Forest matrices around the Laplacian matrix

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Abstract

We study the matrices Q_k of in-forests of a weighted digraph Γ and their connections with the Laplacian matrix L of Γ. The (i,j) entry of Q_k is the total weight of spanning converging forests (*in-forests*) with k arcs such that i belongs to a tree rooted at j. The forest matrices, Q_k, can be calculated recursively and expressed by polynomials in the Laplacian matrix; they provide representations for the generalized inverses, the powers, and some eigenvectors of L. The normalized in-forest matrices are row stochastic; the normalized matrix of maximum in-forests is the eigenprojection of the Laplacian matrix, which provides an immediate proof of the Markov chain tree theorem. A source of these results is the fact that matrices Q_k are the matrix coefficients in the polynomial expansion of $\mathrm{adj}(\lambda I + L)$. Thereby they are precisely Faddeev's matrices for $-L$.

AMS classification: 05C50; 15A48

Keywords: Weighted digraph; Laplacian matrix; Spanning forest; Matrix-forest theorem; Leverrier-Faddeev method; Markov chain tree theorem; Eigenprojection; Generalized inverse

1 Introduction

According to the matrix-tree theorem, the (i,j) cofactor of the Laplacian matrix of a weighted digraph equals the total weight of spanning converging trees rooted at vertex i of the digraph.

Fiedler and Sedláček [25] proved that the principal minor of the Laplacian matrix resulting by the removal of the rows and columns indexed by a set $\mathcal{J}$ is equal to the total weight of in-forests with $|\mathcal{J}|$ trees rooted at the vertices of $\mathcal{J}$.

These results are generalized by the *all minors matrix tree theorem* [17, 10] (see also [53]) which expresses arbitrary minors of the Laplacian matrix in terms of in-forests of the digraph.

We study the matrices, Q_k, of a digraph's in-forests: the (i,j) entry of Q_k is the total weight of in-forests with k arcs where i belongs to a tree converging to j. In this paper, we show that the forest matrices can be recursively calculated and represented by simple polynomials in the Laplacian matrix L; in turn, the powers of L are linear combinations of Q_k's. Further, we demonstrate that the forest matrices are useful to interpret a number

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