beta = n - 3$. Let $K_{p,q}$ be a bipartite graph; a row and column reduction of its laplacian matrix M gives a diagonal matrix

$$E = \text{diag}(1, 1, \underbrace{p, \ldots, p}_{q-2}, \underbrace{q, \ldots, q}_{p-2}, pq, 0).$$

In the case of $K_{2,q}$, the matrix E is its own Smith normal form $F = \text{diag}(1, 1, 2, \ldots, 2, 2q, 0)$; hence $K_{2,q}$ is a simple graph with $\bar{h} = n - 3 = \beta$. Note that $K_{2,q}$ does not belong to the family of graphs described in the previous remark.

Let Γ be a graph with 2 vertices linked by exactly q edges. The graph $K_{2,q}$ is obtained from Γ by 'dividing each edge of Γ in two' and the graph obtained in this way has the property $\bar{h} = \beta$. We generalize these facts in our next proposition. Let G be any graph. By 'dividing an edge e in k edges', we mean the following operation; remove the edge e linked to the vertices v and v' and introduce $k-1$ new vertices $w_1, \ldots, w_{k-1}$ of degree two such that w_i is linked to w_{i+1}, w_1 is linked to v and w_{k-1} is linked to v'.

Proposition 2. *Let G be a connected graph with β independent cycles. Let G_k be the graph obtained by dividing each edge of G in k edges. One has the following exact sequence of abelian groups*:

$$0 \to (\mathbf{Z}/k\mathbf{Z})^{\beta} \to \Phi(G_k) \to \Phi(G) \to 0.$$

In particular, for any positive integer $k > 1$, $\bar{h}(G_k) = \beta(G_k)$.

To prove this proposition, we need to introduce a different description of Φ. Recall that $M = B \cdot B^t$ where B is the incidence matrix of G. We want to use this factorization of M to define Φ in terms of B. This point of view is 'well known from the specialists' in arithmetic geometry. The author learned the following formalism from conversations with Kenneth Ribet.

Let E and V denote respectively the set of edges and the set of vertices of G. $\mathbf{Z}(E)$ and $\mathbf{Z}(V)$ denote the free abelian groups on these two sets. Choose an orientation for each edge of G. This orientation is given by two maps called tip, tail: $E \to V$. Let $j: \mathbf{Z}(E) \to \mathbf{Z}(V)$ be defined by $j(e) = \mathrm{tip}(e) - \mathrm{tail}(e) \in \mathbf{Z}(V)$. The map j is represented by the incidence matrix of G. Let X denote its kernel and $\gamma: X \hookrightarrow \mathbf{Z}(E)$ the natural inclusion map. It is a free $\mathbf{Z}$-module of rank $\beta(G) = m - (n-1)$. For any free $\mathbf{Z}$-module T with basis $t_1, \ldots, t_s$, let $T^* := \mathrm{Hom}(T, \mathbf{Z})$. Recall that one has an isomorphism $\delta: T \xrightarrow{\sim} T^*$ defined by $\delta(t_i)(t_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. One checks that the group $\Phi(G)$ is isomorphic to the cokernel of the map $X \to X^*$ defined by

$$X \overset{\gamma}{\hookrightarrow} \mathbf{Z}(E) \xrightarrow{\sim} \mathbf{Z}(E)^* \xrightarrow{\gamma^*} X^*.$$

It follows immediately from this description of Φ that:

Corollary. *$\Phi(G)$ is minimally generated by at most $\beta(G)$ elements since X^* is free of rank β.*

Let E_k, V_k denote the sets of edges and vertices of G_k; let $j_k: \mathbf{Z}(E_k) \to \mathbf{Z}(V_k)$ be defined as above. We have two natural maps $\pi: E_k \to E$ and $i: V \hookrightarrow V_k$. The orientation of G being fixed, orient each vertex in $\pi^{-1}(e)$, $e \in E$ such that

$$\sum_{f \in \pi^{-1}(e)} \mathrm{tip}(f) - \mathrm{tail}(f) = i(\mathrm{tip}(e)) - i(\mathrm{tail}(e)).$$

We can then define two maps

$$\mu : \mathbf{Z}(E) \hookrightarrow \mathbf{Z}(E_k) \text{ by } \mu(e) = \sum_{\pi(f)=e} f$$

and

$$\eta : \mathbf{Z}(V) \hookrightarrow \mathbf{Z}(V_k) \quad \text{by} \quad \eta(v) = i(v).$$

The following diagram is commutative:

$$\begin{array}{ccc} \mathbf{Z}(E) & \xrightarrow{j} & \mathbf{Z}(V) \\ \mu \downarrow & & \downarrow \eta \\ \mathbf{Z}(E_k) & \xrightarrow{j_k} & \mathbf{Z}(V_k) \end{array}$$

This diagram induces an injection $\alpha : X = \operatorname{Ker} j \to X_k = \operatorname{Ker} j_k$. Since X and X_k have same rank, coker(α) is finite and injects in coker(μ), which is torsion free. Hence α is an isomorphism. We can use this isomorphism to describe $\Phi_k := \Phi(G_k)$ as the cokernel of the map $X \to X^*$ defined by

$$X \to \mathbf{Z}(E) \xrightarrow{\sim} \mathbf{Z}(E)^* \xrightarrow{\text{Mult. by } k} \mathbf{Z}(E)^* \to X^*.$$

Proof of Proposition 2. The above discussion shows how to define a surjective map $\Phi_k \to \Phi$ whose kernel is killed by k. The group Φ_k can be generated by $\beta(G_k) = \beta(G) = \beta$ elements. Hence the kernel of the map $\Phi_k \to \Phi$ can also be generated by β elements. Therefore, in order to conclude the proof of the proposition, we only need to show that this kernel has order k^β. Recall that the order of $\Phi(G)$ equals the number of spanning trees of G; it is easy to check that each spanning tree of G defines k^β distinct spanning trees of G_k. Hence $|\Phi_k| = k^\beta |\Phi|$. □

References

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